ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

MASSIMO VILLARINI

Smooth linearization of centres

Annales de la faculté des sciences de Toulouse 6e série, tome 9, nº 3 (2000), p. 565-570

<http://www.numdam.org/item?id=AFST_2000_6_9_3_565_0>

Annales de la Faculté des Sciences de Toulouse Vol. IX, n° 3, 2000
pp. 565–570

Smooth linearization of centres (*)

MASSIMO VILLARINI (1)

RÉSUMÉ. — Nous étendons le Théorème du Centre de Poincaré-Lyapunov au cas non analytique : plus précisément nous démontrons qu'un champ de vecteurs de classe C^k, $k \geqslant 3$, avec un centre non dégénéré à l'origine est toujours C^{k-2}-orbitalement équivalent à sa partie linéaire dans l'origine.

ABSTRACT. — We extend the Poincaré-Lyapunov Centre Theorem to the smooth case: namely we prove that a C^k vector field, $k \geqslant 3$, having a nondegenerate centre at the origin is always C^{k-2}-orbitally equivalent to its linear part at the origin.

1. Introduction.

The classical Poincaré-Lyapunov Centre Theorem says that a centre-type singularity of an *analytic* vector field on the plane has an analytic first integral, provided that the linear part of the vector field at the singular point generates a (nontrivial) rotation.Equivalently, an analytic vector field on the plane having a singular point at O and a linear part at O generating a nontrivial rotation is C^ω-orbitally equivalent to that linear part if and only if the singular point is a centre. We extend the Centre Theorem to the smooth planar case: we prove the existence of a C^{k-2}-orbital equivalence to its linear part at the singular point for a C^k vector field with a nondegenerate centre: the double loss of regularity is due to one blow up and one blow down process.This result turns to be a an easy and quick application of a theorem

(*) Reçu le 30 avril 1999, accepté le 21 mars 2000
(1) Dipartimento di Matematica Pura e Applicata "G. Vitali", Universitá di Modena e Reggio Emilia, via Campi 213/b, 41100 Modena, Italy.
e-mail: villarini@unimo.it

of Bochner concerning the linearization of the action of a compact group on a manifold near a fixed point for that action.

Aknowledgments: we thank M. Brunella for many stimulating and useful discussions.

2. Linearization of centres.

An isolated singular point O of a planar vector field is a *centre* if there exists a neighbourhood U of O, such that $U \setminus \{O\}$ is filled by closed nontrivial trajectories.A centre at O is *nondegenerate* if the linear part at O of the vector field has eigenvalues $\pm i\omega$.Without loss of generality we will always consider $\omega = 1$. The classical Poincaré-Lyapunov Centre Theorem [6], [4] states

THEOREM 2.1. — *Let X be an analytic vector field in a neighbourhood of the origin, and let its linear part at the origin generate a nontrivial rotation. Then the origin is a singular point of (nondegenerate) centre type for X if and only if X has an analytic first integral of type*

$$F(x,y) = Q(x,y) + \text{higher order terms}$$

where $Q(x,y)$ is a positive definite quadratic form. $\square$

Let us explicitly observe that if the origin of the (x, y) coordinate is a singular point of X, to say that the linear part at the origin of the vector field, $DX(0,0)$ generates a nontrivial rotation is equivalent to say that, up to linear change of coordinate and multiplication by nonzero real factor, we have

$$DX(0,0) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Let $X : U \subset \mathbf{R}^n \mapsto TU$, $Y : V \subset \mathbf{R}^n \mapsto TV$ be two C^k vector fields, $1 \leqslant k \leqslant \omega$: X and Y are *C^k-orbitally equivalent* if there exist a C^k map $\Phi : U \mapsto V$ and a C^k nonvanishing function $f : V \mapsto \mathbf{R}$ such that

$$\Phi_* X \circ \Phi^{-1} = fY.$$

A simple application of Morse Lemma permits to state the Poincaré-Lyapunov Centre Theorem in the following form

THEOREM 2.2. — *Let X be an analytic vector field in a neighbourhood of the origin, and let its linear part at the origin generate a nontrivial rotation. Then the origin is a singular point of nondegenerate centre type for X if and only if X is C^ω-orbitally equivalent to its linear part at the origin.*

We consider vector fields X on the plane, of class C^k, $k \geqslant 3$, having a nondegenerate centre at the origin i.e. with linear part at the origin which generates a rotation.We prove

THEOREM 2.3.— *Let X be of the above specified regularity and form. Then the origin is a nondegenerate centre if and only if X is C^{k-2}-orbitally equivalent to its linear part at the origin.*

To prove this theorem we need a simple lemma related to the lifting of a function *via* polar (spherical) coordinates.

LEMMA 2.4.— *Let*

$$f : \mathbf{R}^m \mapsto \mathbf{R}$$

such that

(i) $f \in C^p(\mathbf{R}^m) \quad , \quad p \geqslant 2$

(ii) $f(x) = o(\|x\|^p)$

Then the function $F : \mathbf{R}^m \mapsto \mathbf{R}$ defined as

$$F(x) = \frac{f(x)}{\|x\|^2}$$

when $\|x\| \neq 0$, and $F(x) = 0$ otherwise, is a C^{p-2} function in a neighbourhood of the origin and $F(x) = o(\|x\|^{p-2})$.

Proof.— The case $p = 2$ is trivial.The general case is proved by a simple inductive argument.Let $\underline{\mathrm{n}} = (n_1, \dots, n_m)$ a multindex of lenght $n = n_1 + \dots + n_m$: we consider, for $x \neq 0$ and for $D^{(\underline{\mathrm{n}})}$ a n-order differential operator

$$D^{(\underline{\mathrm{n}})}(\frac{f(x)}{\|x\|^2}) = \frac{f^{(\underline{\mathrm{n}})}}{\|x\|^{2^{n+1}}}.$$

By induction is easy to prove that

$$f^{(\underline{\mathrm{n}})} = o(\|x\|^{2^{n+1}+p-2-n})$$

and this concludes the proof. $\square$

We are now able to prove the smooth generalization of the Centre Theorem:

Proof (Theorem (2.3)).

We consider first the C^k case, $k < \infty$.Classical results of normal form theory [1] implies that, up to a C^k change of coordinates, in a sufficiently small neighbourhood of the origin the vector field X generates the differential equation

$$\begin{aligned} \dot{x} &= -y(1+g(r^2)) + A(x,y) \\ \dot{y} &= x(1+g(r^2)) + B(x,y) \end{aligned}$$

where $r^2 = x^2 + y^2$, $g(u) = a_1 u + \cdots + a_l u^l$, $2l \leqslant k-1$ and $A(x,y)$, $B(x,y)$ are C^k functions with $A(x,y) = o(r^k)$, $B(x,y) = o(r^k)$. The transformation to polar coordinates

$$\pi : S^1 \times \mathbf{R}^+ \mapsto \mathbf{R}^2$$

where

$$\pi(r,\phi) = (r\cos\phi, r\sin\phi) = (x,y)$$

defines a vector field $\pi_*^{-1}X$ on $S^1 \times \mathbf{R}^+$: such a vector field has a (unique) C^{k-1} extension to $S^1 \times \mathbf{R}_0^+$ [7]:

$$\tilde{X} : S^1 \times \mathbf{R}_0^+ \mapsto T(S^1 \times \mathbf{R}_0^+)$$

such that

$$\pi_*(\tilde{X}) = X.$$

The C^{k-1} differential system generated by $\tilde{X}$ is

$$\begin{aligned} \dot{r} &= \cos\phi A(r\cos\phi, r\sin\phi) + \sin\phi B(r\cos\phi, r\sin\phi) = R(r,\phi) \\ \dot{\phi} &= 1 + g(r^2) + r^{-1}(-\sin\phi A(r\cos\phi, r\sin\phi) \\ &\qquad + \cos\phi B(r\cos\phi, r\sin\phi) = \Theta(r,\phi). \end{aligned}$$

Here $R(r,\phi) = o(r^k)$, $\Theta(r,\phi) = 1 + g(r^2) + o(r^{k-1})$, and $R(r,\phi)$, $\Theta(r,\phi)$ are respectively C^k, and C^{k-1} functions of their arguments. We will show now that $\Theta(r,\phi)$ projects to a function $\Theta \circ \pi^{-1}(x,y)$ which extends to a C^{k-2} function $\Delta(x,y)$ defined in a neighbourhood of the origin.In fact outside the origin

$$\Delta(x,y) = \frac{xB(x,y) - yA(x,y)}{x^2+y^2} + \quad \text{a polynomial in} \quad (x^2+y^2)$$

therefore we just need to prove that

$$\frac{xB(x,y) - yA(x,y)}{x^2+y^2} = \frac{F(x,y)}{x^2+y^2}$$

has a C^{k-2} extension up to the origin:this follows from Lemma (2.4).The vector field $Y = \frac{1}{\Delta}X$ is C^{k-2}-orbitally equivalent to X, and it has an isochronous (*i.e.* all the trajectories have the same period 2π) centre at the origin: in fact, in polar coordinates it generates the equation

$$\begin{aligned} \dot{r} &= \frac{R(r,\phi)}{\Theta(r,\phi)} \\ \dot{\theta} &= 1 \end{aligned}$$

Let us recall now the following classical

THEOREM 2.5 *([5] p. 206).— Let G be a compact group defining a C^k-action on a C^k-manifold M, $1 \leqslant k \leqslant \omega$ and let $p \in M$ be a fixed point for that action. Then there exists a neighbourhood U of p, a choice of coordinates x on U and a C^k map $x \mapsto R(x)$, such that R is tangent to the identity and for every $g \in G$*

$$R \circ L_g(x) = d_pL_g(R(x))$$

where d_pL_g is the differential at p of the induced action diffeomorphism L_g. $\square$

We apply this theorem in the following way: the vector field Y generates *via* its flow a C^{k-2}-action of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ on a neighbourhood of the origin in the plane: by Theorem (2.5) we can linearize this action and therefore we linearize Y, too: this concludes the proof. $\square$

Remark. — as a consequence of this theorem, the *period function* $T : V \setminus \{O\} \mapsto \mathbf{R}^+$ of a planar C^k-regular non degenerate centre at O, $k \geqslant 2$, has a C^{k-2}-extension up to the origin O.

Remark. — in the multidimensional case the above proof does not extend: see [2] for the definition of *multicentre*, the multidimensional analogous of a planar (nondegenerate) centre, and [8] and [2] for linearization results of analytic multicentres. The hypothesis of analyticity of the vector field is crucial in [2] and [8], togheter with a hypothesis on boundness of the period function in a punctured neighbourhood of the singular point. The reason why the above theorem does not extend to the multidimensional case is the (possible) nonexistence of a simple and essentially unique normal form for a multicentre, differently to the planar case. By looking at this dynamical problem in more than two dimensions after we perform one (say spherical) blow up, we observe that it is probably not unrelated to the works (and counterexamples) concerning the stability problem for compact foliations (see *e.g.* [3]): the reader should consult [2] for more details about this topic.

Bibliography

[1] ARNOLD (V.I.). — *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, MIR, Moscow, 1978.

[2] BRUNELLA (M.), VILLARINI (M.). — On Poincaré-Lyapunov Centre Theorem, *Bol. Soc. Mat. Mexicana (3)*, vol. **5**, 1999, 155-161.

[3] EDWARDS (R.), MILLET (K.), SULLIVAN (D.). — Foliations with all leaves compact, *Topology* **16** (1977) 13-22.

[4] LYAPUNOV (A.M.). — *Stability of Motion*, Math. Sci.Engrg. **30**, Academic Press, London-New York, 1966.

[5] MONTGOMERY (D.), ZIPPIN (L.). — *Topological Transformation Groups* , Interscience (1955).

[6] POINCARÉ (H.). — Mémoires sur le courbes définies par une équation différentielles, *J. Math. Pures Appl. (3)* **37** (1881), 375-442.

[7] TAKENS (F.). — Singularities of vector field, *Publ. Math. IHES* **43** (1974) 48-100.

[8] URABE (M.), SIBUYA (Y.). — On Center of Higher Dimensions, *J. Sci. Hirosh. Univ.* **19** 1 (1955) 87-100.