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Large and moderate deviations for moving average processes

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RÉSUMÉ. — Soit $X_n = \sum_{i \in \mathbf{Z}} a_{i+n}\omega_i$, $n \geqslant 1$, le processus à moyenne mobile, $(\omega_i)_{i \in \mathbf{Z}}$ étant une suite de v.a.i.i.d. réelles, et soit $S_n = (X_1 + ... + X_n)$. Dans ce papier, nous établissons un principe de grandes déviations et de déviations modérées pour S_n/n, sous les conditions suivantes : ω_i sont bornées pour tout $i \in \mathbf{Z}$ et $\sum_{n \in \mathbf{Z}} a_n^2 < \infty$.

ABSTRACT. — Let $X_n = \sum_{i \in \mathbf{Z}} a_{i+n}\omega_i$, $n \geqslant 1$, the moving average process, $(\omega_i)_{i \in \mathbf{Z}}$ i.i.d. real random values, and $S_n = (X_1 + ... + X_n)$. In this note, we prove large and moderate deviations principle for S_n/n, under the boundedness of ω_i and $\sum_{n \in \mathbf{Z}} a_n^2 < \infty$.

1. Introduction.

Let $\{\omega_i, \ i \in \mathbf{Z}\}$ be a doubly infinite sequence of independent and identically distributed square integrable real random variables with $\mathbb{E}(\omega_i) = 0$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{a_n, n \in \mathbf{Z}\}$ be a doubly infinite sequence of real numbers such that $\displaystyle\sum_{i \in \mathbf{Z}} a_i^2 < \infty$.

The moving average process X_k, $k \geqslant 1$, is defined by

$$X_k = \sum_{i \in \mathbf{Z}} a_{i+k}\ \omega_i,$$

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and let

$$S_n = \sum_{k=1}^{n} X_k.$$

Numerous works have been made on the problem of large and moderate deviations of S_n/n under the strong condition $\sum_{i \in \mathbf{Z}} |a_i| < \infty$. For example, Burton and Dehling [BD90] have proved a large deviation principle for $\{S_n/n; n \geqslant 1\}$ with speed $\{n; n \geqslant 1\}$ and a good rate function depending only on the moment generating function. Their proof, like many of the referenced papers, relies on the powerful Ellis Theorem. The moderate deviations of $\{S_n/n; n \geqslant 1\}$ are obtained by Jiang and al. [JWR92] under the condition of exponential integrability of ω_0. Jiang and al. [JWR95] proved that the upper bound of large deviations for S_n in a Banach space B holds if and only if the condition $\mathbb{E}\left(e^{q_K(\omega)}\right) < \infty$ is fulfilled for some compact K of B, where q_K is the Minkowski functional of the set K. And the lower bound of large deviations is obtained in [JWR95] without any condition, with a rate function which may not have compact level sets, and which can be different from the rate function of the upper bound.

Remember also the famous work of Donsker and Varadhan [DV85], on large deviations of level-3 for stationary Gaussian processes, under moving average form, which has motivated our study.

In this note, we prove a large deviation principle and a moderate deviation principle for moving average processes, substituting the absolute convergence condition by the continuity of $g(\theta) = \sum_{n \in \mathbf{Z}} a_n e^{in\theta}$ at 0, a well known condition for the Central Limit Theorem of $\{S_n/\sqrt{n}\}$, see ([HH80], Corollary 5.2. pp 135). But we need the boundedness of ω_i instead of the exponential integrability in the works cited above.

2. A large deviation principle
for the moving average processes.

About the language of large deviations, see Dembo and Zeitouni [DZ93], Deuschel and Stroock [DS89]. The main result of this paper is

THEOREM 2.1. — *Let $(\omega_i)_{i \in \mathbf{Z}}$ be a family of $\mathbb{P}$-i.i.d. real valued random variables. Suppose the following conditions*

(H1) $\sum_{i \in \mathbf{Z}} a_i^2 < \infty$, $\mathbb{E}(\omega_i) = 0$, $\mathbb{E}(\omega_i^2) = 1$ *and* $|\omega_i| \leqslant C$ *for all* $i \in \mathbf{Z}$.

(H2) *The function g given by $|g(\theta)|^2 := \left|\sum_n a_n e^{in\theta}\right|^2 = f(\theta)$ (the spectral density function of X_k) is continuous at 0 and belongs to $L^2([-\pi, \pi], d\theta)$.*

Then $\mathbb{P}\left(\dfrac{S_n}{n} \in \cdot\right)$ satisfies a large deviation principle with speed n and the good rate function I given by

$$I(x) = \Lambda^*\left(\frac{x}{g(0)}\right), \quad \forall x \in \mathbb{R}, \tag{2.1}$$

where Λ^* is the Fenchel-Legendre transform of the logarithmic moment generating function $\Lambda(\lambda) := \log \mathbb{E}e^{\lambda\omega_0}$ of the common law of ω_i, $i \in \mathbb{Z}$.

Remark 2.i. — Except the boundedness of ω_i, the assumption **(H1)** is minimal to define X_k.

Remark 2.ii. — Condition **(H2)** is the usual condition for the Central Limit Theorem for S_n, see [HH80]. Notice also that it is much weaker than the condition $f(\theta) \in C([-\pi, \pi])$ used in the pioneering work of Donsker-Varadhan [DV85] for the level-3 large deviations of stationary gaussian processes.

To prove Theorem 2.1 we need the following concentration inequality for S_n, which is a translation of the well known Hoeffding inequality [Hoe63] in our context.

LEMMA 2.2. — *Under condition **(H1)**, we have*

$$\mathbb{P}\left(|S_n| \geqslant t\right) \leqslant 2\, e^{-\frac{t^2}{2C^2\mathbb{E}(S_n^2)}}. \tag{2.2}$$

Proof of Lemma 2.2. — By Hoeffding inequality [Hoe63] (or more exactly its proof), applied to

$$\tilde{S}_n^K = \sum_{|i|\leqslant K} \tilde{X}_i^n$$

where $\tilde{X}_i^n = \sum_{k=1}^n a_{i+k}\omega_i$, we have for all $\lambda \geqslant 0$

$$\mathbb{E}\left(e^{\lambda\tilde{S}_n^K}\right) \leqslant \exp\left(\frac{\lambda^2 C^2}{2}\sum_{|i|\leqslant K}\left(\sum_{k=1}^n a_{i+k}\right)^2\right)$$

$$= \exp\left(\frac{\lambda^2 C^2}{2}\mathbb{E}(\tilde{S}_n^K)^2\right).$$

Letting $K \to \infty$, $\tilde{S}_n^K \longrightarrow S_n$ in $L^2(\mathbb{P})$ and we get by Fatou's lemma:

$$\mathbb{E}(e^{\lambda S_n}) \leqslant e^{\frac{\lambda^2 C^2}{2} \mathbb{E}S_n^2}.$$

Then by Markov inequality, we have that for all $t > 0$

$$\mathbb{P}(S_n \geqslant t) \leqslant e^{-t\lambda + \frac{\lambda^2 C^2}{2} \mathbb{E}S_n^2}, \ \forall \lambda \geqslant 0,$$

and optimizing in λ, we obtain

$$\forall t \geqslant 0, \quad \mathbb{P}(S_n \geqslant t) \leqslant e^{-\frac{t^2}{2C^2 \mathbb{E}S_n^2}}.$$

We have obviously the same inequality for $-S_n$. Thus (2.2) follows.

Remark 2.iii. — Inequality (2.2) can be proved by means of logarithmic Sobolev inequality for convex functions [Led96] (with less better constant), in a way which is also valid for $|S_n|$ and thus give an alternate way to establish *Step 2* in the following proof.

Proof of Theorem 2.1 : we separate its proof into three steps.

Step 1. Let $S_n^K = \sum_{k=1}^{n} X_k^K$ where we have for some fixed K in $\mathbb{N}$

$$X_k^K = \sum_{|j|<K} a_{j+k} \left(1 - \frac{|j|}{K}\right) \omega_j.$$

Then $\mathbb{P}\left(\dfrac{S_n^K}{n} \in \cdot\right)$ satisfies the large deviation principle with speed n and some good rate function I^K, by Sanov's theorem and the contraction principle, or using results of [BD90] which give I^K with the same notations as in Theorem 2.1:

$$I^K(x) = \Lambda^* \left(\frac{x}{\sum_{|j|<K} a_j \left(1 - \frac{|j|}{K}\right)} \right).$$

Step 2. We show that for all $\delta > 0$

$$\limsup_{K \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\left|\frac{S_n}{n} - \frac{S_n^K}{n}\right| > \delta\right) = -\infty. \tag{2.3}$$

By our hypothesis, we can apply Hoeffding inequality for $S_n - S_n^K$, and noting that $\mathbb{E}(S_n - S_n^K) = 0$ we get by lemma 2.2

$$\frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} |S_n - S_n^K| > \delta \right) \leqslant \frac{\log 2}{n} - \frac{n\delta^2}{2C^2 \mathbb{E} \left((S_n - S_n^K)^2 \right)}$$

We have now to control the right term of this inequality. Let f_K denote the spectral density of X_n^K, i.e. $f_K(\theta) = |g_K(\theta)|^2 := \left| \sum_{|j|<K} a_j \left(1 - \frac{|j|}{K} \right) e^{ij\theta} \right|^2$.

Let introduce Fejer's kernel F_K

$$F_K(\theta) = \frac{1}{2\pi K} \left(\frac{\sin \frac{K}{2}\theta}{\sin \frac{1}{2}\theta} \right)^2 .$$

An obvious property is that $\int_{-\pi}^{\pi} F_K(\theta)d\theta = 1$. Moreover, we have $g_K = F_K * g$, where "$*$" denotes the usual convolution product. By the well known theorem ([IL71], Theorem 18.2.1. pp322), we have

$$\frac{\mathbb{E} \left((S_n - S_n^K)^2 \right)}{n} = 2\pi \left[|g - g_K|^2 * F_n \right] (0). \qquad (2.4)$$

For any $\varepsilon > 0$, let $[-\delta, \delta]$ be such that $|g(\theta) - g(0)| < \frac{\varepsilon}{2}$ for $|\theta| < \delta$. For $|\theta| < \frac{\delta}{2}$

$$
\begin{aligned}
|g_K(\theta) - g(\theta)| &= |g * F_K(\theta) - g(\theta)| \\
&= \left| \int_{-\pi}^{\pi} F_K(\theta')g(\theta - \theta')d\theta' - g(\theta) \right| \\
&\leqslant \left| \int_{|\theta'|<\delta/2} F_K(\theta') \left(g(\theta - \theta') - g(\theta) \right) d\theta' \right| \\
&\quad + \left| \int_{|\theta'|\geqslant\delta/2} F_K(\theta') \left(g(\theta - \theta') - g(\theta) \right) d\theta' \right| \\
&\leqslant \varepsilon \int_{|\theta'|<\delta/2} F_K(\theta')d\theta' + C(K,\delta)2\pi \left(\|g\|_{L^2(d\theta')} + |g(\theta)| \right) \\
&\leqslant \varepsilon \int_{|\theta'|<\delta/2} F_K(\theta')d\theta' + C(K,\delta)2\pi \left(\|g\|_{L^2(d\theta')} + |g(0)| + \frac{\varepsilon}{2} \right),
\end{aligned}
$$

where $C(K,\delta) = \sup_{|\theta'|\geqslant\delta/2} |F_K(\theta')| \leqslant (2\pi K \sin^2(\delta/4))^{-1} \to 0$ as $K \to \infty$. We can now control the right hand side of (2.4)

$$\limsup_{K\to\infty} \left[|g - g_K|^2 * F_n \right] (0) = \limsup_{K\to\infty} \int_{-\pi}^{\pi} |g - g_K|^2(\theta)F_n(\theta)d\theta$$

$$\leqslant \limsup_{K\to\infty} \left(\sup_{|\theta|<\delta/2} |g - g_K|^2(\theta) + C(n,\delta) \int_{|\theta'|\geqslant\delta/2} |g - g_K|^2(\theta)d\theta \right)$$

$$\leqslant \varepsilon.$$

We further conclude that

$$\limsup_{K\to\infty} \limsup_{n\to\infty} \frac{\mathbb{E}\left((S_n - S_n^K)^2 \right)}{n} = 0.$$

We then deduce (2.3). Applying the approximation lemma ([DZ93], Th. 4.2.16.), we obtain that S_n satisfies the large deviations principle with speed n and the rate function

$$\tilde{I}(x) := \sup_{\delta>0} \liminf_{K\to\infty} \inf_{y\in B(x,\delta)} I^K(y) = \sup_{\delta>0} \limsup_{K\to\infty} \inf_{y\in B(x,\delta)} I^K(y)$$

where $B(x,\delta)$ is the ball of radius δ centered at x.

Step 3. It remains to show that $\tilde{I}(x) = I(x)$, where I is given by (2.1). We will first prove that $\tilde{I}(x) \leqslant I(x)$. Assume that $I(x) < \infty$ (trivial otherwise). This inequality is obvious for $x = 0$ (as $I^K(0) = 0$). Now for $x \neq 0$, the finiteness of $I(x)$ implies that $g(0) \neq 0$. For each $\delta \geqslant 0$, we have by the convergence of $g_K(0)$ to $g(0)$ that $yg(0) \in B(x,\delta)$ implies $yg_K(0) \in B(x,2\delta)$ for sufficiently large K, so that we have

$$\inf_{z\in B(x,\delta)} \{\Lambda^*(y),\, yg(0) = z\} \geqslant \inf_{z\in B(x,2\delta)} \{\Lambda^*(y),\, yg_K(0) = z\},$$

which yields $\tilde{I}(x) \leqslant I(x)$.

We now have to prove the converse inequality. Assume at first $g(0) \neq 0$, by the lower semi-continuity of I (inf-compact in reality), we have

$$\tilde{I}(x) = \sup_{\delta>0} \limsup_{K\to\infty} \inf_{y\in B(x,\delta)} \Lambda^* \left(\frac{y}{g_K(0)} \right) \geqslant \liminf_{z\to\frac{x}{g(0)}} \Lambda^*(z) \geqslant \Lambda^* \left(\frac{x}{g(0)} \right).$$

Now assume $g(0) = 0$. $\tilde{I}(0) \geqslant I(0)$ (trivial). For $x \neq 0$,

$$\tilde{I}(x) = \sup_{\delta>0} \limsup_{K\to\infty} \inf_{y\in B(x,\delta)} \Lambda^* \left(\frac{y}{g_K(0)} \right) \geqslant \liminf_{|z|\to+\infty} \Lambda^*(z) = +\infty = I(x).$$

So we have that $\tilde{I}(x) = I(x)$, which ends the proof of theorem 2.1.

Remarks. — Under the boundedness "$|\omega_i| \leqslant C$" and the strong condition $\sum_{i\in\mathbb{Z}} |a_i| < \infty$, the level-3 large deviations principle for $(X_n)_{n\in\mathbb{Z}}$ holds.

Indeed, assume without loss of generality that (ω_i) is the coordinates system on the product space $\Omega_C = [-C, C]^{\mathbf{Z}}$ equipped with the product measure $\mathbf{P} = \mu^{\otimes \mathbf{Z}}$, where μ is the common law of ω_i. Let θ_k be the shift operator acting on Ω_C, defined by $(\theta_k \omega)_l = \omega_{l-k}$, $\forall k, l \in \mathbf{Z}$, and let E_n be the empirical process of the i.i.d. sequence, defined on the space $\mathcal{P}(\Omega_C)$ of all probability measures on Ω_C:

$$E_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\theta_k \omega}.$$

By the results of Donsker-Vardahan [DV85], E_n satisfies a level-3 large deviation principle on $\mathcal{P}(\Omega_C)$ equipped with the weak convergence topology, with speed n and the good rate function given by the Donsker-Varadhan level-3 entropy $H(Q)$, see [DV85] for some details on $H(Q)$. Let ϕ be the map given by

$$\phi \quad : \quad \Omega \to \mathbf{R}^{\mathbf{Z}}$$

$$\omega \to \left(\sum_{i \in \mathbf{Z}} a_{i+k} \omega_i \right)_{k \in \mathbf{Z}}.$$

By the absolute summability $\sum_{i \in \mathbf{Z}} |a_i| < +\infty$, ϕ is continuous from Ω_C to $\mathbf{R}^{\mathbf{Z}}$ both equipped with product topology. Let $\mathcal{P}(\mathbf{R}^{\mathbf{Z}})$ be the space of all probability measures on $\mathbf{R}^{\mathbf{Z}}$ equipped with the weak convergence topology. Define on $\mathcal{P}(\mathbf{R}^{\mathbf{Z}})$ the empirical measure

$$R_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{X_{\cdot+k}}.$$

We obviously have $R_n = E_n \circ \phi^{-1}$. By the contraction principle, we conclude that R_n satisfies a level-3 large deviation principle on $\mathcal{P}(\mathbf{R}^{\mathbf{Z}})$ with speed n and the good rate function $I(Q) = \inf \left\{ H(\tilde{Q}); Q = \tilde{Q} \circ \phi^{-1} \right\}$.

3. Moderate deviations.

We are now studying moderate deviations for S_n in the same way as we have proved large deviations in the preceding section, we keep the same conditions on a_i and ω_i. To this purpose, let $(b_n)_{n \geqslant 1}$ be a sequence of positive numbers such that

$$b_n \to +\infty \quad , \quad \frac{b_n}{\sqrt{n}} \to 0. \tag{3.1}$$

THEOREM 3.1. — *Under the conditions* **(H1)** *and* **(H2)**, $\mathbb{P}\left(\dfrac{S_n}{b_n\sqrt{n}} \in \cdot\right)$ *satisfies a large deviation principle with the speed b_n^2 and the good rate function I_M given by*

$$\forall x \in \mathbb{R}, \qquad I_M(x) = \frac{x^2}{2f(0)}. \qquad (3.2)$$

Proof. — We separate its proof into two steps.

Step 1. Let $S_n^K = \displaystyle\sum_{k=1}^{n} X_k^K$ as before.

Then, by [JWR92], $\mathbb{P}\left(\dfrac{S_n^K}{b_n\sqrt{n}} \in \cdot\right)$ satisfies a large deviation principle with speed b_n^2 and the good rate function I_M^K given by

$$I_M^K(x) = \frac{x^2}{2\left(\sum_{|j|<K} a_j \left(1 - \frac{|j|}{K}\right)\right)^2}.$$

Step 2. Since $S_n - S_n^K$ satisfies assumptions of lemma 2.2, we apply the concentration inequality (2.2) to $\dfrac{S_n - S_n^K}{b_n\sqrt{n}}$: $\forall t \geqslant 0$,

$$\frac{1}{b_n^2} \log \mathbb{P}\left(\left|\frac{S_n - S_n^K}{b_n\sqrt{n}}\right| \geqslant t\right) \leqslant -\frac{n\,t^2}{2C^2 \mathbb{E}\left((S_n - S_n^K)^2\right)} + \frac{\log 2}{b_n^2}.$$

But by the proof of Theorem 2.1 we have

$$\limsup_{K\to\infty} \limsup_{n\to\infty} \frac{\mathbb{E}\left((S_n - S_n^K)^2\right)}{n} = 0$$

where it follows

$$\limsup_{K\to\infty} \limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{P}\left(\left|\frac{S_n - S_n^K}{b_n\sqrt{n}}\right| \geqslant t\right) = -\infty.$$

According to the approximation lemma ([DZ93], Th. 4.2.16.), we deduce that S_n satisfies the moderate deviations principle with speed b_n^2 and the rate function

$$\tilde{I}_M(x) := \sup_{\delta>0} \liminf_{K\to\infty} \inf_{y\in B(x,\delta)} I_M^K(y) = \sup_{\delta>0} \limsup_{K\to\infty} \inf_{y\in B(x,\delta)} I_M^K(y).$$

The identification of the rate function is done like in *Step3* of the proof of theorem 2.1.

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