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Stationary states of a gas in a radiation field from a kinetic point of view (*)

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RÉSUMÉ. — Un théorème d'existence est démontré pour une solution stationnaire d'un système d'équations cinétiques décrivant l'interaction d'un faisceau de photons avec un gaz dans un barreau.

ABSTRACT. — An existence theorem is derived for a system of kinetic equations describing the interaction of a radiation field with a gas in a slab in a time-independent frame. Solutions with a given energy and profiles of given indata are found, for Dirac measures as given indata of the photon distribution function.

Introduction

The study of a gas in a radiation field is a subject of interest in astrophysical and laboratory plasmas. So far, the radiative transfer equation for the photons distribution function has been coupled with fluid equations for the gas ([2, 5, 9]). However, many astrophysical and laboratory plasmas show deviations from local thermodynamic equilibrium. This requires a kinetic setting. Kinetic models have been derived in [3, 7, 11]. On the mathematical level, a system of kinetic equations has been studied in [12] for two-level atoms and monochromatic photons. There, a H-theorem is formally obtained, as well as the states of equilibrium. A theorem of existence of a solution to this kinetic model has been derived in [10] in the evolutionary

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case. In this paper, the stationary case is addressed. A theorem of existence of a solution is derived in the slab, for given indata on the boundary. The photons are emitted in beams perpendicular to the walls.

1. The model and the main result

Let a gas of material particles of mass m endowed with only two internal energy levels E_1 and E_2, with $E_1 < E_2$. Denote by A_1 and A_2 particles A at the fundamental level 1 and the excited level 2 respectively, and by $f(x,v)$ and $g(x,v)$ their distribution functions. A time-independent frame is considered, with the geometric setting a slab, i.e. the space variable x is one-dimensional and belongs to $[0,1]$. The velocity variable v belongs to $\mathbb{R}^3$. A radiation field of photons p at a fixed frequency $\nu = \frac{\Delta E}{h}$ interacts with the gas, h being the Planck constant and $\Delta E = E_2 - E_1$. Assume that the gas particles interact elastically among themselves. The interactions between the gas molecules and the photons are, classically, of three types,

$$\begin{aligned} \text{Absorption,} \quad & A_1 + p \rightarrow A_2, \\ \text{Spontaneous emission,} \quad & A_2 \rightarrow A_1 + p, \\ \text{Stimulated emission,} \quad & A_2 + p \rightarrow A_1 + 2p. \end{aligned}$$

Let $\tilde{I}(x,\Omega)$ be the distribution function of the photons, c speed of light and θ the angle between the x-axis and the photon velocity $c\Omega$. Denote by $I(x,\Omega) = ch\nu\tilde{I}(x,\Omega)$ the specific intensity. Let β_{12}, α_{21} and β_{21} be the Einstein coefficients. Then, following [2, 9], the stationary equation for $I(x,\Omega)$ is given by

$$cos\theta \frac{\partial I}{\partial x} = h\nu[(\alpha_{21} + \beta_{21} I) \int g dv - \beta_{12} I \int f dv]. \tag{1.1}$$

Since $\beta_{12} = \beta_{21}$, the subscripts of the Einstein coefficients can be dropped. Denote by ξ the first component of the velocity vector v. The Boltzmann equations for the two particle species A_1 and A_2 can be classically written as

$$\begin{aligned} \xi \frac{\partial f}{\partial x} &= g \int (\alpha + \beta I) d\theta - \beta f \int I d\theta \\ &+ \int_{\mathbb{R}^3 \times S^2} S(v, v_*, \omega)(f' f'_* - f f_*) dv_* d\omega \\ &+ \int_{\mathbb{R}^3 \times S^2} S(v, v_*, \omega)(f' g'_* - f g_*) dv_* d\omega, \end{aligned} \tag{1.2}$$

where S is a given collision kernel,

$$\begin{aligned} f' = f(t,x,v'), \quad f'_* = f(t,x,v'_*), \quad f_* = f(t,x,v_*), \\ v' = v - (v - v_*, \omega)\omega, \quad v'_* = v + (v - v_*, \omega)\omega, \end{aligned}$$

and

$$\xi\frac{\partial g}{\partial x} = -g\int(\alpha+\beta I)d\theta + \beta f\int Id\theta + \int S(v,v_*,\omega)(g'g'_* - gg_*)dv_*d\omega + \int S(v,v_*,\omega)(f'_*g' - f_*g)dv_*d\omega. \tag{1.3}$$

Denote by

$$Q(f,f)(v) = \int S(f'f'_* - ff_*)dv_*d\omega, \quad Q(g,g)(v) = \int S(g'g'_* - gg_*)dv_*d\omega,$$
$$Q_1(f,g)(v) = \int S'(f'g'_* - fg_*)dv_*d\omega, Q_2(f,g)(v) = \int S'(f'_*g' - f_*g)dv_*d\omega.$$

The physical conditions considered here are

$$k_BT \ll mc^2, \quad \Delta E \ll c\sqrt{\frac{8mk_BT}{\pi}}, \tag{1.4}$$

where k_B is the Boltzmann constant and T the temperature of the gas. The first inequality implies that the relativistic effects can be neglected. The velocities of the gas atoms being quite smaller than speed of light, the collisions kernels S and S' in the collision operators are assumed to vanish for $v^2 + v_*^2 \geqslant V$, for some given $V > 0$. Moreover, hard forces interactions are considered. The collision kernel S is defined by

$$S(v,v_*,\omega) = \chi(v^2+v_*^2)|v-v_*|^\beta b(\mu),$$

with $\chi(s) = 0$ if $s \geqslant V$, $0 \leqslant \beta < 2$, $(b,b') \in (L^1_+(0,2\pi))^2$, $b(\mu) \geqslant c > 0$, $b'(\mu) \geqslant c > 0$ a.e. Here, $\omega \in S^2$ is represented by the polar angle μ (with polar axis along $v - v_*$) and the azimuthal angle Φ. The second inequality in (1.4) guarantees that the photon momentum is much smaller than the thermal momentum of the gas, so that any exchange of momentum between photons and molecules can be neglected. The boundary conditions for the gas particles are given indata, i.e.

$$f(0,v) = f_0(v),\ \xi > 0, \quad f(1,v) = f_1(v),\ \xi < 0,$$
$$g(0,v) = g_0(v),\ \xi > 0, \quad g(1,v) = g_1(v),\ \xi < 0.$$

The photons are emitted in beams perpendicular to the walls, i.e.

$$I(0,\theta) = I_0\delta_{\theta=0},\ cos\theta > 0, \quad I(1,\theta) = I_1\delta_{\theta=\pi},\ cos\theta < 0,$$

where I_0 and I_1 are non negative constants. Because of this strong light source, directed along the x-axis, there is much higher intensity in this direction. It is the reason why it is assumed that the stimulated emission for $|cos\theta| < \epsilon$ is negligible compared to the stimulated emission in the other directions. And so, instead of (1.1-3), the distribution functions (f, g, I) must satisfy, for some $\epsilon > 0$,

$$cos\theta \frac{\partial I}{\partial x} = h\nu[(\alpha + \beta I) \int g(v)dv - \beta I \int f(v)dv], \quad |cos\theta| > \epsilon, \tag{1.5}$$

$$cos\theta \frac{\partial I}{\partial x} = h\nu[\alpha \int g(v)dv - \beta I \int f(v)dv], \quad |cos\theta| < \epsilon, \tag{1.6}$$

$$\xi \frac{\partial f}{\partial x} = (\int \alpha d\theta + \int_{|cos\theta|>\epsilon} \beta I d\theta)g - \beta f \int I d\theta + Q(f,f) + Q_1(f,g), \quad , \tag{1.7}$$

$$\xi \frac{\partial g}{\partial x} = -(\int \alpha d\theta + \int_{|cos\theta|>\epsilon} \beta I d\theta)g + \beta f \int I d\theta + Q(g,g) + Q_2(f,g). \tag{1.8}$$

Given a constant $E > 0$, solutions (f, g, I) to (1.5-8) are studied, with

$$\int ((v^2 + E_1)f + (v^2 + E_2)g)(x,v)dxdv + h\nu \int I(x,\theta)d\theta dx = E, \tag{1.9}$$

$$f(0,v) = kf_0(v),\ \xi > 0, \quad f(1,v) = kf_1(v),\ \xi < 0, \tag{1.10}$$

$$g(0,v) = kg_0(v),\ \xi > 0, \quad g(1,v) = kg_1(v),\ \xi < 0, \tag{1.11}$$

$$I(0,\theta) = kI_0\delta_{\theta=0},\ cos\theta > 0, \quad I(1,\theta) = kI_1\delta_{\theta=\pi},\ cos\theta < 0, \tag{1.12}$$

for some constant $k > 0$. The main result of the paper is the following.

THEOREM 1.1.— *Let $E > 0$ be given. Assume that f_0, f_1, g_0 and g_1 are non negative functions satisfying*

$$\int_{\xi>0} ((1 + |lnf_0|)f_0 + (1 + |lng_0|)g_0)(v)dv + \int_{\xi<0} ((1 + |lnf_1|)f_1 + (1 + |lng_1|)g_1)(v)dv < \infty, \tag{1.13}$$

$$\int f_0(v)dv > 0. \tag{1.14}$$

Then there are a constant $k > 0$ and $(f, g, I) \in (L^1_+((0,1) \times \{|v| \leqslant V\}))^2 \times L^1_+(0,1; M(0,2\pi))$ solutions to (1.5-12) in the sense that for any test function φ in $C^1([0,1] \times \mathbb{R}^3)$, with compact support in some $|\xi| \geqslant \delta$ with $\delta > 0$, the weak form of (1.7-8) holds, with $\int I(x,\theta)d\theta$ integrated from (1.5-6)-(1.12). Moreover, $\int \xi^2(f+g)(x,v)dv$ is independent of x and bounded.

Here, $M_+(0,2\pi)$ denotes the set of non negative bounded measures defined on $[0,2\pi]$.

Remarks. — The theorem also holds for more general non negative bounded measures as given indata for the photon distribution function.

Like in [1] for the Boltzmann equation in the slab, a stationary solution having a given profile on the ingoing boundary is here determined.

For the sake of simplicity, the constants α, β and $h\nu$ will be taken equal to 1 in the rest of the paper. Moreover, the interparticles collision terms Q_1 and Q_2 will be skipped. The proof of Theorem 1.1 would also hold with them, with minor adaptations, since the A_1 and A_2 particles are mechanically identical.

2. Approximations with bounded integrands and truncation for small ξ'es

The first approximations bound the integrands in the collision operators. Moreover, a supplementary truncation for $|\xi| \leqslant r$, for some $r > 0$, allows the control of the distribution functions inside the slab by their values at the outgoing boundaries.

Let $r \in (0, 2\epsilon)$, $\alpha \in (0,1)$, and $j \in I\!N^*$ be given. Let $\chi^r \in C^\infty$, $0 \leqslant \chi^r \leqslant 1$ satisfy

$$\chi^r(v, v_*, \omega) = 1 \text{ if } |\xi| > r,\ |\xi_*| > r,\ |\xi'| > r,\ |\xi'_*| > r,$$
$$\chi^r(v, v_*, \omega) = 0 \text{ if } |\xi| < \frac{r}{2} \text{ or } |\xi_*| < \frac{r}{2} \text{ or } |\xi'| < \frac{r}{2} \text{ or } |\xi'_*| < \frac{r}{2}.$$

Let S_j be a positive C^∞ function approximating S when

$$\left|\frac{v - v_*}{|v - v_*| \cdot \omega}\right| > \frac{1}{j} \text{ and } \left|\frac{v - v_*}{|v - v_*| \cdot \omega}\right| < 1 - \frac{1}{j},$$

and such that $S_j(v, v_*, \omega) = 0$ if

$$\left|\frac{v - v_*}{|v - v_*| \cdot \omega}\right| < \frac{1}{2j} \text{ or } \left|\frac{v - v_*}{|v - v_*| \cdot \omega}\right| > 1 - \frac{1}{2j}.$$

Consider $(L^1((0,1) \times \{|v| \leqslant V\}))^2 \times L^1(0,1; M(0,2\pi))$, endowed with its strong topology. For j large enough, a closed and convex subset K of $(L^1_+((0,1) \times \{|v| \leqslant V\}))^2 \times L^1_+(0,1; M(0,2\pi))$ is defined by

$$K := \{(f,g,I); 0 \leqslant f(x,v) \leqslant j^5,\ 0 \leqslant g(x,v) \leqslant j^5,$$

$$\int I(x,\theta)d\theta \leqslant j^2 e^{j^2}, \text{ a.a. } x \in (0,1), |v| \leqslant V,$$

$$\int ((v^2+E_1)f+(v^2+E_2)g)(x,v)dxdv + \int I(x,\theta)d\theta dx = E\}.$$

Let φ be a regular function defined on $(0,1)$. Denote by $\varphi_i(x) = \frac{1}{i}\varphi(\frac{x}{i})$. For $(f,g,I) \in K$ and $\rho \in [0,1]$, let $(\tilde{f},\tilde{g},\tilde{I})$ be the solution to

$$\alpha\tilde{k} + \xi\frac{\partial\tilde{k}}{\partial x} = \frac{\tilde{l}}{1+\frac{\rho\tilde{l}}{j}}(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta) - \frac{\tilde{k}}{1+\frac{\rho\tilde{k}}{j}}\int Id\theta$$
$$+ \int \chi^r S_j \frac{\tilde{k}'}{1+\frac{\rho\tilde{k}'}{j}}\frac{f'_*}{1+\frac{f'_*}{j}} - \frac{\tilde{k}}{1+\frac{\rho\tilde{k}}{j}}\int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}, \quad (2.1)$$

$$\alpha\tilde{l} + \xi\frac{\partial\tilde{l}}{\partial x} = -\frac{\tilde{l}}{1+\frac{\rho\tilde{l}}{j}}(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta) + \frac{\tilde{k}}{1+\frac{\rho\tilde{k}}{j}}\int Id\theta$$
$$+ \int \chi^r S_j \frac{\tilde{l}'}{1+\frac{\rho\tilde{l}'}{j}}\frac{g'_*}{1+\frac{g'_*}{j}} - \frac{\tilde{l}}{1+\frac{\rho\tilde{l}}{j}}\int \chi^r S_j \frac{g_*}{1+\frac{g_*}{j}}, \quad (2.2)$$

$$cos\theta\frac{\partial\tilde{I}}{\partial x} = \int \frac{\tilde{l}}{1+\frac{\rho\tilde{l}}{j}}dv + \tilde{I}\varphi_i * \int (\frac{g}{1+\frac{g}{j}} - \frac{f}{1+\frac{f}{j}})dv, \quad |cos\theta| > \epsilon, \quad (2.3)$$

$$cos\theta\frac{\partial\tilde{I}}{\partial x} = \int \frac{\tilde{l}}{1+\frac{\rho\tilde{l}}{j}}dv - \tilde{I}\varphi_i * \int \frac{f}{1+\frac{f}{j}}dv, \quad |cos\theta| \in (\frac{r}{2},\epsilon), \quad (2.4)$$

together with the boundary conditions

$$\tilde{k}(0,v) = \lambda f_0^j(v),\ \xi > 0, \quad \tilde{k}(1,v) = \lambda f_1^j(v),\ \xi < 0, \quad (2.5)$$
$$\tilde{l}(0,v) = \lambda g_0^j(v),\ \xi > 0, \quad \tilde{l}(1,v) = \lambda g_1^j(v),\ \xi < 0, \quad (2.6)$$
$$\tilde{I}(0,\theta) = \lambda I_0\delta_{\theta=0},\ cos\theta > 0, \quad \tilde{I}(1,\theta) = \lambda I_1\delta_{\theta=\pi},\ cos\theta < 0. \quad (2.7)$$

Here,

$$\lambda = Ee^{\frac{2+4E}{r}}(\int_{\xi>0}((v^2+E_1)f_0^j + (v^2+E_2)g_0^j)(v)dv$$
$$+ \int_{\xi<0}((v^2+E_1)f_1^j + (v^2+E_2)g_1^j)(v)dv)^{-1},$$

and f_0^j (resp. f_1^j, g_0^j, g_1^j) are regularizations of $f_0 \wedge j$ (resp. $f_1 \wedge j$, $g_0 \wedge j$, $g_1 \wedge j$) vanishing for $|\xi| \leqslant \frac{r}{2}$. $a \wedge b$ denotes the minimum of a and b. There is existence of a non negative solution to (2.1-7) in $(L^1_+((0,1)\times\{|v| \leqslant V\}))^2 \times$

$L^1_+(0,1;M(0,2\pi))$. Indeed, let $(\tilde{k}^n)$ and $(\tilde{l}^n)$ be defined by $\tilde{k}^0 = \tilde{l}^0 = 0$ and

$$\alpha\tilde{k}^{n+1} + \xi\frac{\partial\tilde{k}^{n+1}}{\partial x} = \frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}\Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta\Big) - \frac{\tilde{k}^{n+1}}{1+\frac{\rho\tilde{k}^n}{j}}\int Id\theta$$
$$+\int \chi^r S_j \frac{\tilde{k}^{n'}}{1+\frac{\rho\tilde{k}^{n'}}{j}}\frac{f'_*}{1+\frac{f'_*}{j}} - \frac{\tilde{k}^{n+1}}{1+\frac{\rho\tilde{k}^n}{j}}\int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}},$$

$$\alpha\tilde{l}^{n+1} + \xi\frac{\partial\tilde{l}^{n+1}}{\partial x} = -\frac{\tilde{l}^{n+1}}{1+\frac{\rho\tilde{l}^n}{j}}\Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta\Big) + \frac{\tilde{k}^n}{1+\frac{\rho\tilde{k}^n}{j}}\int Id\theta$$
$$+\int \chi^r S_j \frac{\tilde{l}^{n'}}{1+\frac{\rho\tilde{l}^{n'}}{j}}\frac{g'_*}{1+\frac{g'_*}{j}} - \frac{\tilde{l}^{n+1}}{1+\frac{\rho\tilde{l}^n}{j}}\int \chi^r S_j \frac{g_*}{1+\frac{g_*}{j}},$$

together with the boundary conditions (2.5-6). The sequences $(\tilde{k}^n)$ and $(\tilde{l}^n)$ are increasing. Indeed, $\tilde{k}^1 \geqslant 0 = \tilde{k}^0$, $\tilde{l}^1 \geqslant 0 = \tilde{l}^0$, and if $\tilde{k}^n \leqslant \tilde{k}^{n+1}$ and $\tilde{l}^n \leqslant \tilde{l}^{n+1}$, then $\tilde{k}^{n+1} \leqslant \tilde{k}^{n+2}$ and $\tilde{l}^{n+1} \leqslant \tilde{l}^{n+2}$ since

$$\tilde{k}^{n+1}(x,v) = \lambda f_0^j(v) e^{-\frac{\alpha}{\xi}x - \frac{1}{\xi}\int_0^x \frac{\int I(y,\theta)d\theta + \int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}}{1+\frac{\rho\tilde{k}^n(y,v)}{j}}dy}$$
$$+\frac{1}{\xi}\int_0^x \Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta\Big)\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}} + \int \chi^r S_j \frac{\tilde{k}^{n'}}{1+\frac{\tilde{k}^{n'}}{j}}\frac{f'_*}{1+\frac{f'_*}{j}}\Big)$$
$$e^{\frac{\alpha(y-x)}{\xi} - \frac{1}{\xi}\int_y^x \frac{\int I(z,\theta)d\theta + \int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}}{1+\frac{\rho\tilde{k}^n(z,v)}{j}}dz} dy, \quad \xi > 0,$$

and

$$\tilde{k}^{n+1}(x,v) = \lambda f_1^j(v) e^{\frac{\alpha(1-x)}{\xi}x - \frac{1}{|\xi|}\int_x^1 \frac{\int I(y,\theta)d\theta + \int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}}{1+\frac{\rho\tilde{k}^n(y,v)}{j}}dy}$$
$$+\frac{1}{|\xi|}\int_x^1 \Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta\Big)\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}} + \int \chi^r S_j \frac{\tilde{k}^{n'}}{1+\frac{\tilde{k}^{n'}}{j}}\frac{f'_*}{1+\frac{f'_*}{j}}\Big)$$
$$e^{\frac{\alpha(y-x)}{|\xi|} + \frac{1}{\xi}\int_x^y \frac{\int I(z,\theta)d\theta + \int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}}{1+\frac{\rho\tilde{k}^n(z,v)}{j}}dz} dy, \quad \xi < 0,$$

and analogous expressions for $l^{n+1}(x,v)$. Moreover, $(\tilde{k}^n)$ and $(\tilde{l}^n)$ are bounded in L^∞, uniformly with respect to n. Indeed, for j large enough,

$$\frac{\partial}{\partial x}(\tilde{k}^{n+1} + \tilde{l}^{n+1}) + \frac{\alpha}{\xi}(\tilde{k}^{n+1} + \tilde{l}^{n+1}) \leqslant j^3, \quad \xi > \frac{r}{2},$$

and

$$\frac{\partial}{\partial x}(\tilde{k}^{n+1}+\tilde{l}^{n+1})-\frac{\alpha}{|\xi|}(\tilde{k}^{n+1}+\tilde{l}^{n+1})\geqslant -j^3,\quad \xi<-\frac{r}{2},$$

so that

$$\tilde{k}^{n+1}(x,v)\leqslant j^4,\quad \tilde{l}^{n+1}(x,v)\leqslant j^4,\quad a.a.\ x\in(0,1),\ |v|\leqslant V. \tag{2.8}$$

Hence, by Levy's theorem, $(\tilde{k}^n)$ and $(\tilde{l}^n)$ strongly converge in L^1 to some non negative $(\tilde{k},\tilde{l})$ solution of (2.1-7). Then $\tilde{I}$ is obtained by integration of (2.3-4)-(2.7).

There is uniqueness of the solution of (2.1-7) since, if $(\tilde{k}^1,\tilde{l}^1)$ and $(\tilde{k}^2,\tilde{l}^2)$ were two solutions of (2.1-7), let $(\tilde{k},\tilde{l})=(\tilde{k}^1-\tilde{k}^2,\tilde{l}^1-\tilde{l}^2)$. Then

$$\begin{aligned}\alpha\tilde{k}+\xi\frac{\partial\tilde{k}}{\partial x}=&\Big(\int_{|cos\theta|>\frac{r}{2}}d\theta+\int_{|cos\theta|>\epsilon}Id\theta\Big)\frac{\tilde{l}}{(1+\frac{\rho\tilde{l}^1}{j})(1+\frac{\rho\tilde{l}^2}{j})}\\&-\Big(\int Id\theta+\int\chi^r S_j\frac{f_*}{1+\frac{f_*}{j}}\Big)\frac{\tilde{k}}{(1+\frac{\rho\tilde{k}^1}{j})(1+\frac{\rho\tilde{k}^2}{j})}\\&+\int\chi^r S_j\frac{\tilde{k}'}{(1+\frac{\rho\tilde{k}^{1\prime}}{j})(1+\frac{\rho\tilde{k}^{2\prime}}{j})}\frac{f'_*}{1+\frac{f'_*}{j}},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\alpha\tilde{l}+\xi\frac{\partial\tilde{l}}{\partial x}=-&\Big(\int_{|cos\theta|>\frac{r}{2}}d\theta+\int_{|cos\theta|>\epsilon}Id\theta+\int\chi^r S_j\frac{g_*}{1+\frac{g_*}{j}}\Big)\frac{\tilde{l}}{(1+\frac{\rho\tilde{l}^1}{j})(1+\frac{\rho\tilde{l}^2}{j})}\\&+\int Id\theta\frac{\tilde{k}}{(1+\frac{\rho\tilde{k}^1}{j})(1+\frac{\rho\tilde{k}^2}{j})}\\&+\int\chi^r S_j\frac{\tilde{l}'}{(1+\frac{\rho\tilde{l}^{1\prime}}{j})(1+\frac{\rho\tilde{l}^{2\prime}}{j})}\frac{g'_*}{1+\frac{g'_*}{j}},\end{aligned} \tag{2.10}$$

$$\tilde{k}(0,v)=\tilde{l}(0,v)=0,\ \xi>0,\quad \tilde{k}(1,v)=\tilde{l}(1,v)=0,\ \xi<0.$$

Multiply (2.9) by $\mathrm{sgn}\tilde{k}$, (2.10) by $\mathrm{sgn}\tilde{l}$, add them and integrate, so that

$$\begin{aligned}\alpha\int(|\tilde{k}|+|\tilde{l}|)dxdv+&\int_{\xi>0}\xi(|\tilde{k}|+|\tilde{l}|)(1,v)dv\\&+\int_{\xi<0}|\xi|(|\tilde{k}|+|\tilde{l}|)(0,v)dv\leqslant 0.\end{aligned}$$

Hence $\tilde{k}=\tilde{l}=0$.

It follows from

$$\tilde{k}(x,v) \geqslant \lambda f_0^j(v) e^{-\frac{2+4E}{r}}, \ \xi > \frac{r}{2}, \quad \tilde{k}(x,v) \geqslant \lambda f_1^j(v) e^{-\frac{2+4E}{r}}, \ \xi < -\frac{r}{2},$$
$$\tilde{l}(x,v) \geqslant \lambda g_0^j(v) e^{-\frac{2+4E}{r}}, \ \xi > \frac{r}{2}, \quad \tilde{l}(x,v) \geqslant \lambda g_1^j(v) e^{-\frac{2+4E}{r}}, \ \xi < -\frac{r}{2},$$

that

$$\int ((v^2+E_1)\tilde{k} + (v^2+E_2)\tilde{l})(x,v)dxdv \geqslant E. \tag{2.11}$$

Define the map T on $K \times [0,1]$ by $T((f,g,I),\rho) = (\mu(\tilde{k},\tilde{l},\tilde{I}),\mu)$, where

$$\mu = E(\int ((v^2+E_1)\tilde{k} + (v^2+E_2)\tilde{l})(x,v)dxdv + \int \tilde{I}(x,\theta)dxd\theta)^{-1}.$$

The map T is continuous for the strong topology of $(L^1((0,1) \times \{|v| \leqslant V\}))^2 \times L^1(0,1;M(0,2\pi)) \times [0,1]$. Indeed, let $(f_n,g_n,I_n) \in K$ converge to (f,g,I) in $(L^1_+((0,1) \times \{|v| \leqslant V\}))^2 \times L^1_+(0,1;M(0,2\pi))$ and (ρ^n) converge to ρ in $[0,1]$. Let $T((f^n,g^n,I^n),\rho^n) = (\mu^n(\tilde{k}^n,\tilde{l}^n,\tilde{I}^n),\mu^n)$ where

$$\mu^n = E(\int ((v^2+E_1)\tilde{k}^n + (v^2+E_2)\tilde{l}^n)dxdv + \int \tilde{I}^n d\theta dx)^{-1}.$$

Up to a subsequence, (μ^n) converges to some μ, since (μ^n) takes its values in $[0,1]$. Then (2.8) holds for $\tilde{k}^n$ and $\tilde{l}^n$ with similar arguments. Since $\int I^n(x,\theta)d\theta \leqslant j^2 e^{j^2}$, a.a. $x \in (0,1)$, the sequences $(\frac{\partial \tilde{k}^n}{\partial x})$ and $(\frac{\partial \tilde{l}^n}{\partial x})$ are uniformly bounded with respect to n. Moreover, $(\frac{\partial \tilde{k}^n}{\partial v})$ and $(\frac{\partial \tilde{l}^n}{\partial v})$ are solutions to the system

$$\xi \frac{\partial}{\partial x}(\frac{\partial \tilde{k}^n}{\partial v}) + (\alpha + \frac{\int I^n(x,\theta)d\theta + \int \chi^r S_j \frac{f_*^n}{1+\frac{f_*^n}{j}}}{(1+\frac{\rho \tilde{k}^n}{j})^2}) \frac{\partial \tilde{k}^n}{\partial v}$$
$$+ \frac{\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^n(x,\theta)d\theta}{(1+\frac{\rho \tilde{l}^n}{j})^2} \frac{\partial \tilde{l}^n}{\partial v}$$
$$= \int \frac{\partial}{\partial v}(\chi^r S_j) \frac{\tilde{k}^{n'}}{1+\frac{\rho \tilde{k}^{n'}}{j}} \frac{f_*^{n'}}{1+\frac{f_*^{n'}}{j}} - \frac{\tilde{k}^n}{1+\frac{\rho \tilde{k}^n}{j}} \int \frac{\partial}{\partial v}(\chi^r S_j) \frac{f_*^n}{1+\frac{f_*^n}{j}}$$
$$+ \frac{\partial}{\partial v}(\int \chi^r S_j \frac{\tilde{k}^{n'}}{1+\frac{\rho \tilde{k}^{n'}}{j}} \frac{f_*^{n'}}{1+\frac{f_*^{n'}}{j}}),$$

$$\xi \frac{\partial}{\partial x}(\frac{\partial \tilde{l}^n}{\partial v}) + (\alpha + \frac{\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^n(x,\theta)d\theta + \int \chi^r S_j \frac{g_*^n}{1+\frac{g_*^n}{j}}}{(1+\frac{\rho \tilde{l}^n}{j})^2}) \frac{\partial \tilde{l}^n}{\partial v}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad -\frac{\int I^n(x,\theta)d\theta}{(1+\frac{\rho\tilde{k}^n}{j})^2}\frac{\partial\tilde{k}^n}{\partial v}\\
&= \int \frac{\partial}{\partial v}(\chi^r S_j)\frac{\tilde{l}^{n'}}{1+\frac{\rho\tilde{l}^{n'}}{j}}\frac{g_*^{n'}}{1+\frac{g_*^{n'}}{j}} - \frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}\int \frac{\partial}{\partial v}(\chi^r S_j)\frac{g_*^n}{1+\frac{g_*^n}{j}}\\
&\qquad\qquad +\frac{\partial}{\partial v}\Big(\int \chi^r S_j \frac{\tilde{l}^{n'}}{1+\frac{\rho\tilde{l}^{n'}}{j}}\frac{g_*^{n'}}{1+\frac{g_*^{n'}}{j}}\Big),\\
&\frac{\partial\tilde{k}^n}{\partial v}(0,v) = \lambda\frac{\partial f_0^j}{\partial v}(v),\ \xi>0, \quad \frac{\partial\tilde{k}^n}{\partial v}(1,v) = \lambda\frac{\partial f_1^j}{\partial v}(v),\ \xi<0,\\
&\frac{\partial\tilde{l}^n}{\partial v}(0,v) = \lambda\frac{\partial g_0^j}{\partial v}(v),\ \xi>0, \quad \frac{\partial\tilde{l}^n}{\partial v}(1,v) = \lambda\frac{\partial g_1^j}{\partial v}(v),\ \xi<0.
\end{aligned}
$$

The coefficients of $\frac{\partial\tilde{k}^n}{\partial v}$ and $\frac{\partial\tilde{l}^n}{\partial v}$ in the left-hand side of this system are bounded in L^∞, whereas by [8], the right-hand sides of this system are bounded in $L^2((0,1)\times\{|v|\leqslant V\})$. Hence $(\frac{\partial\tilde{k}^n}{\partial v})$ and $(\frac{\partial\tilde{l}^n}{\partial v})$ are bounded in $L^2((0,1)\times\{|v|\leqslant V\})$. And so, the sequences $(\tilde{k}^n)$ and $(\tilde{l}^n)$, bounded in H^1, are compact in L^1. Up to a subsequence they converge to some $\tilde{k}$ and $\tilde{l}$ in L^1. The passage to the limit in (2.1-2) when n tends to infinity is straightforward, so that $(\tilde{k},\tilde{l})$ is the solution to (2.1-2) associated to (f,g,I). By uniqueness of this solution, the whole sequences $(\tilde{k}^n)$ and $(\tilde{l}^n)$ converge to $\tilde{k}$ and $\tilde{l}$ in L^1. Then $(\tilde{I}^n)$ converges in $L^1(0,1;M(0,2\pi))$ to $\tilde{I}$ solution to (2.3-4). Indeed, for any continuous function γ defined on $[0,2\pi]$ and such that $\max_{\theta\in[0,2\pi]}|\gamma(\theta)|\leqslant 1$,

$$
\begin{aligned}
\int \tilde{I}^n(x,\theta)\gamma(\theta)d\theta &= I_0\varphi(0)e^{\int_0^x(\varphi_i*\int(\frac{g^n}{1+\frac{g^n}{j}}-\frac{f^n}{1+\frac{f^n}{j}})dv)(z)dz} +\\
&\qquad I_1\varphi(\pi)e^{\int_x^1(\varphi_i*\int(\frac{g^n}{1+\frac{g^n}{j}}-\frac{f^n}{1+\frac{f^n}{j}})dv)(z)dz} +\\
&+\int_{cos\theta>\epsilon}\frac{\gamma(\theta)}{cos\theta}\int_0^x\int\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}(y,v)dve^{\frac{1}{cos\theta}\int_y^x(\varphi_i*\int(\frac{g^n}{1+\frac{g^n}{j}}-\frac{f^n}{1+\frac{f^n}{j}})dv)(z)dz}dyd\theta\\
&+\int_{cos\theta<-\epsilon}\frac{\gamma(\theta)}{|cos\theta|}\int_x^1\int\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}(y,v)dve^{\frac{1}{|cos\theta|}\int_x^y(\varphi_i*\int(\frac{g^n}{1+\frac{g^n}{j}}-\frac{f^n}{1+\frac{f^n}{j}})dv)(z)dz}dyd\theta\\
&+\int_{\frac{r}{2}<cos\theta<\epsilon}\frac{\gamma(\theta)}{cos\theta}\int_0^x\int\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}(y,v)dve^{-\frac{1}{cos\theta}\int_y^x\varphi_i*\frac{f^n}{1+\frac{f^n}{j}}dv(z)dz}dyd\theta\\
&+\int_{-\epsilon<cos\theta<-\frac{r}{2}}\frac{\gamma(\theta)}{|cos\theta|}\int_x^1\int\frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}(y,v)dve^{-\frac{1}{|cos\theta|}\int_x^y\varphi_i*\int\frac{f^n}{1+\frac{f^n}{j}}dv(z)dz}dyd\theta.
\end{aligned}
$$

Hence,

$$\int_0^1 \sup_{\gamma\in C([0,2\pi]);\max_{\theta\in[0,2\pi]}|\gamma(\theta)|\leqslant 1} |\int (\tilde{I}^n - \tilde{I})(x,\theta)\varphi(\theta)d\theta|dx \to 0 \text{ when } n \to +\infty,$$

by the strong convergence in $L^1(0,1)$ of $(\int \frac{\tilde{l}^n}{1+\frac{\rho\tilde{l}^n}{j}}(x,v)dv)$, $(\varphi_i * \int \frac{f^n}{1+\frac{f^n}{j}}(x,v)dv)$ and $(\varphi_i * \int \frac{g^n}{1+\frac{g^n}{j}}(x,v)dv)$. Consequently, (μ^n) converges to

$$\mu = E(\int ((v^2+E_1)\tilde{k} + (v^2+E_2)\tilde{l})dxdv + \int \tilde{I}d\theta dx)^{-1}$$

when n tends to infinity. Finally, up to a subsequence, $(\tilde{I}^n)$ is a Cauchy sequence in $L^1(0,1;M(0,2\pi))$. And so, $(T((f^n,g^n,I^n),\rho^n))$ converges to $T((f,g,I),\rho)$ when n tends to infinity.

The map T is compact in $(L^1)^2 \times L^1(0,1;M(0,2\pi)) \times [0,1]$. Indeed, for any sequence (f^n,g^n,I^n) and (μ^n) bounded in $(L^1)^2 \times L^1(0,1;M(0,2\pi))$ and $[0,1]$ respectively, the sequences $(\tilde{k}^n)$ and $(\tilde{l}^n)$ are compact in L^1 by similar arguments to the previous proof of the continuity of T. Moreover, μ^n belongs to $[0,1]$. And so, $T((f^n,g^n,I^n),\mu^n)$ is compact in $(L^1)^2 \times L^1(0,1;M(0,2\pi)) \times [0,1]$.

Hence, by the theorem of Schauder, there is a fixed point $((f,g,I),\rho)$ in $K \times [0,1]$ for the map T, which is solution to the system

$$\alpha f + \xi\frac{\partial f}{\partial x} = (\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta)\frac{g}{1+\frac{g}{j}} - \frac{f}{1+\frac{f}{j}}\int Id\theta + \int \chi^r S_j \frac{f'}{1+\frac{f'}{j}}\frac{f'_*}{1+\frac{f'_*}{j}} - \frac{f}{1+\frac{f}{j}}\int \chi^r S_j \frac{f_*}{1+\frac{f_*}{j}}, \quad (2.12)$$

$$\alpha g + \xi\frac{\partial g}{\partial x} = -(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} Id\theta)\frac{g}{1+\frac{g}{j}} + \frac{f}{1+\frac{f}{j}}\int Id\theta + \int \chi^r S_j \frac{g'}{1+\frac{g'}{j}}\frac{g'_*}{1+\frac{g'_*}{j}} - \frac{g}{1+\frac{g}{j}}\int \chi^r S_j \frac{g_*}{1+\frac{g_*}{j}}, \quad (2.13)$$

$$cos\theta\frac{\partial I}{\partial x} = (1+I)\varphi_i * \int \frac{g}{1+\frac{g}{j}}dv - I\varphi_i * \int \frac{f}{1+\frac{f}{j}}dv, \ |cos\theta| > \epsilon, \quad (2.14)$$

$$cos\theta\frac{\partial I}{\partial x} = \int \frac{g}{1+\frac{g}{j}}dv - I\varphi_i * \int \frac{f}{1+\frac{f}{j}}dv, \ |cos\theta| \in (\frac{r}{2},\epsilon) \quad (2.15)$$

$$f(0,v) = \beta f_0^j(v), \ \xi > 0, \quad f(1,v) = \beta f_1^j(v), \ \xi < 0,$$
$$g(0,v) = \beta g_0^j(v), \ \xi > 0, \quad g(1,v) = \beta g_1^j(v), \ \xi < 0,$$
$$I(0,\theta) = \beta I_0 \delta_{\theta=0}, \ cos\theta > 0, \quad I(1,\theta) = \beta I_1 \delta_{\theta=\pi}, \ cos\theta < 0, \quad (2.16)$$

where $\beta = \lambda\rho \leqslant \lambda$. By the exponential forms of f, g and I,

$$(f+g)(x,v) \leqslant c_1(f+g)(1,v),\ \xi > \frac{r}{2},$$
$$(f+g)(x,v) \leqslant c_1(f+g)(0,v),\ \xi < -\frac{r}{2},$$
$$I(x,\theta) \leqslant c_1 I(1,\theta),\ \cos\theta > \frac{r}{2}, \quad I(x,\theta) \leqslant c_1 I(0,\theta),\ \cos\theta < -\frac{r}{2},$$

with c_1 only depending on r. Hence,

$$\begin{aligned} E &= \int ((v^2+E_1)f + (v^2+E_2)g)(x,v)dxdv + \int I(x,\theta)dxd\theta \\ &\leqslant c_2\Big(\int_{\xi>\frac{r}{2}} \xi(f+g)(1,v)dv + \int_{\xi<-\frac{r}{2}} |\xi|(f+g)(0,v)dv \\ &\quad + \int_{cos\theta>\frac{r}{2}} cos\theta I(1,\theta)d\theta + \int_{cos\theta<-\frac{r}{2}} |cos\theta| I(0,\theta)d\theta\Big) \\ &\leqslant c_3\beta, \end{aligned}$$

by integrating the sum of (2.12) and (2.13) on $(0,1)\times\{|v| \leqslant V\}$, and adding (2.13) integrated on $(0,1)\times\{|v| \leqslant V\}$ to (2.14) and (2.15), respectively integrated on $(0,1)\times\{|cos\theta| > \epsilon\}$ and $(0,1)\times\{|cos\theta| \in (\frac{r}{2},\epsilon)\}$. Hence $\beta \geqslant c_4$, with c_4 only depending on r.

The passage to the limit when $i \to +\infty$ can be performed with analogous arguments to the proof of the compactness of T, since it now holds that $(\int \frac{f}{1+\frac{f}{j}}dv)$ and $(\int \frac{g}{1+\frac{g}{j}}dv)$ are strongly compact in $L^1(0,1)$. The passage to the limit when α tends to zero can be performed with analogous arguments to the proof of the continuity of T, after noticing that up to a subsequence $\int I d\theta$ strongly converges in L^1 when α tends to zero. Indeed, an explicit computation of $\int I(x,\theta)d\theta$ from (2.14-16) expresses it in terms of $y \to \int \frac{f}{1+\frac{f}{j}}(y,v)dv$ and $y \to \int \frac{g}{1+\frac{g}{j}}(y,v)dv$, which are compact in L^1 by the averaging lemma. Hence there is a solution (f^j, g^j, I^j) to

$$\begin{aligned} \xi\frac{\partial f^j}{\partial x} &= \Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^j d\theta\Big)\frac{g^j}{1+\frac{g^j}{j}} - \frac{f^j}{1+\frac{f^j}{j}}\int I^j d\theta \\ &\quad + \int \chi^r S_j \frac{f^{j'}}{1+\frac{f^{j'}}{j}}\frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}} - \frac{f^j}{1+\frac{f^j}{j}}\int \chi^r S_j \frac{f^j_*}{1+\frac{f^j_*}{j}}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \xi\frac{\partial g^j}{\partial x} &= -\Big(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^j d\theta\Big)\frac{g^j}{1+\frac{g^j}{j}} + \frac{f^j}{1+\frac{f^j}{j}}\int I^j d\theta \\ &\quad + \int \chi^r S_j \frac{g^{j'}}{1+\frac{g^{j'}}{j}}\frac{g^{j'}_*}{1+\frac{g^{j'}_*}{j}} - \frac{g^j}{1+\frac{g^j}{j}}\int \chi^r S \frac{g^j_*}{1+\frac{g^j_*}{j}}, \end{aligned} \tag{2.18}$$

$$cos\theta\frac{\partial I^j}{\partial x} = (1+I^j)\int \frac{g^j}{1+\frac{g^j}{j}}dv - I^j\int \frac{f^j}{1+\frac{f^j}{j}}dv,\ |cos\theta| > \epsilon, \quad (2.19)$$

$$cos\theta\frac{\partial I^j}{\partial x} = \int \frac{g^j}{1+\frac{g^j}{j}}dv - I^j\int \frac{f^j}{1+\frac{f^j}{j}}dv,\ |cos\theta| \in (\frac{r}{2},\epsilon), \quad (2.20)$$

$$f^j(0,v) = \beta^j f_0^j(v),\ \xi > 0, \quad f^j(1,v) = \beta^j f_1^j(v),\ \xi < 0, \quad (2.21)$$

$$g^j(0,v) = \beta^j g_0^j(v),\ \xi > 0, \quad g^j(1,v) = \beta^j g_1^j(v),\ \xi < 0, \quad (2.22)$$

$$I^j(0,\theta) = \beta^j I_0\delta_{\theta=0},\ cos\theta > 0, \quad I^j(1,\theta) = \beta^j I_1\delta_{\theta=\pi},\ cos\theta < 0, \quad (2.23)$$

for some $\beta^j \in [c_4,\lambda]$. Moreover, (f^j,g^j,I^j) satisfies

$$\int((v^2+E_1)f^j + (v^2+E_2)g^j)(x,v)dxdv + \int I^j(x,\theta)dxd\theta = E. \quad (2.24)$$

3. Approximations with a truncation for small ξ'es.

In this section, the passage to the limit when j tends to infinity is performed in the last system of equations of Section 2. Since β^j belongs to $[c_4,\lambda]$ for all j, where c_4 and λ are independent of j, a subsequence of (β^j) converges to some $\beta \in [c_4,\lambda]$.

LEMMA 3.1.— *The sequences (f^j) and (g^j) are weakly compact in $L^1((0,1)\times\{|v| \leqslant V\})$.*

Adding (2.17) and (2.18) and integrating the sum implies that

$$\int_{\xi>0}\xi(f^j+g^j)(1,v)dv + \int_{\xi<0}|\xi|(f^j+g^j)(0,v)dv \leqslant c. \quad (3.1)$$

Moreover, integrating (2.18) on $(0,1)\times\{|v| \leqslant V\}$, (2.19) on $(0,1)\times\{|cos\theta| > \epsilon\}$, (2.20) on $(0,1)\times\{|cos\theta| \in (\frac{r}{2},\epsilon)\}$, and adding the resulting equations, implies that

$$\int_{\xi>0}\xi g^j(1,v)dv + \int_{\xi<0}|\xi|g^j(0,v)dv$$
$$+\int_{cos\theta>0}cos\theta I^j(1,\theta)d\theta + \int_{cos\theta<0}|cos\theta|I^j(0,\theta)d\theta < c. \quad (3.2)$$

Multiply (2.17) by $ln\frac{f^j}{1+\frac{f^j}{j}}$, (2.18) by $ln\frac{g^j}{1+\frac{g^j}{j}}$, (2.19) by $ln\frac{I^j}{1+I^j}$, (2.20) by lnI^j, add them and integrate. Hence,

$$\int\xi(f^j lnf^j - j(1+\frac{f^j}{j}))ln(1+\frac{f^j}{j})(1,v)dv$$

$$
\begin{aligned}
&- \int \xi(f^j lnf^j - j(1+\frac{f^j}{j}))ln(1+\frac{f^j}{j})(0,v)dv \\
&+ \int \xi(g^j lng^j - j(1+\frac{g^j}{j}))ln(1+\frac{g^j}{j})(1,v)dv \\
&- \int \xi(g^j lng^j - j(1+\frac{g^j}{j}))ln(1+\frac{g^j}{j})(0,v)dv \\
&+\frac{1}{4}e(f^j,f^j)+\frac{1}{4}e(g^j,g^j)+e_1(f^j,g^j,I^j)+e_2(f^j,g^j,I^j) \\
&= \int_{cos\theta>\epsilon} cos\theta((1+I^j)ln(1+I^j)-I^j lnI^j)(1,\theta)d\theta \\
&+ \int_{cos\theta<-\epsilon} |cos\theta|((1+I^j)ln(1+I^j)-I^j lnI^j)(0,\theta)d\theta \\
&+\int_{0<cos\theta<\epsilon} cos\theta I^j(1-lnI^j)(1,\theta)d\theta+\int_{-\epsilon<cos\theta<0}|cos\theta|I^j(1-lnI^j)(0,\theta)d\theta \\
&\leqslant c,
\end{aligned}
$$

by (3.2) and the inequality

$$(1+x)ln(1+x)-xlnx<2x, \quad x>1.$$

Here,

$$e(f,f):=\int \chi^r B(\frac{f'}{1+\frac{f'}{j}}\frac{f'_*}{1+\frac{f'_*}{j}}-\frac{f}{1+\frac{f}{j}}\frac{f_*}{1+\frac{f_*}{j}})ln\frac{\frac{f'}{1+\frac{f'}{j}}\frac{f'_*}{1+\frac{f'_*}{j}}}{\frac{f}{1+\frac{f}{j}}\frac{f_*}{1+\frac{f_*}{j}}}dxdvdv_*d\omega,$$

$$e_1(f,g,I):=\int_{|cos\theta|>\epsilon}((1+I)\frac{g}{1+\frac{g}{j}}-I\frac{f}{1+\frac{f}{j}})ln\frac{(1+I)g(1+\frac{f}{j})}{If(1+\frac{g}{j})}dxdvd\theta,$$

and

$$e_2(f,g,I):=\int_{|cos\theta|\in(\frac{\epsilon}{2}\epsilon)}(\frac{g}{1+\frac{g}{j}}-I\frac{f}{1+\frac{f}{j}})ln\frac{g(1+\frac{f}{j})}{If(1+\frac{g}{j})}dxdvd\theta.$$

Since

$$\frac{7}{8}tlnt-j(1+\frac{t}{j})ln(1+\frac{t}{j})\geqslant 0, \quad j\geqslant 100, \quad t\in(100,j^4),$$

and the sequences (f^j) and (g^j) take their values in $(0,j^4)$, it holds that

$$
\begin{aligned}
&\int_{\xi>0}\xi f^j lnf^j(1,v)dv+\int_{\xi<0}|\xi|f^j lnf^j(0,v)dv \\
&+\int_{\xi>0}\xi g^j lng^j(1,v)dv+\int_{\xi<0}|\xi|g^j lng^j(0,v)dv \\
&+e(f^j,f^j)+e(g^j,g^j)+e_1(f^j,g^j,I^j)+e_2(f^j,g^j,I^j)\leqslant c.
\end{aligned}
$$

By the exponential form of $f^j + g^j$ and the truncation χ^r for $|\xi| \leqslant \frac{r}{2}$,

$$(f^j + g^j)(x,v) \leqslant c(f^j + g^j)(1,v), \ \xi > 0,$$
$$(f^j + g^j)(x,v) \leqslant c(f^j + g^j)(0,v), \ \xi < 0,$$

so that

$$\int (f^j(1 + |lnf^j|) + g^j(1 + |lng^j|))(x,v)dxdv + e_2(f^j, g^j, I^j) \leqslant c. \quad (3.3)$$

Hence, the sequences (f^j) and (g^j) are weakly compact in L^1, and converge, up to subsequences, to some f and g.

LEMMA 3.2. — *The sequence (I^j) is weakly compact in $L^1((0,1) \times \{|cos\theta| < \epsilon\})$. Moreover, $(\int_{|cos\theta|>\epsilon} I^j(x,\theta)d\theta)$ is uniformly bounded with respect to j.*

It follows from the expression of I^j with respect to f^j and g^j from (2.20), (2.23) and the inequality

$$\int (f^j + g^j)(x,v)dxdv \leqslant E(\frac{1}{E_1} + \frac{1}{E_2}),$$

that

$$\int_{|cos\theta|<\epsilon} I^j(x,\theta)dxd\theta < c. \quad (3.4)$$

By (1.14) and the exponential form of f^j, for some subset W of $\{v \in I\!R^3; |v| \leqslant V\}$ of positive measure,

$$I^j(x,\theta) \leqslant \frac{1}{c_5} I^j(x,\theta) f^j(x,v), \quad v \in W.$$

Hence, for any $K > 2$,

$$I^j(x,\theta) \leqslant \frac{K}{c_5} g^j(x,v) + \frac{1}{c_5 lnK}(g^j(x,v)$$
$$-I^j(x,\theta)f^j(x,v))ln\frac{g^j(x,v)}{I^j(x,\theta)f^j(x,v)}, \quad v \in W.$$

The equiintegrability of (I^j) on $(0,1) \times \{|cos\theta| \in (\frac{r}{2}, \epsilon)\}$ then follows from the equiintegrability of (g^j) on $(0,1) \times \{|v| \leqslant V\}$, as well as the bound from above of $e_2(f^j, g^j, I)$ derived in (3.3). Consequently, (I^j) is weakly

compact in $L^1((0,1)\times\{|cos\theta|<\epsilon\})$. Moreover, by integration of (2.19), (2.23),

$$\int_{|cos\theta|>\epsilon} I^j(x,\theta)d\theta = \beta^j I_0 e^{\int_0^x \int (\frac{g^j}{1+\frac{g^j}{j}} - \frac{f^j}{1+\frac{f^j}{j}})(z,v)dvdz}$$

$$+\beta^j I_1 e^{\int_x^1 \int (\frac{g^j}{1+\frac{g^j}{j}} - \frac{f^j}{1+\frac{f^j}{j}})(z,v)dvdz}$$

$$+\int_0^x (\int \frac{g^j}{1+\frac{g^j}{j}}(y,v)dv) \int_{cos\theta>\epsilon} e^{\frac{1}{cos\theta}\int_y^x \int (\frac{g^j}{1+\frac{g^j}{j}} - \frac{f^j}{1+\frac{f^j}{j}})(z,v)dvdz} \frac{d\theta}{cos\theta}dy$$

$$+\int_x^1 (\int \frac{g^j}{1+\frac{g^j}{j}}(y,v)dv) \int_{cos\theta<-\epsilon} e^{\frac{1}{|cos\theta|}\int_x^y \int (\frac{g^j}{1+\frac{g^j}{j}} - \frac{f^j}{1+\frac{f^j}{j}})(z,v)dvdz} \frac{d\theta}{|cos\theta|}dy$$

$$\leqslant c,$$

by (2.24).

Passage to the limit when $j \to \infty$ in (2.17-24).

It follows from the weak L^1 compactness of (f^j) and (g^j) and from

$$\int \xi^2 (f^j + g^j)(x,v)dv \leqslant c,$$

that

$$(f^j \int \chi^r S \frac{f_*^j}{1+\frac{f_*^j}{j}} dv_* d\omega) \text{ and } (g^j \int \chi^r S \frac{g_*^j}{1+\frac{g_*^j}{j}} dv_* d\omega)$$

are weakly compact in L^1. Then, it classically follows from the boundedness of $(e(f^j, f^j))$ and $(e(g^j, g^j))$ that

$$(\int \chi^r S \frac{f^{j'}}{1+\frac{f^{j'}}{j}} \frac{f_*^{j'}}{1+\frac{f_*^{j'}}{j}} dv_* d\omega) \text{ and } (\int \chi^r S \frac{g^{j'}}{1+\frac{g^{j'}}{j}} \frac{g_*^{j'}}{1+\frac{g_*^{j'}}{j}} dv_* d\omega)$$

are weakly compact in L^1. Since (I^j) is weakly compact in $L^1((0,1)\times\{|cos\theta| \in (\frac{r}{2},\epsilon)\})$, and $(\int_{|cos\theta|>\epsilon} I^j(x,\theta)d\theta)$ is bounded, the sequences $(\xi\frac{\partial}{\partial x} ln(1+f^j))$ and $(\xi\frac{\partial}{\partial x} ln(1+g^j))$ are weakly compact in L^1. And so, the averaging lemma applies, which allows to pass to the limit when j tends to infinity. And so, there is a family (f^r, g^r, I^r, k_r) of solutions to

$$\xi\frac{\partial f^r}{\partial x} = (\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^r d\theta) g^r - f^r \int I^r d\theta$$

$$+\int \chi^r S(f^{r'} f_*^{r'} - f^r f_*^r) dv_* d\omega, \qquad (3.5)$$

$$\xi \frac{\partial g^r}{\partial x} = -(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^r d\theta) g^r + f^r \int I^r d\theta$$

$$+\int \chi^r S(g^{r'} g_*^{r'} - g^r g_*^r) dv_* d\omega, \qquad (3.6)$$

$$cos\theta \frac{\partial I^r}{\partial x} = (1 + I^r) G^r - I^r F^r, \quad |cos\theta| > \epsilon, \qquad (3.7)$$

$$cos\theta \frac{\partial I^r}{\partial x} = G^r - I^r F^r, \quad |cos\theta| \in (\frac{r}{2}, \epsilon), \qquad (3.8)$$

$$f^r(0,v) = k_r f_0(v), \ \xi > 0, \quad f^r(1,v) = k_r f_1(v), \ \xi < 0, \qquad (3.9)$$

$$g^r(0,v) = k_r g_0(v), \ \xi > 0, \quad g^r(1,v) = k_r g_1(v), \ \xi < 0, \qquad (3.10)$$

$$I^r(0,\theta) = k_r I_0 \delta_{\theta=0}, \ cos\theta > 0, \quad I^r(1,\theta) = k_r I_1 \delta_{\theta=\pi}, \ cos\theta < 0, \qquad (3.11)$$

$$\int ((v^2 + E_1) f^r + (v^2 + E_2) g^r) dx dv + \int I^r d\theta dx = E. \qquad (3.12)$$

Here, $F(x) = \int f(x,v)dv$ and $G(x) = \int g(x,v)dv$.

4. Removal of the truncation for small ξ'es

This section is devoted to the passage to the limit when r tends to zero in the previous system (3.5-12). First,

$$f^r(x,v) \geqslant c k_r f_0(v), \ \xi > 1, \quad f^r(x,v) \geqslant c k_r f_1(v), \ \xi < -1,$$

where c is a constant independent of r. Hence

$$E \geqslant \int (v^2 + E_1) f^r dx dv \geqslant c k_r,$$

so that $\sup_{r \leqslant r_0} k_r = k_0 < +\infty$. Let us prove that

$$e(f^r, f^r) < c k_r, \quad e(g^r, g^r) < c k_r,$$

where

$$e(f,f) := \int_0^1 \int \chi^r S(f' f_*' - f f_*) ln \frac{f' f_*'}{f f_*} dx dv dv_* d\omega.$$

Denote by $\tilde{f}^r := \frac{f^r}{k_r}$ and $\tilde{g}^r := \frac{g^r}{k_r}$. They satisfy

$$\xi \frac{\partial \tilde{f}^r}{\partial x} = (\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^r d\theta) \tilde{g}^r - \tilde{f}^r \int I^r d\theta$$

$$+k_r \int \chi^r S(\tilde{f}^{r'} \tilde{f}_*^{r'} - \tilde{f}^r \tilde{f}_*^r) dv_* d\omega, \qquad (4.1)$$

$$\xi \frac{\partial \tilde{g}^r}{\partial x} = -(\int_{|cos\theta|>\frac{r}{2}} d\theta + \int_{|cos\theta|>\epsilon} I^r d\theta)\tilde{g}^r + \tilde{f}^r \int I^r d\theta$$

$$+k_r \int \chi^r S(\tilde{g}^{r'} \tilde{g}_*^{r'} - \tilde{g}^r \tilde{g}_*^r) dv_* d\omega, \qquad (4.2)$$

$$\frac{1}{k_r} cos\theta \frac{\partial I^r}{\partial x} = (1 + I^r)\tilde{G}^r - I^r \tilde{F}^r, \quad |cos\theta| > \epsilon, \qquad (4.3)$$

$$\frac{1}{k_r} cos\theta \frac{\partial I^r}{\partial x} = \tilde{G}^r - I^r \tilde{F}^r, \quad |cos\theta| \in (\frac{r}{2}, \epsilon), \qquad (4.4)$$

$$\tilde{f}^r(0,v) = f_0(v),\ \xi > 0, \quad \tilde{f}^r(1,v) = f_1(v),\ \xi < 0,$$
$$\tilde{g}^r(0,v) = g_0(v),\ \xi > 0, \quad \tilde{g}^r(1,v) = g_1(v),\ \xi < 0,$$
$$I^r(0,\theta) = k_r I_0 \delta_{\theta=0},\ cos\theta > 0, \quad I^r(1,\theta) = k_r I_1 \delta_{\theta=\pi},\ cos\theta < 0.$$

Multiply (4.1) by $ln\tilde{f}^r$, (4.2) by $ln\tilde{g}^r$, (4.3) by $ln\frac{I^r}{1+I^r}$, and (4.4) by lnI^r, integrate and use that

$$\int \xi(\tilde{f}^r + \tilde{g}^r)(1,v)dv - \int \xi(\tilde{f}^r + \tilde{g}^r)(0,v)dv = 0.$$

Hence,

$$\int \xi \tilde{f}^r ln\tilde{f}^r(1,v)dv - \int \xi \tilde{f}^r ln\tilde{f}^r(0,v)dv$$
$$+ \int \xi \tilde{g}^r ln\tilde{g}^r(1,v)dv - \int \xi \tilde{g}^r ln\tilde{g}^r(0,v)dv$$
$$+\frac{1}{k_r} \int_{|cos\theta|>\epsilon} cos\theta(I^r lnI^r - (1+I^r)ln(1+I^r))(1,\theta)d\theta$$
$$+\frac{1}{k_r} \int_{|cos\theta|<\epsilon} cos\theta(I^r lnI^r - I^r)(1,\theta)d\theta$$
$$-\frac{1}{k_r} \int_{|cos\theta|>\epsilon} cos\theta(I^r lnI^r - (1+I^r)ln(1+I^r))(0,\theta)d\theta$$
$$-\frac{1}{k_r} \int_{|cos\theta|<\epsilon} cos\theta(I^r lnI^r - I^r)(0,\theta)d\theta$$
$$= -k_r e(\tilde{f}^r, \tilde{f}^r) - k_r e(\tilde{g}^r, \tilde{g}^r) - e_1(\tilde{f}^r, \tilde{g}^r, I^r) - e_2(\tilde{f}^r, \tilde{g}^r, I^r),$$

where

$$e_1(\tilde{f}, \tilde{g}, I) = \int_{|cos\theta|>\epsilon} ((1+I)\tilde{g} - I\tilde{f}) ln \frac{(1+I)\tilde{g}}{I\tilde{f}},$$
$$e_2(\tilde{f}, \tilde{g}, I) = \int_{|cos\theta|\in(\frac{r}{2},\epsilon)} (\tilde{g} - I\tilde{f}) ln \frac{\tilde{g}}{I\tilde{f}}.$$

From

$$\int_{\xi>0} \xi g^r(1,v)dv + \int_{\xi<0} |\xi| g^r(0,v)dv$$
$$+ \int_{cos\theta>0} cos\theta I^r(1,\theta)d\theta + \int_{cos\theta<0} |cos\theta| I^r(0,\theta)d\theta < ck_r,$$
$$(1+x)ln(1+x) - xlnx < 2x, \quad x > 1,$$

and the bounded domains of integration, it holds that

$$k_r e(\tilde{f}^r, \tilde{f}^r) + k_r e(\tilde{g}^r, \tilde{g}^r) + e_1(\tilde{f}^r, \tilde{g}^r, I^r) + e_2(\tilde{f}^r, \tilde{g}^r, I^r)$$
$$\leqslant \int_{\xi>0, \tilde{f}^r<1} \xi \tilde{f}^r |ln\tilde{f}^r|(1,v)dv + \int_{\xi<0, \tilde{f}^r<1} |\xi| \tilde{f}^r |ln\tilde{f}^r|(0,v)dv$$
$$+ \int_{\xi>0, \tilde{g}^r<1} \xi \tilde{g}^r |ln\tilde{g}^r|(1,v)dv + \int_{\xi<0, \tilde{g}^r<1} |\xi| \tilde{g}^r |ln\tilde{g}^r|(0,v)dv + c$$
$$\leqslant c,$$

Hence,

$$\frac{1}{k_r}(e(f^r, f^r) + e(g^r, g^r)) + e_1(\tilde{f}^r, \tilde{g}^r, I^r) + e_2(\tilde{f}^r, \tilde{g}^r, I^r)$$
$$= k_r(e(\tilde{f}^r, \tilde{f}^r) + e(\tilde{g}^r, \tilde{g}^r)) + e_1(\tilde{f}^r, \tilde{g}^r, I^r) + e_2(\tilde{f}^r, \tilde{g}^r, I^r) \leqslant c.$$

And so,

$$e(f^r, f^r) < ck_r, \quad e(g^r, g^r) < ck_r.$$

Moreover,

$$\int \xi(f^r + g^r)(1,v)dv - \int \xi(f^r + g^r)(0,v)dv = 0,$$

so that

$$\int_{\xi>0} \xi(f^r + g^r)(1,v)dv + \int_{\xi<0} |\xi|(f^r + g^r)(0,v)dv$$
$$= k_r\left(\int_{\xi<0} |\xi|(f_1 + g_1)(v)dv + \int_{\xi>0} \xi(f_0 + g_0)(v)dv\right) \leqslant ck_r.$$

Then,

$$\int \xi^2(f^r + g^r)(x,v)dv = \int \xi^2(f^r + g^r)(1,v)dv$$
$$= k_r \int_{\xi<0} \xi^2(f_1 + g_1)(v)dv + \int_{\xi>0} \xi^2(f^r + g^r)(1,v)dv$$
$$\leqslant ck_r + V \int_{\xi>0} \xi(f^r + g^r)(1,v)dv \leqslant ck_r, \quad x \in (0,1). \tag{4.5}$$

Hence,

$$\int_{|\xi|>\frac{1}{10}} f^r(x,v)dxdv \leqslant 100ck_r. \tag{4.6}$$

Then,

$$f^r(x,v_*) \geqslant ck_r, \quad 1 \leqslant |\xi_*| \leqslant 10.$$

Then, for v such that $|\xi| < \frac{1}{10}$ and v_* such that $\xi\xi_* < 0$ and $1 \leqslant |\xi_*| \leqslant 10$, there is a set of $\omega \in S^2$ of measure (say) $\frac{|S^2|}{100}$, depending on x, v, v_*, such that $|\xi'| > c$ and $|\xi'_*| > c$. Hence, for $L > 2$,

$$\begin{aligned} cb(\theta)f^r(x,v) \leqslant cf^r(x,v)\frac{f^r(x,v_*)}{k_r} \leqslant \frac{cL}{k_r}(\xi'^2 + \xi'^2_*)f^r(x,v')f^r(x,v'_*) \\ + \frac{c}{k_r lnL}b(\theta)|v-v_*|^\beta(f^r f^r_* - f^{r'}f^{r'}_*)ln\frac{f^r f^r_*}{f^{r'}f^{r'}_*}. \end{aligned}$$

Hence,

$$\int_{|\xi|<\frac{1}{10}} f^r(x,v)dxdv \leqslant cLk_r + \frac{c}{lnL}.$$

Together with (4.6) it implies that

$$\int f^r(x,v)dxdv \leqslant cLk_r + \frac{c}{lnL}. \tag{4.7}$$

Analogously,

$$\int g^r(x,v)dxdv \leqslant cLk_r + \frac{c}{lnL}. \tag{4.8}$$

Since

$$\begin{aligned} \int I^r(x,\theta)d\theta dx = k_r(I_0 e^{\int_0^x (G^r-F^r)(z)dz} + I_1 e^{\int_x^1 (G^r-F^r)(z)dz}) \\ + \int_{cos\theta>\epsilon}\int_0^x G^r(y)e^{\frac{1}{cos\theta}\int_y^x (G^r-F^r)(z)dz}dydx\frac{d\theta}{cos\theta} \\ + \int_{cos\theta\in(0,\epsilon)}\int_0^x G^r(y)e^{-\frac{1}{cos\theta}\int_y^x F^r(z)dz}dydx\frac{d\theta}{cos\theta} \\ + \int_{cos\theta<-\epsilon}\int_x^1 G^r(y)e^{\frac{1}{cos\theta}\int_y^x (G^r-F^r)(z)dz}dydx\frac{d\theta}{|cos\theta|} \\ + \int_{cos\theta\in(-\epsilon,0)}\int_x^1 G^r(y)e^{\frac{1}{cos\theta}\int_y^x F^r(z)dz}dydx\frac{d\theta}{|cos\theta|}, \end{aligned}$$

and

$$E = \int (v^2 + E_1) f^r dx dv + \int (v^2 + E_2) g^r dx dv + \int I^r dx d\theta,$$

it follows that

$$E \leqslant cLk_r + \frac{c}{lnL} + \int_{|cos\theta| \in (\frac{r}{2}, \epsilon)} I^r dx d\theta. \tag{4.9}$$

Assume that $\underline{lim}_{r \to 0} k_r = 0$. Choose $L = |lnk_r|$. Then, by (4.7-8),

$$\lim_{r \to 0} \int_0^1 F^r(y) dy = \lim_{r \to 0} \int_0^1 G^r(y) dy = 0. \tag{4.10}$$

Since

$$cos\theta I^r_x = G^r - I^r F^r, \quad |cos\theta| \in (\frac{r}{2}, \epsilon),$$

$$I^r(0, \theta) = 0, \ cos\theta \in (\frac{r}{2}, \epsilon), \quad I^r(1, \theta) = 0, \ cos\theta \in (-\epsilon, -\frac{r}{2}),$$

it holds that

$$|cos\theta| I^r(x, \theta) \leqslant \int_0^1 G^r(y) dy, \quad |cos\theta| \in (\frac{r}{2}, \epsilon).$$

By (4.10),

$$\lim_{r \to 0} I^r(x, \theta) = 0, \quad \text{a.a. } x \in (0, 1), \quad |cos\theta| \in (\frac{r}{2}, \epsilon). \tag{4.11}$$

Moreover, it follows from (4.5) that for a.a. $x \in (0, 1)$, there is a subset $W(x)$ of $\{v \in I\!R^3; |v| \leqslant V, |\xi| \geqslant 1\}$ with measure at least half the measure of $\{v \in I\!R^3; |v| \leqslant V, |\xi| \geqslant 1\}$, such that

$$g^r(x, v) \leqslant c_1 k_r, \quad x \in (0, 1), \quad v \in W(x).$$

Then, by the exponential form of f^r,

$$f^r(x, v) \geqslant c_2 k_r, \quad x \in (0, 1), \quad v \in W(x).$$

And so, for a.a. $x \in (0, 1)$ and $|cos\theta| \in (\frac{r}{2}, \epsilon)$,

$$I^r(x, \theta) \leqslant c_2 I^r(x, \theta) \frac{f^r(x, v)}{k_r}, \quad v \in W(x),$$

$$\leqslant c_2 (2 \frac{g^r(x, v)}{k_r} + \frac{1}{k_r ln2} (I^r(x, \theta) f^r(x, v) - g^r(x, v)) ln \frac{I^r(x, \theta) f^r(x, v)}{g^r(x, v)}), \quad v \in W(x),$$

$$\leqslant c_2 (2c_1 + \frac{c_3}{ln2}).$$

Together with (4.11) and the Lebesgue theorem, it implies that

$$\lim_{r\to 0}\int_{|cos\theta|\in(\frac{\pi}{2},\epsilon)} I^r(x,\theta)dxd\theta = 0.$$

This contradicts (4.9), when taking $L = |lnk_r|$. Hence $\underline{lim}_{r\to 0}k_r > 0$. We may now choose $r_1 > 0$ and $k_1 > 0$ so that

$$k_1 \leqslant k_r \leqslant k_0, \quad 0 < r \leqslant r_1.$$

LEMMA 4.1. — *For $\delta > 0$, the families $(f^r)_{r\leqslant r_1}$ and $(g^r)_{r\leqslant r_1}$ are weakly compact in $L^1((0,1)\times\{v\in I\!R^3; |v|\leqslant V, |\xi|\geqslant\delta\})$.*

First,

$$\sup_{x\in(0,1)}\int_{|\xi|\geqslant\delta} f^r(x,v)dv \leqslant \frac{c}{\delta^2}, \quad \sup_{x\in(0,1)}\int_{|\xi|\geqslant\delta} g^r(x,v)dv \leqslant \frac{c}{\delta^2}.$$

Then,

$$\begin{aligned}
&\int_{|\xi|>\delta}(f^r lnf^r + g^r lng^r)dxdv = \\
&\int_{\xi>\delta}\xi\int_{-\frac{1}{\xi}}^{0}(f^r lnf^r + g^r lng^r)(1+s\xi,v)dsdv \\
&+\int_{\xi<-\delta}|\xi|\int_{\frac{1}{\xi}}^{0}(f^r lnf^r + g^r lng^r)(s\xi,v)dsdv \\
&\leqslant c(\int_{\xi>\delta}\xi(f^r+g^r)(1,v)dv + \int_{\xi<-\delta}|\xi|(f^r+g^r)(0,v)dv) \\
+c(\int_{\xi>\delta}&\xi(f^r lnf^r + g^r lng^r)(1,v)dv + \int_{\xi<-\delta}|\xi|(f^r lnf^r + g^r lng^r)(0,v)dv) \\
&\leqslant c.
\end{aligned}$$

Denote by $\chi^j := \chi^{r_j}$, $f^j := f^{r_j}$ and $g^j := g^{r_j}$, where (r_j) is a sequence tending to 0. By Lemma 4.1, there are subsequences, still denoted by (f^j), (g^j) and (k^j), with $\lim_{j\to+\infty} f^j = f$, $\lim_{j\to+\infty} g^j = g$ in weak $L^1((0,1)\times\{|v|\leqslant V, |\xi|\geqslant\delta\})$ for all $\delta > 0$ and $\lim_{j\to+\infty} k^j = k$. Let φ be a test function vanishing on $|\xi|\geqslant\delta$, for some $\delta > 0$. In order to prove that $(\int I^j(x,\theta)f^j(x,v)\varphi(x,v)d\theta dxdv)$, $(\int I^j(x,\theta)g^j(x,v)\varphi(x,v)d\theta dxdv)$, $(\int Q_j^{\pm}(f^j,f^j)\varphi(x,v)dxdv)$ and $(\int Q_j^{\pm}(g^j,g^j)\varphi(x,v)dxdv)$ have the respective limits $\int I(x,\theta)f(x,v)\varphi(x,v)d\theta dxdv$, $\int I(x,\theta)g(x,v)\varphi(x,v)d\theta dxdv$, $\int Q^{\pm}(f,f)\varphi(x,v)dxdv$ and $\int Q^{\pm}(g,g)\varphi(x,v)dxdv$ when j tends to infinity, we first prove the two following lemmas, with similar arguments to some proven in [1].

LEMMA 4.2

$$\lim_{\rho\to 0} \sup_{S\subset(0,1);|S|<\rho,j>\frac{1}{\rho}} \int_{S\times\{|v|\leqslant V\}} (f^j+g^j)(x,v)dxdv = 0.$$

It follows from the bound from above of $\int \xi^2(f^j+g^j)(x,v)dv$ that

$$\int_{|\xi|>\frac{1}{j}} (f^j+g^j)(x,v)dv \leqslant c_{10}j^2.$$

If the lemma does not hold, then there are $\eta>0$ and a subsequence of (say) (f^j), still denoted by (f^j), such that for each j there is a subset S_j of $(0,1)$ with $|S_j|<\frac{\eta}{2c_{10}j^2}$ and

$$\int_{S_j\times\{|v|\leqslant V\}} f^j(x,v)dxdv \geqslant \eta.$$

Hence,

$$\int_{S_j\times\{|v|\leqslant V, r_j\leqslant|\xi|\leqslant\frac{1}{j}\}} f^j(x,v)dxdv \geqslant \frac{\eta}{2}.$$

Here, at least half the integral comes from the set of (x,v) with $f^j(x,v)\geqslant c_{10}j^2$. Let $V_*:=\{v_*\in\mathbb{R}^3; |v_*|\leqslant V, 1\leqslant|\xi_*|\leqslant 2\}$. By the exponential form of f^j,

$$f^j(x,v_*)\geqslant c_{11}, \quad v_*\in V_*. \tag{4.12}$$

Then, from the geometry of the velocities involved, and from

$$\int_{|\xi'|\geqslant 1} f^j(x,v')dv' \leqslant c_{12},$$

given v such that $|v|\leqslant V$, $r_j\leqslant|\xi|\leqslant\frac{1}{j}$ and $f^j(x,v)\geqslant c_{10}j^2$, it holds for v_* in a subset of V_* of measure $\frac{|V_*|}{2}$ and for $\omega\in S^2$ in a subset of measure $\frac{|S^2|}{100}$, that

$$|\xi'|\geqslant\frac{1}{4},\ |\xi'_*|\geqslant\frac{1}{4},\ f^j(x,v')\leqslant c_{13},\ f^j(x,v'_*)\leqslant c_{13}.$$

It follows that, for some constants c_{14} and c_{15} independent of such v, $v_*\in V_*$, ω and for j large,

$$c_{14}f^j(x,v)\leqslant f^j(x,v)f^j(x,v_*)-f^j(x,v')f^j(x,v'_*),$$

$$c_{15}j\leqslant\frac{f^j(x,v)f^j(x,v_*)}{f^j(x,v')f^j(x,v'_*)}.$$

And so, using the entropy dissipation estimate,

$$\int_{S_j \times \{|v| \leqslant V, r_j \leqslant |\xi| \leqslant \frac{1}{j}\}} f^j(x,v)dxdv \leqslant \frac{c_{16}}{lnj} < \frac{\eta}{4},$$

for j large enough. The lemma follows by contradiction.

LEMMA 4.3. — *Given $\rho > 0$, there is j_0 such that for $j > j_0$ and outside a j-dependent set in x of measure less than ρ, $(\int_{|\xi|<\frac{1}{i}} f^j(x,v)dv)$ and $(\int_{|\xi|<\frac{1}{i}} g^j(x,v)dv)$ tend to 0 when i tends to infinity, uniformly with respect to x and j.*

Let us prove Lemma 4.3 for $(\int_{|\xi|<\frac{1}{i}} f^j(x,v)dv)$. Given $0 < \eta \ll 1$ and x, j, either

$$\int_{|\xi|<\frac{1}{i}} f^j(x,v)dv \leqslant \eta^2 < \eta,$$

or

$$\int_{|\xi|<\frac{1}{i}} f^j(x,v)dv > \eta^2.$$

In the latter case,

$$\int_{|\xi| \leqslant \frac{1}{i}, f^j(x,v) > \frac{\eta^2 i}{2V^2}} f^j(x,v)dv \geqslant \frac{\eta^2}{2}.$$

For each (x,v) such that $|\xi| \leqslant \frac{1}{i}$ and $f^j(x,v) > \frac{\eta^2 i}{2V^2}$, take v_* in $V_* := \{v_* \in I\!R^3; |v_*| \leqslant V, 1 \leqslant |\xi_*| \leqslant 2\}$, so that by (4.8),

$$f^j(x,v_*) \geqslant c_{11}, \quad v_* \in V_*.$$

Given v, it holds for v_* in a subset of V_* of measure $\frac{|V_*|}{2}$ and for $\omega \in S^2$ in a subset of measure $\frac{|S^2|}{100}$, that

$$|\xi'| \geqslant 1, \ |\xi'_*| \geqslant 1, \ f^j(x,v') \leqslant c_{17}, \ f^j(x,v'_*) \leqslant c_{17}.$$

It follows that, for some constants c_{18} and c_{19} independent of such v, $v_* \in V_*$, ω and for j large,

$$f^j(x,v) \leqslant c_{18} f^j(x,v) f^j(x,v_*)$$
$$\leqslant \frac{c_{19}}{lni} B(f^j(x,v)f^j(x,v_*) - f^j(x,v')f^j(x,v'_*)) \frac{f^j(x,v)f^j(x,v_*)}{f^j(x,v')f^j(x,v'_*)}.$$

Since there is a constant $c_{20} > 0$ such that, uniformly with respect to j, the integral

$$\int S(f^j(x,v)f^j(x,v_*) - f^j(x,v')f^j(x,v'_*))ln\frac{f^j(x,v)f^j(x,v_*}{f^j(x,v')f^j(x,v'_*)}dvdv_*d\omega$$

is bounded by c_{20} outside a j-dependent set S_j in x, of measure ρ, it follows that for $x \in S_j^c$,

$$\int_{|\xi|\leqslant\frac{1}{i}} f^j(x,v)dv \leqslant \frac{c_{19}c_{20}}{lni} + \eta < 2\eta,$$

for i large enough.

LEMMA 4.4. — *Let φ be a test function vanishing on $|\xi| \leqslant \delta$, for some $\delta > 0$. Then $(\int I^j(x,\theta)f^j(x,v)\varphi(x,v)d\theta dxdv)$ and $(\int I^j(x,\theta)g^j(x,v)\varphi(x,v)$ $d\theta dxdv)$ respectively converge to $\int I(x,\theta)f(x,v)\varphi(x,v)d\theta dxdv$ and $\int I(x,\theta)$ $g(x,v)\varphi(x,v)d\theta dxdv$ when j tends to infinity, where I is the solution to*

$$cos\theta\frac{\partial I}{\partial x} = (1+I)G - IF, \ |cos\theta| > \epsilon, \quad cos\theta\frac{\partial I}{\partial x} = G - IF, \ |cos\theta| \leqslant \epsilon,$$
$$I(0,\theta) = kI_0\delta_{\theta=0}, \ cos\theta > 0, \quad I(1,\theta) = kI_1\delta_{\theta=\pi}, \ cos\theta < 0.$$

Let us prove the first part of the lemma. By the expressions of $\int I^j(x,\theta)d\theta$ and $\int I(x,\theta)d\theta$ derived in (4.1),

$$\int I^j(x,\theta)f^j(x,v)\varphi(x,v)d\theta dxdv - \int I(x,\theta)f(x,v)\varphi(x,v)d\theta dxdv$$

splits into the sum of

$$(k^j - k)I_0\int f^j\varphi e^{\int_0^x(G^j-F^j)(z)dz}, \quad (k^j-k)I_1\int f^j\varphi e^{\int_x^1(G^j-F^j)(z)dz},$$

which tend to zero when j tends to infinity, by (3.12), and

$$X_{1j} := kI_0\int(f^j e^{\int_0^x(G^j-F^j)(z)dz} - fe^{\int_0^x(G-F)(z)dz})\varphi dxdv,$$
$$X_{2j} := kI_1\int(f^j e^{\int_x^1(G^j-F^j)(z)dz} - fe^{\int_x^1(G-F)(z)dz})\varphi dxdv,$$
$$X_{3j} := \int f^j\varphi(\int_{cos\theta>\epsilon}\int_0^x G^j(y)e^{\frac{1}{cos\theta}\int_y^x(G^j-F^j)(z)dz}dy\frac{d\theta}{cos\theta})dxdv$$
$$- \int f\varphi(\int_{cos\theta>\epsilon}\int_0^x G(y)e^{\frac{1}{cos\theta}\int_y^x(G-F)(z)dz}dy\frac{d\theta}{cos\theta})dxdv$$

$$X_{4j} := \int f^j\varphi(\int_{cos\theta<-\epsilon}\int_x^1 G^j(y)e^{\frac{1}{cos\theta}\int_y^x (G^j-F^j)(z)dz}dy\frac{d\theta}{|cos\theta|})dxdv$$
$$-\int f\varphi(\int_{cos\theta<-\epsilon}\int_x^1 G(y)e^{\frac{1}{cos\theta}\int_y^x (G-F)(z)dz}dy\frac{d\theta}{|cos\theta|})dxdv,$$
$$X_{5j} := \int f^j\varphi(\int_{cos\theta\in(0,\epsilon)}\int_0^x G^j(y)e^{\frac{-1}{cos\theta}\int_y^x F^j(z)dz}dy\frac{d\theta}{cos\theta})dxdv$$
$$-\int f\varphi(\int_{cos\theta\in(0,\epsilon)}\int_0^x G(y)e^{\frac{-1}{cos\theta}\int_y^x F(z)dz}dy\frac{d\theta}{cos\theta})dxdv,$$

and

$$X_{6j} := \int f^j\varphi(\int_{cos\theta\in(-\epsilon,0)}\int_x^1 G^j(y)e^{\frac{1}{cos\theta}\int_y^x F^j(z)dz}dy\frac{d\theta}{|cos\theta|})dxdv$$
$$-\int f\varphi(\int_{cos\theta\in(-\epsilon,0)}\int_x^1 G(y)e^{\frac{1}{cos\theta}\int_y^x F(z)dz}dy\frac{d\theta}{|cos\theta|})dxdv.$$

Then,

$$|X_{1j}| \leqslant kI_0|\int (f^j-f)\varphi e^{\int_0^x (G-F)(z)dz}dxdv|$$
$$+\frac{c}{\delta^2}\int_0^1\Big(|\int_0^x (G^j-G)(z)dz| + |\int_0^x (F^j-F)(z)dz|\Big)dx,$$

since

$$|\int (f\varphi)(x,v)dv| \leqslant \frac{c}{\delta^2}, \quad \text{a.a. } x\in(0,1).$$

The first term in the right-hand side tends to zero when j tends to infinity, since $(f^j-f)\varphi$ converges weakly to zero in L^1 and $\int_0^x (G-F)(z)dz$ is a bounded function. The second term in the right-hand side tends to zero when j tends to infinity. Indeed, let $\eta>0$ be given. Since $f\in L^1$ and by Lemma 4.2, there is $\rho_0>0$ such that for every $j>\frac{1}{\rho_0}$, for any subset S of $(0,1)$ with $|S|<\rho_0$,

$$\int_{S\times\{|v|\leqslant V\}} (f^j+f)(z,w)dzdw < \eta.$$

By Lemma 4.3 applied to ρ_0, for j large enough, there are $X_j\subset(0,1)$ with $|X_j^c|<\rho_0$ and $i_0\in I\!N^*$ such that

$$\int_0^1\int_{|\xi|<\frac{1}{i_0}} (f^j+f)(z,w)dzdw<\eta, \quad z\in X_j.$$

Then, for a.a. $x \in (0,1)$, by the weak L^1 convergence of (f^j) to f on $(0,1) \times \{|v| \leqslant V, |\xi| > \frac{1}{i_0}\}$, there is $j_0(x)$ such that

$$|\int_0^x \int_{|\xi|>\frac{1}{i_0}} (f^j - f)(z,w)dzdw| < \eta, \quad j > j_0(x).$$

and so, for a.a. $x \in (0,1)$

$$|\int_0^x \int (f^j - f)(z,w)dzdw| < 3\eta, \quad j > j_0(x).$$

Hence, for almost all $x \in (0,1)$,

$$\lim_{j\to+\infty} \int_0^x (F^j - F)(z)dz = 0.$$

And so, the second term in the right-hand side tends to zero, by applying the theorem of the dominated convergence. All other terms, i.e. $X_{2,j}$, ..., $X_{6,j}$ can be treated analogously.

LEMMA 4.5. — *Let φ be a test function vanishing on $|\xi| \leqslant \delta$, for some $\delta > 0$. Then, $(\int Q^{\pm}(f^j, f^j)\varphi(x,v)dxdv)$ and $(\int Q^{\pm}(g^j, g^j)\varphi(x,v)dxdv)$ respectively tend to $\int Q^{\pm}(f,f)\varphi(x,v)dxdv$ and $\int Q^{\pm}(g,g)\varphi(x,v)dxdv$ when j tends to infinity.*

Split φ into its positive and negative parts respectively, so that φ can be considered as non negative in the rest of the proof. Let us first prove that

$$\lim_{j\to+\infty} \int Q^-(f^j, f^j)\varphi(x,v)dxdv = \int Q^-(f,f)\varphi(x,v)dxdv.$$

Let $\gamma > 0$ be given. By the weak L^1 compactness of $(Q^-(f^j, f^j)\varphi)$ and the integrability of $Q^-(f,f)\varphi$, there is a number $\eta > 0$ and $j_0 \in \mathbb{N}$, such that for any subset A of $(0,1) \times \{|v| \leqslant V\}$ with $|A| < \eta$,

$$|\int_A Q^-(f^j, f^j)\varphi dxdv| < \gamma, \ j > j_0, \text{ and } |\int_A Q^-(f,f)\varphi dxdv| < \gamma. \quad (4.13)$$

By Lemma 4.3, there is $j_1 > j_0$ such that for $j > j_1$ and outside a j-dependent set S_j^1 in x of measure less than $\frac{\eta}{4\pi V^3}$, $(\int_{|\xi|\leqslant\mu} f^j(x,v)dv)$ tends to zero when μ tends to zero, uniformly with respect to x and j. Moreover,

$$\int S(f^j\varphi)(x,v)dvd\omega \leqslant c_{21} \int (f^j\varphi)(x,v)dv$$

is bounded from above by a constant c_{22} outside of a set S_j^2 in x of measure less than $\frac{\eta}{4\pi V^3}$, by the averaging lemma and Egoroff's theorem. Then, $\int Sf(x,v_*)dv_*d\omega$, which is smaller than $c_{22}\int f(x,v_*)dv_*$, is bounded from above by a constant c_{23} outside of a set S_j^3 in x of measure less than $\frac{\eta}{4\pi V^3}$. Denote by
$S_j := S_j^1 \cup S_j^2 \cup S_j^3$. By (4.10),

$$\int_{S_j\times\{|v|\leqslant V\}} (Q^-(f^j,f^j)+Q^-(f,f))\varphi dxdv < 2\gamma, \quad j > j_0.$$

Then,

$$\begin{aligned}
&|\int_{S_j^c\times\{|v|\leqslant V\}} (Q^-(f^j,f^j)-Q^-(f,f))\varphi dxdv| \\
&\leqslant |\int_{S_j^c} (f^j-f)\varphi(x,v)(\int Sf(x,v_*)dv_*d\omega)dxdv| \\
&+|\int_{S_j^c}(\int_{|\xi_*|<\mu}(f^j-f)(x,v_*)(\int S(f^j\varphi)(x,v)dvd\omega)dv_*)dx| \\
&+|\int_{S_j^c}(\int_{|\xi_*|>\mu}(f^j-f)(x,v_*)(\int S(f^j\varphi)(x,v)dvd\omega)dv_*)dx|.
\end{aligned}$$

By the weak L^1 compactness of $(f^j\varphi)$ and the boundedness of $\int Sf(x,v_*)dv_*d\omega$ on $S_j^c\times\{|v|\leqslant V\}$, the first term in the right-hand side tends to zero when j tends to infinity. Choose then μ small enough so that

$$|\int_{S_j^c}(\int_{|\xi_*|<\mu}(f^j-f)(x,v_*)(\int S(f^j\varphi)(x,v)dvd\omega)dv_*)dx|,$$

which is smaller than

$$c_{23}\int_{S_j^c}\int_{|\xi_*|<\mu}(f^j+f)(x,v_*)dv_*dx,$$

be smaller than ϵ for j bigger than some $j_2 > j_1$. For such a μ, $(\int_{|\xi_*|>\mu}(f^j-f)(x,v_*)dv_*)$ strongly converges to zero in $L^1((0,1))$. Since $(\int S(f^j\varphi)(x,v)dvd\omega)$ is bounded by c_{22} on $S_j^c\times\{|v_*|\leqslant V\}$, the third term in the right-hand side tends to zero when j tends to infinity. This ends the proof of the convergence of $(\int Q^-(f^j,f^j)\varphi(x,v)dxdv)$ to $\int Q^-(f,f)\varphi(x,v)dxdv$ when j tends to infinity. Let us finally prove that

$$\lim_{j\to+\infty}\int Q^+(f^j,f^j)\varphi(x,v)dxdv = \int Q^+(f,f)\varphi(x,v)dxdv.$$

Let $\rho > 0$ and $\mu > 0$, $\mu_* > 0$ be given. As a consequence of the proof of Lemma 4.3, for some j_0 there is a sequence of subsets $(X_{j,\rho})$ of $(0,1)$ such that $|X_{j,\rho}| < \rho$ and $(\int_{|\xi|<\mu} f^j(x,v)dv)$ (resp. $(\int_{|\xi_*|<\mu_*} f^j(x,v_*)S\chi^j\varphi(x,v')$ $dv_*d\omega)$) converges to 0 with μ (resp. μ_*), uniformly with respect to $x \in X^c_{j,\rho}$, $j \geqslant j_0$ and $|v| \leqslant V$. For $K \geqslant 2$,

$$\begin{aligned}
&\int_{X_{j,\rho}} Q^+(f^j,f^j)\varphi(x,v)dxdv \\
&\leqslant K\int_{X_{j,\rho}} Q^-(f^j,f^j)\varphi(x,v)dxdv + \frac{c}{lnK} \\
&\leqslant cK\int_{X_{j,\rho}} f^j(x,v_*)dv_*dx + \frac{c}{lnK}, \qquad (4.14)
\end{aligned}$$

which by Lemma 4.2 tends to 0 when $K \to +\infty$, then $\rho \to 0$, uniformly with respect to $j > \frac{1}{\rho}$. By the averaging lemma and Egoroff's theorem, there is for any $\mu_* > 0$ a subset $Y_{\mu_*} \subset (0,1)$ of measure smaller than ρ, such that $\int_{|\xi_*|>\mu_*} f^j(x,v_*)S\chi^j\varphi(x,v')dv_*d\omega$ converges to $\int_{|\xi_*|>\mu_*} f(x,v_*)S\varphi(x,v')dv_*d\omega$ when $j \to +\infty$, uniformly with respect to $x \in Y^c_{\mu_*}$, and is bounded. Split $\int_{X^c_{j,\rho}} Q^+(f^j,f^j)\varphi(x,v)dxdv$ into

$$\begin{aligned}
&\int_{X^c_{j,\rho}} Q^+(f^j,f^j)\varphi(x,v)dxdv \\
&= \int_{X^c_{j,\rho}} f^j(x,v)(\int_{|\xi_*|\leqslant\mu_*} f^j(x,v_*)S\chi^j\varphi(x,v')dv_*d\omega)dvdx \\
&+ \int_{Y^c_{\mu_*}} f^j(x,v)(\chi_{X^c_{j,\rho}}(x)\int_{|\xi_*|\geqslant\mu_*} f^j(x,v_*)S\chi^j\varphi(x,v')dv_*d\omega)dvdx \\
&+ \int_{Y_{\mu_*}\cap X^c_{j,\rho}} f^j(x,v)(\int_{|\xi_*|\geqslant\mu_*} f^j(x,v_*)S\chi^j\varphi(x,v')dv_*d\omega)dvdx.
\end{aligned}$$

By Lemma 4.3, the first term in the right-hand side tends to zero when $\mu_* \to 0$, uniformly with respect to j and ρ. For μ_* fixed, the second term in the right-hand side tends to $\int_{Y^c_{\mu_*}} f(x,v)(\int_{|\xi_*|>\mu_*} f(x,v_*)S\varphi(x,v')dv_*d\omega)dvdx$ when $j \to +\infty$ and $\rho \to 0$. The third term in the right-hand side is bounded by $c\int_{Y_{\mu_*}} f^j(x,v)dvdx$, which tends to 0 when $\rho \to 0$, uniformly with respect to j by Lemma 4.2. Hence,

$$\lim_{j\to\infty}\int Q^+(f^j,f^j)\varphi(x,v)dxdv = \int Q^+(f,f)\varphi(x,v)dxdv,$$

by letting first $\mu_* \to 0$, then $j \to \infty$ and $\rho \to 0$. This ends the proof of Lemma 4.5.

Bibliography

[1] ARKERYD (L.), NOURI (A.). — "The stationary Boltzmann equation in the slab with given weighted mass for hard and soft forces", Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), Vol. XXVII, 533-556, 1998.

[2] BARDOS (C.), GOLSE (F.), PERTHAME (B.), SENTIS (R.). — "The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximation", J. Functional Analysis, 77: 434, 1998.

[3] BURGERS (J.M.). — Flow equations for composite gases, Academic Press, New-York, 1969.

[4] CERCIGNANI (C.). — The Boltzmann equation and its applications, Springer Verlag, New-York, 1988.

[5] CHANDRASEKHAR (S.). — Radiative transfer, Dover Publications, New-York, 1960.

[6] DI PERNA (R.J.), LIONS (P.L.). — "On the Cauchy problem for Boltzmann equations: Global existence and weak stability", Ann. of Math. 130, 321-366, 1989.

[7] EU (B.C.), MAO (F.). — "Kinetic theory approach to irreversible thermodynamics of radiation and matter", Physica A, 180:65, 1992.

[8] LIONS (P.L.). — "Compactness in Boltzmann's equation via Fourier integral operator and applications. I", J. Math. Kyoto Univ. 34, 391-427,1994.

[9] MONACO (R.), POLEWCZAK (J.), ROSSANI (A.). — "Kinetic theory of atoms and photons: an application to the Milne-Chandrasekhar problem", Transport theory and Statistical Physics, 27 (5-7): 625, 1998.

[10] NOURI (A.). — "The evolution of a gas on a radiation field from a kinetic point of view", Preprint 2000.

[11] OXENIUS (J.). — Kinetic theory of particles and photons, Springer Verlag, New-York, 1986.

[12] ROSSANI (A.), SPIGA (G.), MONACO (R.). — "Kinetic approach for two-level atoms interacting with monochromatic photons", Mech. Research Comm., 24:237, 1996.