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Holomorphic group actions with few compact orbits (*)

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RÉSUMÉ. — Pour une grande classe de variétés compactes complexes, contenant les images méromorphes des espaces de Kähler compacts, nous montrons le théorème suivant: Soit G un groupe de Lie complexe agissant holomorphiquement sur X tel qu'il y a qu'un nombre fini positif d'orbites compactes. Alors X est un fibré $G \times_I F$, où G/I est le tore d'Albanese de X et F est une fibre de l'application d'Albanese. De plus, F est connexe et son premier nombre de Betti est nul.

ABSTRACT. — For a large class of compact complex manifolds, including for example meromorphic images of compact Kähler spaces, we prove the following theorem: Let G be a complex Lie group acting holomorphically on the manifold X and suppose there is at least one, but only finitely many compact orbits. Then X is a fibre bundle $G \times_I F$, where G/I is the Albanese torus of X and F is a fibre of the Albanese map. Furthermore F is connected and has vanishing first Betti number.

1. Introduction

Holomorphic group actions on a compact complex manifold X with an open set of compact orbits are studied in [GeWu]. There it is proved that X consists only of compact orbits and is itself of a product structure, reflecting the single orbit decomposition of Borel and Remmert, provided there are no

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equivariantly embedded complex tori in the fibres of the Albanese mapping. This condition especially holds true if X is Kähler.

The authors would like to mention that they learned that this result was, in the Kähler case, already proved by D. Snow in 1981 ([Sn]). He used methods, which – in contrast to those of [GeWu] – depend on the fact that the connected automorphism group of a compact Kähler space acts compactifiably (cf. [F], [L]). In the algebraic case parts of this result were obtained by J. Konarski in 1981 ([K]).

Furthermore, Snow showed that the set C of compact orbits of a holomorphic action is analytic, if X is Kähler. In this note we consider the case that C is extremely small, i.e. there are only finitely many compact orbits. For manifolds in Fujikis class $\mathcal{C}$ (see [F]), which contains for example Kähler and Moišhezon manifolds, the following observation can be made:

THEOREM. — *Let X be in $\mathcal{C}$, $G \times X \to X$ a holomorphic action of a connected complex Lie group such that there is at least one, but only finitely many compact orbits. Then the following holds:*

X is a fibre bundle $G \times_I F$, where G/I is the Albanese torus of X and F is a fibre of the Albanese map.

Furthermore F is connected and has vanishing first Betti number.

Remark. — Looking at a $\mathbb{C}^n$-action as a multidimensional generalization of a holomorphic dynamical system, the case that I is discrete, corresponds to the "suspension" of the discrete dynamical system I on F in a continuous system, the $\mathbb{C}^n$-action on X.

2. Proofs

Let G always denote a connected complex Lie group. If $G \times X \to X$ is a holomorphic action on a complex space, we refer to X as a holomorphic G-space. In this situation C_G will be the set of compact G-orbits in X.

For smooth X one has the Albanese map $\Psi_X : X \to \operatorname{Alb} X$ (see [Bl]). The equivariance of ψ_X implies that there is an induced Lie group homomorphism $\lambda_X : \operatorname{Aut}_{\mathcal{O}}(X) \to \operatorname{Aut}_{\mathcal{O}}(\operatorname{Alb} X)$. The kernel of λ_X restricted to the connected component of the identity in $\operatorname{Aut}_{\mathcal{O}}(X)$ is called $L(X)$, the "linear" automorphisms of X.

Using the compactness of the irreducible components of the space of analytic cycles in complex spaces in class $\mathcal{C}$ resp. Kähler spaces, Fujiki respectively Lieberman prove that $L(X)$ carries the natural structure of a linear algebraic group (Cor. 5.8 in [F], Thm. 3.12 in [L]). For a subgroup H of $L(X)$, $\overline{H}$ will denote the Zariski closure with respect to this structure.

Consider now a smooth holomorphic G-space X. We may assume that G acts without ineffectivity. Let $I = G \cap L(X)$ and $\bar{I}$ the closure of I. Furthermore fix a Levi-Malcev decomposition $\bar{I}^\circ = R_{\bar{I}^\circ} \cdot S_{\bar{I}^\circ}$ of $\bar{I}^\circ$.

LEMMA 1. — *Let X be a smooth holomorphic G-space in $\mathcal{C}$ and x in X. Assume C_G is not empty. Then the following statements are equivalent:*

(1) *$G(x)$ is compact.*

(2) *$I(x)$ is compact.*

(3) *$\bar{I}(x)$ is compact.*

(4) *$R_{\bar{I}^\circ}$ fixes x and $S_{\bar{I}^\circ}(x)$ is compact.*

Proof. — Without loss of generality, we assume throughout this proof that G acts effectively on X.

(1) $\Longleftrightarrow$ (2) Since $\mathrm{Aut}_{\mathcal{O}}(\mathrm{Alb}\, X)^\circ = \mathrm{Alb}\, X$ the isotropy groups of the induced G-action on $\mathrm{Alb}\, X$ is equal to I for all points in $\mathrm{Alb}\, X$. Since C_G is not empty, this implies that all G-orbits on $\mathrm{Alb}\, X$ are compact. Thus, $G(x)$ is compact iff $I(x)$ is compact.

(2) $\Longrightarrow$ (3) The group of all g in $L(X)$ which stabilize the compact analytic set $I(x)$ is a Zariski closed subgroup of $L(X)$ (Lemma 2.4 in [F], Prop. 3.4 in [L]). Hence $\bar{I}$ stabilizes $I(x)$ and consequently $\bar{I}(x) = I(x)$.

(3) $\Longrightarrow$ (4) Since the radical $R_{\bar{I}^\circ}$ is a connected solvable subgroup of $L(X)$ the Borel Fixed Point Theorem for class $\mathcal{C}$ (cf. the proof of Prop. 6.9 in [F] resp. [So] in the Kähler case) shows that it has a fixed point on each component of $\bar{I}(x)$. Since $R_{\bar{I}^\circ}$ is normal in $\bar{I}$, it acts trivially on $\bar{I}(x)$. Therefore $S_{\bar{I}^\circ}$ acts transitively on the $\bar{I}$-components.

(4) $\Longrightarrow$ (2) Since $G/I = \lambda_X(G)$ is a subtorus of $\mathrm{Alb}\, X$, I/I° is an abelian discrete group. Thus the commutator group $I' \subset I^\circ$ and $I'' \subset (I^\circ)'$. By a result of Chevalley (Thm. 13 and Thm. 15 of paragraph 14, Chapter II in [Ch]) $(I^\circ)' = (\overline{I^\circ})'$ and this group is Zariski closed in $\bar{I}$.

It follows that the Zariski closure of I'' is contained in I. Since I is Zariski dense in $\bar{I}$, the same holds true for I'' in $(\bar{I})''$. Therefore $(\bar{I})'' \subset I$ and a fortiori $S_{\bar{I}^\circ} \subset I^\circ$. This fact shows that $I^\circ(x)$ equals $S_{\bar{I}^\circ}(x)$. Since $\bar{I}$

has only finitely many connected components, the assumptions imply that $\bar{I}(x)$ is compact. Obviously the same holds now for $I(x)$.

The proof of ((2) $\Longrightarrow$ (3)) shows that indeed $I(x) = \bar{I}(x)$. $\square$

COROLLARY 1 (Snow). — *C_G is analytic.*

Proof. — If the set C_G of compact G-orbits is not empty Lemma 1 shows $C_G = C_{S_{\bar{I}^\circ}} \cap (\operatorname{Fix} R_{\bar{I}^\circ})$. Since a fixed point set is obviously analytic it suffices to consider connected semisimple groups S.

A compact S-orbit in a complex space X in class $\mathcal{C}$ is a homogeneous-rational manifold S/P (Thm. on p. 255 in [F] respectively [BoRe] in the Kähler case). Thus an arbitrary, but fixed Borel subgroup B of S has non-empty fixed point set on each compact orbit.

It follows that the map

$$\phi : S \times (\operatorname{Fix} B) \to X \ , \qquad \phi(s,x) := s \cdot x$$

has image C_S. Factorizing ϕ via

$$\overline{\phi} : S/B \times (\operatorname{Fix} B) \to X \ , \qquad \overline{\phi}(sB,x) := s \cdot x$$

we realize C_S as the image of an analytic space under a proper holomorphic map. The proper mapping theorem of Remmert (see e.g. [CAS]) yields the analyticity of C_S. $\square$

COROLLARY 2. — *If C_G is not empty and A a closed G-invariant analytic set, then C_G intersects A in a non-empty set.*

Proof. — By Lemma 1 it is enough to show that $\bar{I}$, which stabilizes A, has a compact orbit on A. This follows from the fact that for all x in X, $\bar{I}(x)$ is a constructible set with respect to the analytic Zariski topology of X since $\bar{I}$ acts meromorphically/compactifiably (Lemma 2.4 in [F], Remark 3.7 in [L]). $\square$

Remark. — Lieberman states that $G \cdot A$ is Zariski open in its Zariski closure if G acts compactifiably and A is analytic. Obviously this is wrong even in the algebraic case. What is meant is that $G \cdot A$ contains a Zariski open subset of its (analytic) Zariski closure in X.

Proof of the Theorem. — Without loss of generality we assume that G acts effectively on X.

Step 1. Fibre bundle structure of the Albanese map.

By assumption C_G is not empty. Thus the induced G-action on $\mathrm{Alb}\, X$ has only compact orbits, all of them isomorphic to G/I, where $I = G \cap L(x)$. Applying Cor. 2 to the analytic sets

$$\Psi_X^{-1}\big(\lambda_X(G)(\Psi_X(x))\big) = G \cdot \Psi_X^{-1}\big(\{\Psi_X(x)\}\big)$$

it follows that G has at least as many compact orbits in X as in $\Psi_X(X)$. Since $\Psi_X(X)$ is connected the assumptions of the theorem imply that $\Psi_X(X)$ is only one G-orbit, which is a subtorus of $\mathrm{Alb}\, X$. Universality of the Albanese torus implies that Ψ_X is surjective.

Denoting $\Psi_X^{-1}(\{0\})$ by F it is easily checked that the map

$$G \times_I F \to X \ , \qquad [g, f] \mapsto g \cdot f$$

is G-equivariant and biholomorphic.

Step 2. The topology of the Ψ_X-fibre F.

Stein factorization of Ψ_X together with the universality of Ψ_X yields the connectivity of the Ψ_X-fibres.

Since I stabilizes F the same follows for the Zariski closure $\bar{I}$ in $L(X)$. By Lemma 1 an $\bar{I}$-orbit in F is closed iff the resp. I-orbit is closed iff the resp. G-orbit is closed. Thus we have only finitely many compact $\bar{I}$-orbits in F. Since $\bar{I}/\bar{I}^\circ$ is finite the same holds true for $\bar{I}^\circ$. Applying Step 1 to the action $\bar{I}^\circ \times F \to F$, $\mathrm{Alb}\, F$ turns out to be $\bar{I}^\circ$-homogeneous.

In [GeWu] it is shown that the Borel Fixed Point Theorem implies that the restriction morphism

$$\mathrm{Stab}_F \mathrm{Aut}_{\mathcal{O}}(X) \to \mathrm{Aut}_{\mathcal{O}}(F)$$

maps $L(X)^\circ$ into $L(F)$ (for F the Albanese fibre as above).

Thus $\mathrm{Alb}\, F$ is homogeneous under a subgroup of $L(F)$, which clearly says that $\mathrm{Alb}\, F$ reduces to a point. By the equality $\frac{1}{2} b_1(Y) = \dim_{\mathbb{C}}(\mathrm{Alb}\, Y)$ for Y in class $\mathcal{C}$ (Cor. 1.7 in [F]) the first Betti number of F must vanish. □

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