

LUIGI D'ONOFRIO

LUIGI GRECO

On the regularity of solutions to a nonvariational elliptic equation

Annales de la faculté des sciences de Toulouse 6^e série, tome 11, n° 1 (2002), p. 47-56

http://www.numdam.org/item?id=AFST_2002_6_11_1_47_0

© Université Paul Sabatier, 2002, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (<http://picard.ups-tlse.fr/~annales/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

On the regularity of solutions to a nonvariational elliptic equation ^(*)

LUIGI D'ONOFRIO ⁽¹⁾ AND LUIGI GRECO ^{(1),(2)}

RÉSUMÉ. — Nous réalisons une étude détaillée du degré de régularité de toutes les solutions d'une équation elliptique aux dérivées partielles non-variationnelle. Cette équation a souvent été une source de contre-exemples importants en théorie des équations aux dérivées partielles.

ABSTRACT. — We make a detailed study of the degree of regularity of all solutions of a nonvariational elliptic partial differential equation. Such equation has often turned out to be a source of counter examples which are critical to the theory of PDE's.

1. Introduction

In this paper we study solutions to the elliptic partial differential equation of nondivergence form

$$Lu = \sum_{ij=1}^n a_{ij} D_{ij}^2 u = 0 \quad (1.1)$$

with the coefficient matrix

$$A = (a_{ij}) = \frac{1}{K} I + \left(K - \frac{1}{K} \right) \frac{x \otimes x}{|x|^2}. \quad (1.2)$$

^(*) Reçu le 23 octobre 2001, accepté le 11 décembre 2002

⁽¹⁾ Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", via Cintia, 80126 Napoli, Italy.

e-mail : ldonofri@unina.it ; luigreco@unina.it

⁽²⁾ Was supported by M.U.R.S.T., P.R.I.N. 2000 "Calcolo delle Variazioni".

Here I denotes the identity matrix of order n , $x \otimes x$ is the matrix whose entries are $x_i x_j$, and $K \geq 1$. We recall that the coefficient matrix A at (1.2) is often used to provide interesting examples in the theory of PDE's, both variational and nonvariational ones [6], [10], [11], see also [3]. The entries a_{ij} are not continuous at the origin 0; they are weakly differentiable with derivatives in $L^{n,\infty} - L^n$ near 0, compare with [5], [9]. In [1] equations with weakly differentiable coefficients whose derivatives are in $L^{n,\infty}$ are considered. However, for K large, Equation (1.1) does not fall under the scope of the results of [1]. We note also that for large K (and $n \geq 3$) Equation (1.1) is not of Cordes type; for equations of Cordes type we refer to [13].

We consider strong solutions, namely twice weakly differentiable functions u satisfying the Equation (1.1) a.e., and examine their behavior near the origin 0 of \mathbb{R}^n in the spirit of the results of [8], aiming to show that all solutions of Equation (1.1) have a definite degree of regularity at 0, which is precisely among those exhibited by certain particular solutions. The work [8] concerns a variational equation. As it is well known, the theory of non-divergence equations with discontinuous coefficients in dimension $n \geq 3$ is less complete than the one in the divergence case. The argument used in [8] does not extend to our context, hence our approach here is completely different.

We consider the Dirichlet problem for equation (1.1) in a ball centered at 0 and investigate how solutions depend on the boundary data. It is not restrictive to consider the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$. So we consider the problem

$$\begin{cases} Lu = 0 & \text{in } B \\ u = \varphi & \text{on } \partial B \end{cases} \quad (1.3)$$

where φ is a given function on $\partial B = \{x : |x| = 1\}$. We shall show that for each $\varphi \in C^0(\partial B)$, problem (1.3) has a unique solution of class $W_{\text{loc}}^{2,n}(B) \cap C^0(\bar{B})$. Of course, uniqueness is a consequence of the celebrated maximum principle of Aleksandrov-Bakelman-Pucci, see e.g. [7]. Concerning the existence of a solution, as may be expected, there are a number of ways to get it. Here we use the familiar method of superposition of solutions [4], which is both elementary and yields a representation formula for the solution.

The preliminary step in doing so is to find a decomposition for the boundary data, and for this we recall [12] that a complete orthogonal system in

On the regularity of solutions to a nonvariational elliptic equation

$L^2(\partial B)$ can be formed by (the restriction to ∂B of) harmonic homogeneous polynomials. Thus we find the expansion

$$\varphi = \sum_{l=0}^{\infty} H_l, \quad (1.4)$$

where H_l is a harmonic homogeneous polynomial of degree $l = 0, 1, \dots$. Note that these polynomials in the decomposition (1.4) are uniquely determined. Then, for each l we consider the Dirichlet problem

$$\begin{cases} Lv_l = 0 & \text{in } B \\ v_l = H_l & \text{on } \partial B \end{cases} \quad (1.5)$$

It can be shown (see Lemma 2.1 below) that problem (1.5) has a solution v_l of the form

$$v_l(x) = |x|^{\lambda_l} H_l(x), \quad (1.6)$$

where λ_l is a nonpositive number depending only on n, K and l , not on H_l . The final step will be then to prove that for $\varphi \in C^0(\partial B)$, the function

$$u = \sum_{l=0}^{\infty} |x|^{\lambda_l} H_l \quad (1.7)$$

belongs to $W_{\text{loc}}^{2,n}(B) \cap C^0(\overline{B})$ and solves (1.3).

Using formula (1.7), as mentioned we extend the results of [8] proved in the variational case, to the nonvariational equation (1.1). Essentially, these results read as follows. Defining the moment of u by setting

$$w(r) = \int_{\partial B_r} u H \, ds, \quad 0 < r \leq 1, \quad (1.8)$$

where H is an arbitrary harmonic homogeneous polynomial of degree $l \in \mathbb{N}$, if $w(1) \neq 0$, then the solution u of $Lu = 0$ has at the origin 0 at most the same regularity as the function $v = |x|^{\lambda_l} H$. Note that $w(1)$ depends on the boundary value φ of u : $w(1) = \int_{\partial B} \varphi H \, ds$.

2. The Dirichlet problem

In this section we solve the Dirichlet problem (1.3) for $\varphi \in C^0(\partial B)$, using the method of superposition of solution, hence we begin by finding a solution to the problem

$$\begin{cases} Lv = 0 & \text{in } B \\ v = H & \text{on } \partial B \end{cases} \quad (2.1)$$

for H a harmonic homogeneous polynomial. Let l be the degree of H . Clearly, for all $\lambda \in \mathbb{R}$, the function $|x|^\lambda H$ satisfies the boundary condition. On the other hand, an elementary calculation shows that for $x \neq 0$

$$L(|x|^\lambda H) = K|x|^{\lambda-2} H \left[\lambda^2 + \left(2l + \frac{n-1}{K^2} - 1 \right) \lambda + \left(1 - \frac{1}{K^2} \right) l(l-1) \right].$$

Then we find a solution to problem (2.1) of the following type:

$$v = |x|^{\lambda_l} H \tag{2.2}$$

with $\lambda_0 = \lambda_1 = 0$, and for $l \geq 2$

$$\lambda_l = \lambda_l(n, K) = -l - \frac{n-1}{2K^2} + \frac{1}{2} + \frac{1}{2} \sqrt{\left(\frac{n-1}{K^2} - 1 \right)^2 + 4l \frac{n-2+l}{K^2}}. \tag{2.3}$$

Thus for $l = 0$ and $l = 1$, $v = H \in C^\infty(\mathbb{R}^n)$. For all l , we have $v \in C^\infty(\mathbb{R}^n - \{0\})$. To study the regularity of v at 0 for $l \geq 2$, we examine the quantity

$$l + \lambda_l = -\frac{n-1}{2K^2} + \frac{1}{2} + \frac{1}{2} \sqrt{\left(\frac{n-1}{K^2} - 1 \right)^2 + 4l \frac{n-2+l}{K^2}}. \tag{2.4}$$

Note that the function $l \mapsto l + \lambda_l$ is increasing and diverging at $+\infty$ as $l \rightarrow +\infty$. The following is trivially seen

LEMMA 2.1. — *For all $l \geq 2$ we have $l + \lambda_l > 1$ and*

$$\sigma = \inf \left\{ \frac{l + \lambda_l - 1}{l} : l \geq 2 \right\} > 0.$$

Hence $v \in C^1(\mathbb{R}^n)$, $\forall l \geq 2$. Moreover, the regularity of v increases with $l \geq 2$.

We shall use the following property of harmonic homogeneous polynomials, see e.g. [12], Appendix C on pg. 274-276; if H has degree $l \geq 1$, we have:

$$\sup_B |D^k H| \leq C(n, k) l^{n/2+k} \left\{ \int_{\partial B} |H|^2 ds \right\}^{1/2} \quad k = 0, 1, \dots \tag{2.5}$$

Now we prove some estimates for functions of the form

$$v = |x|^\lambda H,$$

H being a harmonic polynomial homogeneous of degree $l \geq 2$, and $l + \lambda - 1 \geq \sigma l > 0$.

On the regularity of solutions to a nonvariational elliptic equation

LEMMA 2.2. — *We have the following*

$$\|v\|_{L^2(B)} \leq \|H\|_{L^2(\partial B)}, \quad (2.6)$$

$$|D^2v| \leq 5|x|^\lambda |D^2H| \quad (2.7)$$

and for all $\rho \in]0, 1[$

$$\|D^2v\|_{L^n(B_\rho)} \leq C(n, \sigma) \rho^{\sigma l} l^{n+1} \|H\|_{L^2(\partial B)}. \quad (2.8)$$

Proof. — Integrating in spherical coordinates we find

$$\|v\|_{L^2(B)}^2 = \|H\|_{L^2(\partial B)}^2 \int_0^1 r^{2(l+\lambda)} dr$$

which immediately gives (2.6).

To prove (2.7) we compute

$$D^2v = HD^2|x|^\lambda + (DH \otimes D|x|^\lambda + D|x|^\lambda \otimes DH) + |x|^\lambda D^2H$$

and hence

$$|D^2v| \leq |H| |D^2|x|^\lambda| + 2|DH| |D|x|^\lambda| + |x|^\lambda |D^2H|.$$

Moreover

$$|D|x|^\lambda| = |\lambda| |x|^{\lambda-1}, \quad |D^2|x|^\lambda| \leq |\lambda| (|\lambda - 2| + 1) |x|^{\lambda-2}$$

and taking into account homogeneity of H

$$l|H| \leq |DH| |x|, \quad (l-1)|DH| \leq |D^2H| |x|, \quad l(l-1)|H| \leq |D^2H| |x|^2.$$

We then conclude, recalling that $1 - l < \lambda \leq 0$

$$|D^2v| \leq \left(\frac{(l-1)(l+2)}{l(l-1)} + 2 \frac{l-1}{l-1} + 1 \right) |x|^\lambda |D^2H| \leq 5|x|^\lambda |D^2H|.$$

To prove (2.8) we use (2.7) and get

$$\|D^2v\|_{L^n(B_\rho)} \leq \frac{5 \rho^{\sigma l}}{(n\sigma l)^{1/n}} \|D^2H\|_{L^n(\partial B)}. \quad (2.9)$$

Inequality (2.8) now follows easily using (2.5) for $k = 2$.

Recalling the expansion at (1.4) it is natural to consider the series at (1.7), that is

$$\sum_{l=0}^{\infty} v_l = \sum_{l=0}^{\infty} |x|^{\lambda_l} H_l. \quad (2.10)$$

For each $l = 0, 1, \dots$, here v_l is the solution to problem (2.1) and is defined by formulas (2.2)-(2.3). By means of the estimates of Lemma 2.2, we can easily see the following

LEMMA 2.3. — *The series at (2.10) converges in $L^2(B)$ and in $W_{\text{loc}}^{2,n}(B)$. Its sum u solves the equation $Lu = 0$.*

For the proof, it is enough to note that by Bessel inequality we have

$$\sum_{l=0}^{\infty} \|H_l\|_{L^2(\partial B)}^2 < +\infty,$$

hence applying (2.6) and (2.8) to each term v_l proves the claim.

Remark 2.1. — Actually, the series at (2.10) converges in $L^2(\partial B_r)$, for all $r \in]0, 1]$.

We are now in a position to state the main result of this section

PROPOSITION 2.1. — *For every $\varphi \in C^0(\partial B)$ the Dirichlet problem*

$$\begin{cases} Lu = 0 & \text{in } B \\ u = \varphi & \text{on } \partial B \end{cases}$$

has a unique solution of class $W_{\text{loc}}^{2,n}(B) \cap C^0(\bar{B})$. Moreover the solution is expressed by

$$u = \sum_{l=0}^{\infty} |x|^{\lambda_l} H_l. \quad (2.11)$$

What remains is to show the continuity of $u = \sum_{l=0}^{\infty} v_l$ on \bar{B} . Clearly, by Lemma 2.3 and the Sobolev embedding theorem, u is continuous in B . On the other hand, the series at (1.4) need not converge uniformly even for $\varphi \in C^0(\partial B)$. For this reason, to prove continuity up to the boundary we do not rely directly on the expansion (1.4). Instead, we recall how (1.4) is usually proved. We approximate φ uniformly by polynomials φ_m , $m = 1, 2, \dots$. Each φ_m on ∂B coincides with a finite sum of harmonic homogeneous polynomials

$$\varphi_m = \sum_{l=0}^{\delta_m} H_{m,l} \quad \text{on } \partial B,$$

see [12], pg. 70. Here $H_{m,l}$ has degree l . Then it is easily seen that the polynomials occurring in the expansion at (1.4) are given by $H_l = \lim_m H_{m,l}$ in $L^2(\partial B)$. (We mean that $H_{m,l} = 0$ if $l > \delta_m$.) This implies that $u = \lim_m w_m$ in $L^2(B)$, where

$$w_m = \sum_{l=0}^{\delta_m} |x|^{\lambda_l} H_{m,l}, \quad m = 1, 2, \dots$$

Note that $w_m \in C^0(\overline{B})$. On the other hand w_m solves the problem

$$\begin{cases} Lw_m = 0 & \text{in } B \\ w_m = \varphi_m & \text{on } \partial B \end{cases}$$

As $\{\varphi_m\}$ converges uniformly, by the maximum principle $\{w_m\}$ converges uniformly on \overline{B} . Clearly the limit is u , which therefore belongs to $C^0(\overline{B})$.

Remark 2.2. — Actually, the series at (2.10) is converging in a much stronger way than merely in the sense of $W_{\text{loc}}^{2,n}(B)$. For all $k \in \mathbb{N}$, there exist $l_k \in \mathbb{N}$ such that $l + \lambda_l > k$ and hence $v_l = |x|^{\lambda_l} H_l \in C^k(B)$, if $l \geq l_k$. Then the series

$$\sum_{l=l_k}^{\infty} |x|^{\lambda_l} H_l \tag{2.12}$$

converges in the sense of $C^k(\overline{B}_\rho)$, for any $\rho \in]0, 1[$. To see this, we note that (2.7) can be generalized as follows

$$|D^k v| \leq C(k) |x|^\lambda |D^k H|$$

and then by (2.5), as in the proof of inequality (2.8) we find

$$\sup_{B_\rho} |D^k v| \leq C(k, n) (\rho^{\sigma_k})^l l^{n/2+k} \|H\|_{L^2(\partial B)}, \tag{2.13}$$

where $\sigma_k = \inf \{(l + \lambda_l - k)/l : l \geq l_k\} > 0$. Applying (2.13) to each term $v_l = |x|^{\lambda_l} H_l$ in the series (2.12) we obtain the desired result.

We close this Section mentioning that in [2] equations with radially homogeneous coefficients are studied. However, the results presented there in this more general context are not suitable for the purpose of this paper, as we rely heavily on the representation formula (2.11).

3. The degree of regularity of solutions

Here, we examine the regularity of solutions to Equation (1.1) with $k > 1$ in a neighborhood of the origin $0 \in \mathbb{R}^n$. Of course, solutions are of class C^∞ away from the origin, so we are concerned with their behavior at 0. The regularity we want to study is best expressed in terms of the spaces of Hölder continuous functions. We recall the notation used in [8]. For a given open set $\Omega \subset \mathbb{R}^n$, $k = 1, 2, \dots$ and $0 < \alpha \leq 1$, $C_{\text{loc}}^{k,\alpha}(\Omega)$ denotes the space of functions whose k -th order partial derivatives are locally Hölder continuous with exponent α . The completion of $C^\infty(\Omega)$ is denoted by $C^{k+\alpha}(\Omega)$. For $\alpha = 1$, it coincides with $C^{k+1}(\Omega)$, while if $0 < \alpha < 1$, $v \in C_{\text{loc}}^{k+\alpha}(\Omega)$ means that

$$|D^k v(x) - D^k v(y)| = o(|x - y|^\alpha)$$

uniformly on compact subsets of $\Omega \times \Omega$. Also, note the embedding

$$C_{\text{loc}}^{k+\beta}(\Omega) \subsetneq C_{\text{loc}}^{k,\alpha}(\Omega) \subsetneq C_{\text{loc}}^{k+\alpha}(\Omega)$$

for $0 < \alpha < \beta \leq 1$.

Now we introduce the so-called momenta of a function $u \in C^0(B_R - \{0\})$, where $R > 0$; for a given harmonic homogeneous polynomial H , we set

$$w(r) = w_H(r) = \int_{\partial B_r} u H \, ds, \quad 0 < r < R.$$

We can now state our first result, which provide an upper bound for the degree of regularity of the solutions at 0.

THEOREM 3.1. — *Let $u \in C^2(B_R - \{0\})$ be a solution to Equation (1.1). For a given harmonic polynomial H homogeneous of degree $l \geq 2$, if the moment w_H does not vanish, then $u \notin C_{\text{loc}}^{l+\lambda_l}(B_R)$, the number λ_l being defined at (2.3).*

Proof. — By rescaling, we may assume $R > 1$. Also, we may assume $u \in C^1(\overline{B})$. Then we show that u coincides with the solution we constructed in Section 2 to the Dirichlet problem for L , with boundary value $\varphi = u|_{\partial B}$. Let us denote by U this solution. We observe that $u, U \in C^2(B - \{0\}) \cap C^1(\overline{B})$ both solve $Lu = LU = 0$ in $B - \{0\}$ and $u = U$ on ∂B . If $K \leq \sqrt{n-1}$, then the identity $u = U$ follows by an extended maximum principle (in the sense of [6]) proved by Pucci, see Theorem XI on pg. 157 and the subsequent remark in [10]. In the case $K > \sqrt{n-1}$ we recall that the continuous function $|x|^{1-(n-1)/K^2}$ solves the equation in $B - \{0\}$. Applying the maximum principle of Aleksandrov on $B - \{0\}$, we see that

$$u(x) = U(x) + (u(0) - U(0))(1 - |x|^{1-(n-1)/K^2})$$

Moreover this function is C^1 if and only if $u(0) = U(0)$, that is, $u = U$.

By the results of the previous Section, u has the representation

$$u = \sum_{k=0}^{\infty} |x|^{\lambda_k} H_k \quad (3.1)$$

with H_k a harmonic polynomial homogenous of degree k . Hence by orthogonality

$$w_H(r) = \int_{\partial B_r} u H \, ds = r^{\lambda_l} \int_{\partial B_r} H_l H \, ds = w(1) r^{2l+\lambda_l+n-1}.$$

We write $l + \lambda_l = m + \alpha$, with m integer, $1 \leq m \leq l - 1$ and $0 \leq \alpha < 1$, and denote by Q the MacLaurin polynomial of order m of u . Recall that Q and H are orthogonal to each other. Assuming $u \in C_{\text{loc}}^{l+\lambda_l} = C_{\text{loc}}^{m+\alpha}$, we would have as $x \rightarrow 0$

$$u(x) - Q(x) = o(|x|^{l+\lambda_l})$$

and in turn

$$w(r) = o(r^{2l+\lambda_l+n-1})$$

which is false as $w(1) \neq 0$.

By means of the momenta of u we can find also a “lower bound” for the degree of regularity.

THEOREM 3.2. — *Every solution $u \in W_{\text{loc}}^{2,n}(B_R)$ of Equation (1.1) belongs $C_{\text{loc}}^{1,1+\lambda_2}(B_R)$. If there exists an integer $l > 2$ such that*

$$w_H(r) = \int_{\partial B_r} u H \, ds = 0$$

for every harmonic polynomial H homogenous of degree k with $1 < k < l$, then

$$u \in C_{\text{loc}}^{m,\alpha}(B_R).$$

Here $l + \lambda_l = m + \alpha$, as above.

Proof. — For the first part, we note that in the series (3.1) the term $|x|^{\lambda_2} H_2$ is that with the lowest degree of regularity. To complete the proof, we see that, by the assumptions, the representation (3.1) reduces to

$$u(x) = H_0 + H_1 + \sum_{k \geq l} |x|^{\lambda_k} H_k, \quad (3.2)$$

that is, the terms corresponding to k for all $1 < k < l$ vanish. In the series (3.2), the worst term is now $|x|^{\lambda_l} H_l$ (assuming that $H_l \neq 0$). The thesis $u \in C_{\text{loc}}^{m,\alpha}(B_R)$ can be then proved by the argument used in the Remark 2.2.

As a complement to the above theorems we have

PROPOSITION 3.1. — *Let $u \in W_{\text{loc}}^{2,n}(B_R)$ be a solution to Equation (1.1). If $w_H \equiv 0$, for all harmonic homogenous polynomials H , then $u \equiv 0$.*

The claim follows directly by our representation (2.11) of solutions of class $W_{\text{loc}}^{2,n}$.

Bibliography

- [1] ALVINO (A.) and TROMBETTI (G.). — *Second Order Elliptic Equations whose Coefficients Have Their First Derivates Weakly L^n* , Ann. Mat. Pura Appl., (4) 138 (1985), 331-340.
- [2] BASS (R.F.). — *The Dirichlet problem for radially homogeneous elliptic operator*, Trans. Amer. Mat. Soc., 320, no. 2 (1990), 593-614.
- [3] BÉNILAN (P.) and BOUHSISS (F.). — *Une remarque sur l'unicité des solutions pour l'opérateur de Serrin*, C. R. Acad. Sci. Paris, t. 325, Série I, (1997), 611-616.
- [4] COURANT (R.) and HILBERT (D.). — *Methods of Mathematical Physics*, Interscience Publishers, Inc., New York, 1953.
- [5] FRANCIOSI (M.) and FUSCO (N.). — *A $W^{2,2}$ bound for a class of elliptic equations with discontinuous coefficients*, Ricerche Mat., 31 (1982), 207-218.
- [6] GILBARG (D.) and SERRIN (J.). — *On isolated singularities of solutions of second order elliptic differential equations*, J. Analyse Math., 4 (1955-56), 309-340.
- [7] GILBARG (D.) and TRUDINGER (N.S.). — *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer-Verlag, Berlin, 1983.
- [8] GRECO (L.) and SBORDONE (C.). — *Sharp Upper Bounds for the Degree of Regularity of the Solutions to an Elliptic Equation*, Comm. Partial Differential Equations, (5&6) 27 (2002), 945-952.
- [9] MIRANDA (C.). — *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale a coefficienti discontinui*, Ann. Mat. Pura Appl., (4) 63 (1963), 353-386.
- [10] PUCCI (C.). — *Operatori ellittici estremanti*, Ann. Mat. Pura Appl., (4) 72 (1966), 141-170.
- [11] SERRIN (J.). — *Pathological solutions of elliptic differential equations*, Ann. Scuola Norm. Sup. Pisa, (3) 18 (1964), 385-387.
- [12] STEIN (E.M.). — *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
- [13] TALENTI (G.). — *Sopra una classe di equazioni ellittiche a coefficienti misurabili*, Ann. Mat. Pura Appl., (4) 69 (1965), 285-304.