ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

MUSTAPHA MOKHTAR-KHARROUBI
AHMED ZEGHAL

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*Annales de la faculté des sciences de Toulouse 6e série*, tome 9, nº 3 (2000), p. 487-507

<http://www.numdam.org/item?id=AFST_2000_6_9_3_487_0>

Annales de la Faculté des Sciences de Toulouse Vol. IX, n° 3, 2000
pp. 487–507

# Inverse problems for periodic transport equations (*)

MUSTAPHA MOKHTAR-KHARROUBI (1), AHMED ZEGHAL (2)

RÉSUMÉ. — Dans [14], [15], une classe de problèmes inverses a été introduite et étudiée pour des conditions de flux rentrant nul. Le problème consiste à déterminer explicitement des termes de sources à partir de moments (en vitesses) de la solution à l'aide de mesures signées adéquates. Nous étendons ces résultats au tore et montrons leur optimalité.

ABSTRACT. — In [14], [15] a class of inverse problems has been introduced and studied for nonincoming boundary conditions. The problem consists in determining explicitly the internal source from (velocity) moments of the solution by means of appropriate signed measures. We extend these results to the torus and show their optimality.

## 1. Introduction

There is an important literature devoted to inverse problems in transport theory. The reader is referred to the reviews [9], [10] by McCormick for a great deal of references up to 1986 and to [14] Chap. 11 for more recent works (see also the bibliography of the present paper).

(*) Reçu le 14 mars 2000, accepté le 19 octobre 2000

(1) Université de Franche-Comté, équipe de mathématiques-UMR CNRS 6623, 16 Route de Gray, 25030 Besançon Cedex, France,
Email:mokhtar@Math.univ-fcomte.fr

(2) Université Cadi Ayyad, Faculté des Sciences et Techniques de Béni-Mellal, Département de mathématiques Appliquées et Informatique, BP. 523, Béni-Mellal, Morocco,
Email:a.zeghal@fstbm.ac.ma

A new class of inverse problems was considered in [14] Chap. 11 and the results were partially announced in [15], [16]. The problems consist in the *explicit determination* of the "spatial parts" of internal sources from suitable (velocity) moments of the solution of integro-differential transport equations for the classical *vacuum* boundary conditions. Typically, for collisionless transport equations and velocity-independent sources $S$, results of the following type are given in [14], [15] :

There exists a class $C$ of bounded Radon measures $d\mu$ on the velocity space $V$ such that for each $d\mu \in C$, there exists a measure $d\mu'$ (given explicitly in terms of $d\mu$) such that the knowledge of the (velocity) moments of the solution $f$

$$\varphi_1(x) \ := \ \int_V f(x,v)\ d\mu(v) \ \text{ and } \ \varphi_2(x) \ := \ \int_V f(x,v)\ d\mu'(v)$$

determines explicitly the internal source $S$ by the formula

$$\Delta\varphi_1 = c_N S + \varphi_2$$

where $c_N$ is a constant depending on the dimension $N$ and $\Delta$ denotes the Laplacian operator. In general, $\varphi_1$ is related to the source $S$ by a compact operator (of convolution type)

$$T_\mu \ : \ S \in L^2(D) \to \varphi_1 \in L^2(D).$$

The determination of $S$ amounts to some *"deconvolution"* procedure. The kernel of $T_\mu$ is, *in general*, weakly singular, i.e. of order

$$\frac{1}{|x|^{N-1}}.$$

A basic idea is that if the measure $d\mu$ is chosen *appropriately*, then the singularity of the kernel of $T_\mu$ is weakened to

$$\frac{1}{|x|^{N-2}}$$

for $N > 2$ and to

$$\operatorname{Log}\,(|x|)$$

for $N = 2$. Hence a connection with the fundamental solution of the Laplacian is derived and lies behind this kind of inverse results.
The present paper extends the results above to the $N$-dimensional torus by taking full advantage of Fourier Analysis. Moreover, sources with *$M$-degenerate dependence on velocities* can be recovered at the cost of $2M$

velocity moments of the solution. We also show how this tool applies to the determination of the "spatial part" of scattering kernels. Finally, the various assumptions on the class $C$ (of Radon measures on $V$) are shown to be necessary and sufficient for the validity of our inverse results. Useful remarks on time dependent problems are also given at the end of the article.

We would like to thank the referee for his helpful remarks and suggestions.

## 2. Inverse problems in one dimension

We first consider the following periodic transport equation in a purely absorbing medium

$$\begin{cases} \mu\dfrac{\partial\psi}{\partial x} + \sigma(\mu)\psi(x,\mu) = S(x,\mu) \\ \psi(0,\mu) = \psi(2\pi,\mu) \end{cases} \tag{2.1}$$

where $x \in (0, 2\pi)$, $\mu \in (-1, 1)$.

Let $d\alpha$ be a (not necessarily positive) bounded measure on $(-1, 1)$ satisfying the following properties

$$d\alpha \text{ is invariant by symmetry with respect to zero} \tag{$H1$}$$

$$\int_0^1 \frac{d|\alpha|(\mu)}{\mu^2} < \infty \tag{$H2$}$$

where $d|\alpha|$ is the absolute value of the measure $d\alpha$. Let

$$X = L^2\Big((0, 2\pi) \times (-1, 1)\ ;\ dx \otimes d|\alpha|\Big),$$

where $dx$ is the Lebesgue measure on $(0, 2\pi)$ and let

$$W = \Big\{\psi \in X \mid \mu\frac{\partial\psi}{\partial x} \in X\Big\}.$$

In view of Eq (2.1) and the inverse problem, we assume that

$$\sigma \in L^\infty\Big((-1,1); d|\alpha|\Big) \ \text{ and } \ \mathrm{d}|\alpha| \text{ ess inf } \sigma = \lambda^* > 0 \tag{$H3$}$$

$$\sigma \ \text{ and } \ \mathrm{S(x,.)} \ \text{ are even in } \mu \ \ \mathrm{d}|\alpha| \text{ a.e.} \tag{$H4$}$$

$$S \in X. \tag{$H5$}$$

It is easy to see that under Assumptions $(H3)$ and $(H5)$, Eq (2.1) has a unique solution $\psi \in W_{\text{per}}$ where

$$W_{\text{per}} = \{\psi \in W \mid \psi(0,.) = \psi(2\pi,.)\}.$$

We state now the main result of this section

THEOREM 2.1.— *Let $(H1)-(H4)$ be satisfied and assume that*

$$S \in L^2\Big((0,2\pi)\times(-1,1); dx \otimes \frac{d|\alpha|}{\mu^2}\Big). \tag{$H6$}$$

*Then*

(i) $$\psi,\ \mu\frac{\partial\psi}{\partial x} \in L^2\Big((0,2\pi)\times(-1,1); dx \otimes \frac{d|\alpha|}{\mu^2}\Big)$$

(ii) $$\varphi_1(.) = \int_{-1}^{1} \psi(.,\mu)d\alpha(\mu) \in H^2(0,2\pi)$$

*and satisfies*

$$\varphi_1''(x) = -\int_{-1}^{1} \sigma(\mu)S(x,\mu)\frac{d\alpha(\mu)}{\mu^2} + \varphi_2(x)$$

*where* $\varphi_2(.) = \displaystyle\int_{-1}^{1} \psi(.,\mu)\sigma(\mu)^2\frac{d\alpha(\mu)}{\mu^2}$. □

Before giving the proof, we derive several practical consequences.

COROLLARY 2.1.— *Let the source $S$ be of the form $S(x,\mu) = S_1(x)S_2(\mu)$. Then the knowledge of $\displaystyle\int_{-1}^{1} \sigma(\mu)S_2(\mu)\frac{d\alpha(\mu)}{\mu^2}$ and of the two moments of the solution $\psi$ of Eq (2.1) with respect to $d\alpha$ and $\sigma(\mu)^2\dfrac{d\alpha}{\mu^2}$ yields the spatial part of the source :*

$$S_1(x) = \Big[\varphi_2(x) - \varphi_1''(x)\Big]\Big(\int_{-1}^{1} \sigma(\mu)S_2(\mu)\frac{d\alpha(\mu)}{\mu^2}\Big)^{-1}. \qquad \square$$

*Remark* 2.1.— If the source $S$ is $M$-degenerate with respect to velocities, i.e.

$$S(x,\mu) = \sum_{j=1}^{M} S_j^1(x)\ S_j^2(\mu),$$

then Theorem 2.1 provides us with a linear combination of $S_j^1$ $(1 \leqslant j \leqslant M)$

$$\sum_{j=1}^{M} S_j^1(x) \int_{-1}^{+1} \sigma(\mu)\ S_j^2(\mu)\ \frac{d\alpha(\mu)}{\mu^2} = \varphi_2 - \varphi_1''.$$

Clearly, if $\{S_j^2(.)\ ;\ 1 \leqslant j \leqslant M\}$ are known, the determination of $\{S_j^1(.)\ ;\ 1 \leqslant j \leqslant M\}$ requires more measures $d\alpha$. Thus , we easily obtain the following result. □

COROLLARY 2.2.— *Let $\{d\alpha_i\ ;\ 1 \leqslant i \leqslant M\}$ be a set of $M$ signed measures satisfying Assumptions (H1), (H2) and let $S$ be a $M$-degenerate source, i.e.*

$$S(x,\mu) = \sum_{j=1}^{M} S_j^1(x)\ S_j^2(\mu),$$

*satisfying (H4) and (H2) for each $d\alpha_i$ $(1 \leqslant i \leqslant M)$. Define the moments*

$$\varphi_1^i = \int_{-1}^{+1} \varphi(.\,,\mu)\ d\,\alpha_i(\mu) \text{ and } \varphi_2^i = \int_{-1}^{+1} \varphi(.\,,\mu)\ \sigma(\mu)^2\ \frac{d\,\alpha_i(\mu)}{\mu^2}\ (1 \leqslant i \leqslant M).$$

*Then $\varphi_1^i \in H^2([0,2\pi])$ and*

$$\sum_{j=1}^{M} \beta_{ij}\ S_j^1(x) = \varphi_2^i - (\varphi_1^i)''\ (1 \leqslant i \leqslant M)$$

*where*

$$\beta_{ij} = \int_{-1}^{+1} \sigma(\mu)\ S_j^2(\mu)\ \frac{d\,\alpha_i(\mu)}{\mu^2}.$$

*In particular, if $\{S_j^2(.)\ ;\ 1 \leqslant j \leqslant M\}$ are known, then $\{S_j^1(.)\ ;\ 1 \leqslant j \leqslant M\}$ are recovered from the moments $\{\varphi_1^i\ ;\ 1 \leqslant i \leqslant M\}$ and $\{\varphi_2^i\ ;\ 1 \leqslant i \leqslant M\}$ provided the matrix $\{\beta_{ij}\}$ is invertible.* □

*Remark* 2.2.— Note that, after recovering $\{S_j^1(.)\ ;\ 1 \leqslant j \leqslant M\}$, the solution $\psi$ itself is recovered from Eq (2.1). Thus, for known $\{S_j^2(.)\ ;\ 1 \leqslant j \leqslant M\}$, the solution $\psi$ to Eq (2.1) is recovered from 2M (velocity) moments $\{\varphi_1^i,\ \varphi_2^i\ 1 \leqslant i \leqslant M\}$. □

*Proof of Theorem* 2.1.— Note that $(i)$ is a consequence of the existence theory for Eq (2.1) when we replace $d|\alpha|$ by $\frac{d|\alpha|}{\mu^2}$. To deal with the second part of Theorem 2.1., we expand $\psi$ and $S$ into (spatial) Fourier series of *periodic distributions*

$$\begin{cases} \psi = \sum_{k\in\mathbb{Z}} f_k(\mu)e^{ixk}\ , \quad f_k(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \psi(x,\mu)e^{-ixk}dx \\ S = \sum_{k\in\mathbb{Z}} g_k(\mu)e^{ixk}\ , \quad g_k(\mu) = \frac{1}{2\pi}\int_0^{2\pi} S(x,\mu)e^{-ixk}dx \end{cases} \tag{2.2}$$

where $(x,\mu) \in (0,2\pi)\times(-1,1)$ and where $f_k,\ g_k \in L^2\Big((-1,1);\frac{d|\alpha|}{\mu^2}\Big)$. Observe that Eq (2.1) yields

$$\psi(x,\mu) = \sum_{k\in\mathbb{Z}} \frac{g_k(\mu)}{\sigma(\mu)+i\mu k}e^{ixk} \tag{2.3}$$

and

$$f_k(\mu) = \frac{g_k(\mu)}{\sigma(\mu)+i\mu k}\ .. \tag{2.4}$$

Let

$$\varphi_1(x) = \int_{-1}^{1} \psi(x,\mu)d\alpha(\mu).$$

Then

$$\varphi_1(x) = \sum_{k\in\mathbb{Z}}\Big(\int_{-1}^{1}\frac{g_k(\mu)}{\sigma(\mu)+i\mu k}d\alpha(\mu)\Big)e^{ixk},$$

and by differentiating in the sense of periodic distributions

$$\varphi_1'' = -\sum_{k\in\mathbb{Z}} k^2\Big(\int_{-1}^{1}\frac{g_k(\mu)}{\sigma(\mu)+i\mu k}d\alpha(\mu)\Big)e^{ixk}.$$

In view of $(H1)$ and the eveness of $\sigma(.)$ and $g_k(.)$, it follows that

$$\begin{aligned} k^2\Big|\int_{-1}^{1}\frac{g_k(\mu)}{\sigma(\mu)+i\mu k}d\alpha(\mu)\Big| &= k^2\Big|\int_0^1 \frac{2\sigma(\mu)g_k(\mu)\ d\alpha(\mu)}{\sigma(\mu)^2+\mu^2k^2}\Big| \\ &\leqslant 2\ \|\sigma(.)\|_{L^\infty}\int_0^1\frac{k^2\mu^2}{\sigma(\mu)^2+\mu^2k^2}|g_k(\mu)|\frac{d|\alpha|(\mu)}{\mu^2} \\ &\leqslant 2\ \|\sigma(.)\|_{L^\infty}\Big(\int_0^1\frac{d|\alpha|(\mu)}{\mu^2}\Big)^{1/2}\Big(\int_0^1|g_k(\mu)|^2\frac{d|\alpha|(\mu)}{\mu^2}\Big)^{1/2}. \end{aligned}$$

By noting that (Parseval formula)

$$\frac{1}{2\pi}\int_0^1\frac{d|\alpha|(\mu)}{\mu^2}\int_0^{2\pi}|S(x,\mu)|^2dx = \sum_{k\in\mathbb{Z}}\Big(\int_0^1|g_k(\mu)|^2\frac{d|\alpha|(\mu)}{\mu^2}\Big) < \infty,$$

it follows that

$$\sum_{k\in\mathbb{Z}} |k^2 \int_{-1}^{1} \frac{g_k(\mu)}{\sigma(\mu)+i\mu k} d\alpha(\mu)|^2 < \infty.$$

Hence $\varphi'' \in L^2(0,2\pi)$ by Parseval formula. By the eveness assumptions $(H1)$, $(H4)$

$$\begin{aligned}\int_{-1}^{1} k^2 \frac{g_k(\mu)}{\sigma(\mu)+i\mu k} d\alpha(\mu) &= \int_{-1}^{1} \frac{k^2\mu^2}{\sigma(\mu)^2+\mu^2k^2} g_k(\mu)\sigma(\mu)\frac{d\alpha(\mu)}{\mu^2} \\ &= \int_{-1}^{1} \sigma(\mu) g_k(\mu) \frac{d\alpha(\mu)}{\mu^2} \\ &\quad - \int_{-1}^{1} \frac{\sigma(\mu)^3}{\sigma(\mu)^2+\mu^2k^2} g_k(\mu)\frac{d\alpha(\mu)}{\mu^2}.\end{aligned}$$

According to (2.4)

$$\begin{aligned}\int_{-1}^{1} \frac{\sigma(\mu)^3}{\sigma(\mu)^2+\mu^2k^2} g_k(\mu)\frac{d\alpha(\mu)}{\mu^2} &= \int_{-1}^{1} \frac{\sigma(\mu)^2}{\sigma(\mu)+i\mu k} g_k(\mu) d\alpha(\mu) \\ &= \int_{-1}^{1} f_k(\mu)\sigma(\mu)^2 \frac{d\alpha(\mu)}{\mu^2} .\end{aligned}$$

Thus

$$\varphi_1''(x) = -\int_{-1}^{1} \sigma(\mu) S(x,\mu) \frac{d\alpha(\mu)}{\mu^2} + \int_{-1}^{1} \psi(x,\mu)\sigma(\mu)^2 \frac{d\alpha(\mu)}{\mu^2}$$

which finishes the proof. □

*Remark* 2.3. — We can also deal with inverse problems for transport equations involving (partially known) collission operators with scattering kernels of the form

$$k(x,\mu,\mu') = \sum_{i=1}^{M} k_i^2(\mu)\ k_i^1(x,\mu').$$

Indeed, consider the transport equation

$$\begin{aligned}&\mu \frac{\partial \psi}{\partial x} + \sigma(\mu)\ \psi(x,\mu) \\ &\quad = \sum_{i=1}^{M} k_i^2(\mu) \int_{-1}^{+1} k_i^1\ (x,\mu')\psi(x,\mu')\ d\lambda(\mu') + \sum_{j=1}^{M'} S_j^1(x)\ S_j^2(\mu)\end{aligned}$$

where $d\lambda$ is a positive measure on $[-1,+1]$. We assume that $\{S_j^2(.)\ ;\ 1 \leqslant j \leqslant M'\}$ and $\{k_i^2(.)\ ;\ 1 \leqslant i \leqslant M\}$ are known. One sees that we fail withing the frame of Corollary 2.2 where the right hand side $R(x,\mu)$ of the equation is $(M+M')$-degenerate with respect to velocities. By introducing suitable $(M+M')$ signed measures $d\alpha_i$ $(1 \leqslant i \leqslant M+M')$, the knowledge of $2(M+M')$ velocity moments of the solution allows the recontruction of the "spatial part" of the right hand side term, i.e.

$$\{S_j^1(.)\ ;\ 1 \leqslant j \leqslant M'\} \text{ and } \left\{\int_{-1}^{+1} k_i^1(x,\mu')\ \psi(x,\mu')\ d\lambda(\mu')\ ;\ 1 \leqslant i \leqslant M\right\}.$$

We leave the formal statement of this result to the interested reader. Observe that the right hand side $R(x,\mu)$ being recovered, the solution itself is recovered (Remark 2.2) so that, if $k_i^1(x,\mu')$ is separable, i.e.

$$k_i^1(x,\mu') = \hat{k}_i^1(x)\ \tilde{k}_i^1(\mu')$$

and if $\{\tilde{k}_i^1(.)\ ;\ 1 \leqslant i \leqslant M\}$ are known, then we recover the terms

$$\int_{-1}^{+1} \tilde{k}_i^1(\mu')\ \psi(x,\mu')\ d\lambda(\mu')$$

and then the "spatial part" of the scattering kernel $\left\{\hat{k}_i^1(.)\ ;\ 1 \leqslant i \leqslant M\right\}$ satisfies the linear equation

$$\sum_{i=1}^{M} k_i^2(\mu)\ \hat{k}_i^1(x) \int_{-1}^{+1} \tilde{k}_i^1(\mu')\ \psi(x,\mu')d\lambda(\mu') + \sum_{j=1}^{M'} S_j^1(x)\ S_j^2(\mu) = R(x,\mu).$$

We note that, in the case $M=1$, we can recover the cross-section $\hat{k}^1(x)$. □

## 3. Inverse problems in N-dimensions ($N \geqslant 2$)

The first part of this section is devoted to isotropic sources. More precisely, we consider the multidimensional transport equation with periodic boundary conditions in a purely absorbing medium, where the source is *independent of the velocity*

$$\begin{cases} v.\dfrac{\partial \psi}{\partial x} + \sigma(|v|)\psi(x,v) = S(x) & (x,v) \in D \times V \\ \psi_{|x_i=0} = \psi_{|x_i=2\pi} & (1 \leqslant i \leqslant N) \end{cases} \tag{3.1}$$

where $D$ is the cube $(0,2\pi)^N$ and $V = \{v \in \mathbb{R}^N / |v| < 1\}$.

Let $d\mu(v) = d\alpha(\rho) \otimes dS(w)$ be a measure on $V$, where $dS$ is the Lebesgue measure on $S^{N-1}$ (the unit sphere of $\mathbb{R}^N$) and where $d\alpha$ is a bounded measure on $[0,1)$ satisfying the following properties

$$\int_0^1 \frac{d\alpha(\rho)}{\rho} = 0 \tag{H7}$$

$$\int_0^1 \frac{d|\alpha|(\rho)}{\rho^2} < \infty .. \tag{H8}$$

Let $S \in L^2(D; dx)$ and

$$X = L^2\Big(D \times V;\ dx \otimes d|\alpha| \otimes dS\Big).$$

Note that if

$$\sigma \in L^\infty\Big([0,1); d|\alpha|\Big) \quad \text{and} \quad \mathrm{d}|\alpha| \ \operatorname{ess\,inf} \sigma = \lambda^* > 0 \tag{H9}$$

then Eq (3.1) has a unique solution $\psi \in W_{\text{per}}$ where

$$W_{\text{per}} = \Big\{\psi \in X / v.\frac{\partial \psi}{\partial x} \in X\ ;\ \ \psi_{|x_i=0} = \psi_{|x_i=2\pi}\ (1 \leqslant i \leqslant N)\Big\}. \qquad \square$$

In the sequel, we use the following hypotheses

$$S \in L^2(D; dx) \tag{H10}$$

$$\sigma \in W^{1,\infty}\Big([0,1]\Big). \tag{H11}$$

In view of the statement of our results we define the following function

$$P(\rho) = \int_0^\rho \frac{d\alpha(s)}{s} \tag{3.2}$$

and the following bounded measure

$$d\beta(\rho) = (N-3)\Big[\sigma(\rho)(\rho\sigma'(\rho) - \sigma(\rho))\frac{P(\rho)}{\rho^2}\Big]d\rho + \rho d\Big(\frac{\sigma(\rho)^2 P(\rho)}{\rho^2}\Big) \tag{3.3}$$

where $d\rho$ is the Lebesgue measure on $[0,1]$.

Now we state a basic result of this section

THEOREM 3.1.— *We assume that* $(H7) - (H11)$ *are satisfied. Let* $\psi$ *be the solution of* (3.1). *Let*

$$\varphi_1(x) = \int_0^1 d\alpha(\rho) \int_{S^{N-1}} \psi(x, \rho w) dS(w)$$

*and*

$$\varphi_2(x) = \int_0^1 d\beta(\rho) \int_{S^{N-1}} \psi(x, \rho w) dS(w).$$

*Then*

(i) $$\Delta\varphi_1 \in L^2(D; dx) \quad (\textit{consequently } \varphi_1 \in H^2(D))$$

(ii) $$\Delta\varphi_1 = C_N S(x) + \varphi_2(x)$$

*where* $C_N = (N-2)|S^{N-1}| \int_0^1 \frac{\sigma(\rho)}{\rho^2} d\alpha(\rho)$ *and where* $d\beta$ *is the measure defined by* (3.3). □

*Remark* 3.1. — Note that we can recover the source term only for $N \neq 2$ (and $\int_0^1 \frac{\sigma(\rho)}{\rho^2}\, d\beta(\rho) \neq 0$). The same curious phenomenon occurs for non-incoming boundary conditions (see [14]) and also in the problem of recovering the collision kernel from the albedo operator on the boundary (see [7]). □

*Remark* 3.2. — We recall the useful formula (see [17]) for $f \in \mathcal{C}([-1,1])$

(i) $$\int_{S^{N-1}} f(\omega.\omega_0) d\, S(\omega) = |S^{N-2}| \int_{-1}^1 f(t)(1-t^2)^{\frac{N-3}{2}}\, dt \ (\omega_0 \in S^{N-1}).$$

Note that $f \equiv 1$ yields

(ii) $$|S^{N-1}| = 2\ |S^{N-2}| \int_0^1 (1-t^2)^{\frac{N-3}{2}} dt.$$

Integrating by parts $\int_0^1 (1-t^2)^{\frac{N-3}{2}} dt$, rearranging terms and using (i) we obtain the identity

(iii) $$2(N-3)\ |S^{N-2}| \int_0^1 (1-t^2)^{\frac{N-5}{2}}\, dt = (N-2)|S^{N-1}|$$

which will be used in the sequel. □

*Proof of Theorem 3.1 for* $N > 2$. — As in the proof of Theorem 2.1, we use (spatial) Fourier series of the periodic distributions $\psi$ and $S$

$$\begin{cases} \psi = \sum_{k \in \mathbb{Z}^N} f_k(v) e^{i(x.k)} \\ S = \sum_{k \in \mathbb{Z}^N} g_k e^{i(x.k)} \end{cases} \tag{3.4}$$

where $(x, v) \in D \times V$. Eq (3.1) yields

$$\psi(x, v) = \psi(x, \rho w) = \sum_{k \in \mathbb{Z}^N} \frac{g_k}{\sigma(\rho) + i\rho(w.k)} e^{i(x.k)} \tag{3.5}$$

where $v = \rho w,\ \rho \in [0, 1)$ and $w \in S^{N-1}$. Note that

$$\varphi_1(x) = \int_0^1 d\alpha(\rho) \int_{S^{N-1}} \psi(x, \rho w) dS(w)$$

expands as

$$\varphi_1(x) = \sum_{k \in \mathbb{Z}^N} g_k \Big( \int_0^1 d\alpha(\rho) \int_{S^{N-1}} \frac{dS(w)}{\sigma(\rho) + i\rho(w.k)} \Big) e^{i(x.k)} \ . \tag{3.6}$$

According to remark 3.2 we can write

$$\varphi_1(x) = |S^{N-2}| \sum_{k \neq 0} g_k \Big( \int_0^1 d\alpha(\rho) \int_{-1}^1 \frac{(1-t^2)^{\frac{N-3}{2}}}{\sigma(\rho) + i\rho|k|t} dt \Big) e^{i(x.k)}$$
$$+ g_0 |S^{N-1}| \int_0^1 \frac{d\alpha(\rho)}{\sigma(\rho)}.$$

Let

$$G_0 = g_0 |S^{N-1}| \int_0^1 \frac{d\alpha(\rho)}{\sigma(\rho)} \ . \tag{3.7}$$

We write $\varphi_1$ in the form
$\varphi_1(x) = 2|S^{N-2}| \times$

$$\sum_{k \neq 0} g_k \Big( \int_0^1 d\alpha(\rho) \int_0^1 (1-t^2)^{\frac{N-3}{2}} \frac{\sigma(\rho)}{\sigma(\rho)^2 + \rho^2 |k|^2 t^2} dt \Big) e^{i(x.k)} + G_0. \tag{3.8}$$

We will assume that $N > 3$ (the proof for $N = 3$ is easier and is omitted).

An integration by parts yields

$$\varphi_1(x) = 2(N-3)|S^{N-2}| \sum_{k \neq 0} \frac{g_k}{|k|} \Big( \int_0^1 t(1-t^2)^{\frac{N-5}{2}} dt$$
$$\int_0^1 \arctan\big(\frac{\rho|k|t}{\sigma(\rho)}\big) \frac{d\alpha(\rho)}{\rho} \Big) e^{i(x.k)} + G_0.$$

Using the identity $arctan\ s + arctan\ \frac{1}{s} = \frac{\pi}{2}$ $(s > 0)$ we get

$$\varphi_1(x) = -\,2(N-3)|S^{N-2}|\sum_{k\neq 0}\frac{g_k}{|k|}\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt$$

$$\int_0^1 \arctan(\frac{\sigma(\rho)}{\rho|k|t})\frac{d\alpha(\rho)}{\rho}\Big)e^{i(x.k)}$$

$$+\,\pi(N-3)|S^{N-2}|\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt\Big)\Big(\int_0^1\frac{d\alpha(\rho)}{\rho}\Big)$$

$$\sum_{k\neq 0}\frac{g_k}{|k|}e^{i(x.k)} + G_0. \tag{3.9}$$

In view of (H7)

$$\varphi_1(x) = -2(N-3)|S^{N-2}|$$

$$\sum_{k\neq 0}\frac{g_k}{|k|}\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt\int_0^1 \arctan(\frac{\sigma(\rho)}{\rho|k|t})\frac{d\alpha(\rho)}{\rho}\Big)e^{i(x.k)} + G_0.$$

Integrating by parts

$$P(\rho) = \int_0^\rho \frac{d\alpha(s)}{s},$$

in the sense of Stieljes measures gives

$$\int_0^1 \arctan(\frac{\sigma(\rho)}{\rho|k|t})dP(\rho)$$

$$= -\frac{1}{|k|t}\int_0^1 P(\rho)(\frac{\sigma(\rho)}{\rho})'\frac{\rho^2|k|^2t^2}{\sigma(\rho)^2+\rho^2|k|^2t^2}d\rho + \Big[P(\rho)\arctan(\frac{\sigma(\rho)}{\rho|k|t})\Big]_{\rho=0}^{\rho=1}$$

Note that, in view of $(H7)$, the last term vanishes.

Let

$$Z(\rho) = P(\rho)(\frac{\sigma(\rho)}{\rho})', \tag{3.10}$$

then

$$\varphi_1(x) = 2(N-3)|S^{N-2}|$$

$$\sum_{k\neq 0}\frac{g_k}{|k|^2}\Big(\int_0^1 (1-t^2)^{\frac{N-5}{2}}dt\int_0^1 Z(\rho)\frac{\rho^2|k|^2t^2}{\sigma(\rho)^2+\rho^2|k|^2t^2}d\rho\Big)e^{i(x.k)} + G_0.$$

Clearly $\varphi_1(x)$ may be decomposed as follows

$$\varphi_1(x) = 2(N-3)|S^{N-2}|\Big(\int_0^1 Z(\rho)d\rho\Big)\Big(\int_0^1 (1-t^2)^{\frac{N-5}{2}}dt\Big)\sum_{k\neq 0}\frac{g_k}{|k|^2}e^{i(x.k)}$$

$$
\begin{aligned}
&- 2(N-3)|S^{N-2}| \\
&\sum_{k\neq 0} \frac{g_k}{|k|^2}\Big(\int_0^1 (1-t^2)^{\frac{N-5}{2}} dt \int_0^1 \frac{\sigma(\rho)^2 Z(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2} d\rho\Big) e^{i(x.k)} + G_0 \\
&= A_1(x) + A_2(x) + G_0 \ . \qquad (3.11)
\end{aligned}
$$

Let us consider the first term of (3.11). Since

$$
\int_0^1 Z(\rho)d\rho = -\int_0^1 \frac{\sigma(\rho)}{\rho^2} d\alpha(\rho) \qquad (3.12)
$$

and, in view of remark 3.2, then

$$
\begin{aligned}
\Delta A_1(x) = (N-2)|S^{N-1}|\Big(\int_0^1 \frac{\sigma(\rho)}{\rho^2} d\alpha(\rho)\Big) S(x) - (N-2)|S^{N-1}| \\
\Big(\int_0^1 \frac{\sigma(\rho)}{\rho^2} d\alpha(\rho)\Big) g_0 \ . \qquad (3.13)
\end{aligned}
$$

We consider now the second term of (3.11). We note that

$$
\begin{aligned}
A_2(x) = &- 2(N-3)|S^{N-2}| \\
&\sum_{k\neq 0} \frac{g_k}{|k|^2}\Big(\int_0^1 (1-t^2)^{\frac{N-3}{2}} dt \int_0^1 \frac{\sigma(\rho)^2 Z(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2} d\rho\Big) e^{i(x.k)} \\
&- 2(N-3)|S^{N-2}| \\
&\sum_{k\neq 0} \frac{g_k}{|k|^2}\Big(\int_0^1 t^2(1-t^2)^{\frac{N-5}{2}} dt \int_0^1 \frac{\sigma(\rho)^2 Z(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2} d\rho\Big) e^{i(x.k)} \\
= \quad & A_2^1(x) + A_2^2(x) \ .
\end{aligned}
$$

Let

$$
d\gamma = (N-3)\Big[\sigma(\rho)(\rho\sigma'(\rho) - \sigma(\rho))\frac{P(\rho)}{\rho^2}\Big] d\rho \ . \qquad (3.14)
$$

Then

$$
\begin{aligned}
\Delta A_2^1(x) = &\, 2(N-3)|S^{N-2}| \\
&\sum_{k\neq 0} g_k \Big(\int_0^1 (1-t^2)^{\frac{N-3}{2}} dt \int_0^1 \frac{\sigma(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2} d\gamma(\rho)\Big) e^{i(x.k)}.
\end{aligned}
$$

Comparing to (3.8) and using remark 3.2

$$
\Delta A_2^1(x) = -g_0|S^{N-1}| \int_0^1 \frac{d\gamma(\rho)}{\sigma(\rho)} + \int_0^1 d\gamma(\rho) \int_{S^{N-1}} \psi(x,\rho w) dS(w). \quad (3.15)
$$

On the other hand

$$A_2^2(x) = -2(N-3)|S^{N-2}|$$

$$\sum_{k\neq 0} \frac{g_k}{|k|^2}\Big(\int_0^1 t^2(1-t^2)^{\frac{N-5}{2}}dt \int_0^1 P(\rho)\sigma(\rho)^2 \frac{(\frac{\sigma(\rho)}{\rho})'}{\sigma(\rho)^2+\rho^2|k|^2t^2}d\rho\Big)e^{i(x.k)}.$$

An integration by parts yields

$$\begin{aligned}\int_0^1 P(\rho)\sigma(\rho)^2 \frac{(\frac{\sigma(\rho)}{\rho})'}{\sigma(\rho)^2+\rho^2|k|^2t^2}d\rho &= \frac{1}{|k|t}\int_0^1 \arctan(\frac{\rho|k|t}{\sigma(\rho)})d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\\ &\quad -\frac{1}{|k|t}\Big[\frac{P(\rho)}{\rho^2}\sigma(\rho)^2\arctan(\frac{\rho|k|t}{\sigma(\rho)})\Big]_{\rho=0}^{\rho=1}\\ &= \frac{1}{|k|t}\int_0^1 \arctan(\frac{\rho|k|t}{\sigma(\rho)})d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\end{aligned}$$

because $P(1)=0$ and $\lim_{\rho\to 0}\dfrac{P(\rho)}{\rho}=0$ (in view of $(H7)$). Thus

$$A_2^2(x) = -2(N-3)|S^{N-2}|$$

$$\sum_{k\neq 0} \frac{g_k}{|k|^3}\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt \int_0^1 \arctan(\frac{\rho|k|t}{\sigma(\rho)})d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\Big)e^{i(x.k)}$$

and

$$\Delta A_2^2(x) = 2(N-3)|S^{N-2}|$$

$$\sum_{k\neq 0} \frac{g_k}{|k|}\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt \int_0^1 \arctan(\frac{\rho|k|t}{\sigma(\rho)})d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\Big)e^{i(x.k)}.$$

Comparing to the expression of $\varphi_1$ given just before (3.9) shows that

$$-|S^{N-1}|\Big(\int_0^1 \frac{\rho}{\sigma(\rho)}d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\Big)g_0$$

$$+\int_0^1 \rho d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)\int_{S^{N-1}}\psi(x,\rho w)dS(w).$$

Finally, one easily checks that

$$(N-2)\int_0^1 \frac{\sigma(\rho)}{\rho^2}d\alpha(\rho)+\int_0^1\frac{d\gamma(\rho)}{\sigma(\rho)}+\int_0^1\frac{\rho}{\sigma(\rho)}d\Big(\frac{P(\rho)\sigma(\rho)^2}{\rho^2}\Big)=0$$

which finishes the proof for $N\geqslant 3$.

*Proof of Theorem 3.1 for $N = 2$.* — We recall that

$$\varphi_1(x) = 4\sum_{k\neq 0} g_k\Big(\int_0^1 d\alpha(\rho)\int_0^1 \frac{\sigma(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2}\frac{dt}{\sqrt{1-t^2}}\Big)e^{i(x.k)} + G_0 \tag{3.16}$$

which may be decomposed as follows

$$\begin{aligned}\varphi_1(x) = {} & 4\sum_{k\neq 0} g_k\Big(\int_0^1 d\alpha(\rho)\int_0^1 \frac{\sigma(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2}\sqrt{1-t^2}dt\Big)e^{i(x.k)}\\ & + 4\sum_{k\neq 0} g_k\Big(\int_0^1 d\alpha(\rho)\int_0^1 \frac{\sigma(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2}\frac{t^2}{\sqrt{1-t^2}}dt\Big)e^{i(x.k)} + G_0\\ = {} & I_1(x)+I_2(x)+G_0 \ .\end{aligned} \tag{3.17}$$

We have seen, in the proof for $N \geqslant 3$, that

$$\begin{aligned}\Delta I_1(x) = {} & \int_0^1\Big[\sigma(\rho)(\rho\sigma'(\rho)-\sigma(\rho))\frac{P(\rho)}{\rho^2}\Big]d\rho\int_{S^1}\psi(x,\rho w)dS(w) - 2\pi g_0\\ & \int_0^1 P(\rho)(\frac{\sigma(\rho)}{\rho})'d\rho\\ & +2\pi\Big(\int_0^1\frac{\sigma(\rho)}{\rho^2}d\alpha(\rho)\Big)S(x) - 2\pi g_0\int_0^1\frac{\sigma(\rho)}{\rho^2}d\alpha(\rho).\end{aligned} \tag{3.18}$$

Consider now the second term in (3.17)

$$\begin{aligned}I_2(x) &= 4\sum_{k\neq 0}\frac{g_k}{|k|^2}\Big(\int_0^1\frac{dt}{\sqrt{1-t^2}}\int_0^1\frac{\rho^2|k|^2t^2}{\sigma(\rho)^2+\rho^2|k|^2t^2}\sigma(\rho)\frac{d\alpha(\rho)}{\rho^2}\Big)e^{i(x.k)}\\ &= 4\Big(\int_0^1\frac{dt}{\sqrt{1-t^2}}\Big)\Big(\int_0^1\sigma(\rho)\frac{d\alpha(\rho)}{\rho^2}\Big)\sum_{k\neq 0}\frac{g_k}{|k|^2}e^{i(x.k)}\\ &\quad - 4\sum_{k\neq 0}\Big(\frac{g_k}{|k|^2}\int_0^1\frac{dt}{\sqrt{1-t^2}}\int_0^1\frac{\sigma(\rho)}{\sigma(\rho)^2+\rho^2|k|^2t^2}\sigma(\rho)^2\frac{d\alpha(\rho)}{\rho^2}\Big)e^{i(x.k)}.\end{aligned}$$

Consequently, in view of (3.16),

$$\begin{aligned}\Delta I_2(x) = {} & -2\pi\Big(\int_0^1\sigma(\rho)\frac{d\alpha(\rho)}{\rho^2}\Big)S(x) + 2\pi g_0\int_0^1\sigma(\rho)\frac{d\alpha(\rho)}{\rho^2}\\ & + \int_0^1\sigma(\rho)^2\frac{d\alpha(\rho)}{\rho^2}\int_{S^1}\psi(x,\rho w)dS(w) - 2\pi g_0\int_0^1\sigma(\rho)\frac{d\alpha(\rho)}{\rho^2}\end{aligned} \tag{3.19}$$

which finishes the proof of Theorem 3.1 for $N = 2$. □

*Remark* 3.3. — According to (3.9)

$$\begin{aligned}\varphi_1(x) = &-2(N-3)|S^{N-2}| \\ &\sum_{k\neq 0}\frac{g_k}{|k|}\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt\int_0^1 \arctan(\frac{\sigma(\rho)}{\rho|k|t})\frac{d\alpha(\rho)}{\rho}\Big)e^{i(x.k)} \\ &+\pi(N-3)|S^{N-2}|\Big(\int_0^1 t(1-t^2)^{\frac{N-5}{2}}dt\Big) \\ &\Big(\int_0^1\frac{d\alpha(\rho)}{\rho}\Big)\sum_{k\neq 0}\frac{g_k}{|k|}e^{i(x.k)}+G_0.\end{aligned}$$

In the proof of Theorem 3.1, we disregarded the second term thanks to the basic assumption (H7), i.e.

$$\int_0^1\frac{d\alpha(\rho)}{\rho}=0,$$

and we showed that the distributional Laplacian of the first term belongs to $L^2$. If $\int_0^1\frac{d\alpha(\rho)}{\rho}\neq 0$ then, by Parseval identity, the Laplacian of the second term *does not* belong to $L^2$, unless $\{|k|\ g_k\}_k\in\ell^2$, i.e. unless $S\in H^1$. Thus, *both* the $H^2$ regularity of $\varphi_1$ and the possibility to recover $S$ by means of the Laplacian operator are definitely connected to (H7). $\square$

We end this section with the treatment of some velocity dependent sources. More precisely we extend the previous results to a class of degenerate sources. We start with the following (basic) example

$$\begin{cases} v.\dfrac{\partial\psi}{\partial x}+\sigma(|v|)\psi(x,v)=S(x,v) \\ \psi_{|x_i=0}=\psi_{|x_i=2\pi} \qquad\qquad (1\leqslant i\leqslant N)\end{cases} \tag{3.20}$$

where $S(x,v)=\overline{S}(x)T(|v|)$, $\overline{S}\in L^2(D)$, and $T\in L^2\Big([0,1);d|\alpha|\Big)$.

Let $d\alpha$ be a bounded measure such that $T(\rho)d\alpha(\rho)$ satisfies the properties $(H7)$ and $(H8)$. We assume that

$$T \text{ is not identically equal to zero} \tag{$H12$}$$

and define the following function of bounded variation

$$P(\rho)=\int_0^\rho T(s)\frac{d\alpha(s)}{s} \tag{3.21}$$

and the bounded measure

$$d\beta(\rho) = (N-3)\chi_{A(T)}(\rho)\Big[\sigma(\rho)(\rho\sigma'(\rho) - \sigma(\rho))\frac{P(\rho)}{\rho^2}\Big]d\rho \\ +\rho\chi_{A(T)}(\rho)d\Big(\frac{\sigma(\rho)^2P(\rho)}{\rho^2}\Big) \qquad (3.22)$$

where $A(T) = \text{support } (T)$.

As a consequence of Theorem 3.1 we deduce the following

COROLLARY 3.1. — *Let $(H9)$, $(H11)$ and $(H12)$ be satisfied. Let $\psi$ be the solution of* (3.20) *and* $\varphi_1(x) = \int_0^1 d\alpha(\rho)\int_{S^{N-1}}\psi(x,\rho w)dS(w)$.

*Then*

(i) $$\Delta\varphi_1 \in L^2(D;dx) \quad (\textit{consequently } \varphi_1 \in H^2(D))$$

(ii) $$\Delta\varphi_1 = \tilde{C}_N\overline{S}(x) + \varphi_2(x)$$

*where* $\tilde{C}_N = (N-2)|S^{N-1}|\int_0^1\sigma(\rho)T(\rho)\frac{d\alpha(\rho)}{\rho^2}$ *and where*

$$\varphi_2(x) = \int_0^1\frac{d\beta(\rho)}{T(\rho)}\int_{S^{N-1}}\psi(x,\rho w)dS(w). \qquad \square$$

*Proof.* — First, it is easy to see that $\psi(x,v) = 0$ when $T(|v|) = 0$. Let

$$\phi(x,v) = \begin{cases} 0 & \text{if } T(|v|) = 0 \\ \dfrac{\psi(x,v)}{T(|v|)} & \text{otherwise} \end{cases}$$

Then $\phi$ satisfies the following equation on $D \times A(T)$

$$\begin{cases} v.\dfrac{\partial\phi}{\partial x} + \sigma(|v|)\phi(x,v) = \overline{S}(x) & (x,|v|) \in D \times A(T) \\ \phi_{|x_i=0} = \phi_{|x_i=2\pi} & (1 \leqslant i \leqslant N) \end{cases}$$

Thus

$$\begin{aligned}\varphi_1(x) &= \int_0^1 d\alpha(\rho)\int_{S^{N-1}}\psi(x,\rho w)dS(w) \\ &= \int_0^1 T(\rho)d\alpha(\rho)\int_{S^{N-1}}\phi(x,\rho w)dS(w).\end{aligned}$$

Since the measure $T(\rho)d\alpha(\rho)$ satisfies $(H7)$ and $(H8)$, then by Theorem 3.1

$$\Delta\varphi_1(x) = \tilde{C}_N\overline{S}(x) + \int_0^1 \frac{d\beta(\rho)}{T(\rho)} \int_{S^{N-1}} \psi(x,\rho w)dS(w). \qquad \square$$

We discuss now the more general case

$$S(x,v) = \sum_{i=1}^{r} S_i(x)T_i(|v|) \ (r > 1),$$

where $S_i \in L^2(D)$ and $T_i \in L^2\Big([0,1); d|\alpha|\Big)$ $(1 \leqslant i \leqslant r)$. We need the important *technical hypothesis*

$$A(T_i) \cap A(T_j) = \emptyset \quad i \neq j \tag{$H13$}$$

where $A(T_i)$ = support $(T_i)$.

Let $d\alpha$ be a bounded measure such that $T_i(\rho)d\alpha(\rho)$ satisfies the properties $(H7)$ and $(H8)$ for $1 \leqslant i \leqslant r$. We assume that $T_i$ satisfies $(H12)$. We define the following functions of bounded variation

$$P_i(\rho) = \int_0^\rho T_i(s)\frac{d\alpha(s)}{s} \tag{3.23}$$

and the measure $d\beta$

$$\int_0^1 g(\rho)d\beta(\rho) = \sum_{i=1}^{r} \int_{A(T_i)} \frac{g(\rho)}{T_i(\rho)} d\beta_i(\rho) \tag{3.24}$$

where $d\beta_i$ is the measure

$$d\beta_i(\rho) = (N-3)\chi_{A(T_i)}(\rho)\Big[\sigma(\rho)(\rho\sigma'(\rho) - \sigma(\rho))\frac{P_i(\rho)}{\rho^2}\Big]d\rho$$
$$+\rho\chi_{A(T_i)}(\rho)d\Big(\frac{\sigma(\rho)^2 P_i(\rho)}{\rho^2}\Big). \tag{3.25}$$

Then, we have the following

COROLLARY 3.2. — *Let* $(H9)$, $(H11)$ *and* $(H12)$ *be satisfied. Let* $\psi$ *be the solution of* (3.20) *and* $\varphi_1(x) = \int_0^1 d\alpha(\rho) \int_{S^{N-1}} \psi(x,\rho w)dS(w)$. *Then*

$$\Delta\varphi_1 = \sum_{i=1}^{r} C_{N,i}S_i(x) + \int_0^1 d\beta(\rho) \int_{S^{N-1}} \psi(x,\rho w)dS(w)$$

*where* $C_{N,i} = (N-2)|S^{N-1}| \int_0^1 \sigma(\rho) T_i(\rho) \frac{d\alpha(\rho)}{\rho^2}$. $\square$

*Proof.* — Let $\phi_i$ $(1 \leqslant i \leqslant r)$ be the solution of the equation

$$\begin{cases} v.\dfrac{\partial \phi_i}{\partial x} + \sigma(|v|)\phi_i(x,v) = S_i(x)T_i(|v|) \\ \phi_{i|x_j=0} = \phi_{i|x_j=2\pi} \qquad (1 \leqslant j \leqslant N) \end{cases}$$

Then $\psi(x,v) = \sum_{i=1}^{r} \phi_i(x,v)$ (by uniqueness). Thanks to $(H13)$

$$\psi(x,v) = \psi(x,\rho w) = \begin{cases} \phi_i(x,\rho w) & \text{if } \rho \in A(T_i) \\ 0 \quad \text{if } \rho \notin \cup_{i=1}^{r} A(T_i) \end{cases}$$

Thus, by Corollary 3.1,

$$\Delta\varphi_1(x) = \sum_{i=1}^{r} C_{N,i} S_i(x) + \sum_{i=1}^{r} \int_{A(T_i)} \frac{d\beta_i(\rho)}{T_i(\rho)} \int_{S^{N-1}} \phi_i(x,\rho w)\, dS(w)$$

Thanks to $(H13)$, the last term is nothing else but

$$\sum_{i=1}^{r} \int_{A(T_i)} \frac{d\beta_i(\rho)}{T_i(\rho)} \int_{S^{N-1}} \psi(x,\rho w)\, dS(w) = \int_0^1 d\beta(\rho) \int_{S^{N-1}} \psi(x,\rho w)\, dS(w).$$

*Remark* 3.4. — Corollary 3.2 shows that, if $N \neq 2$, then the knowledge of the moments of the solution with respect to the measures $d\alpha(\rho) \otimes dS(w)$ and $d\beta(\rho) \otimes dS(w)$ yields a linear combination of $S_i$. Thus, to recover all $S_i$ $(1 \leqslant i \leqslant r)$, it is necessary to use more measures. More precisely we have the following $\square$

COROLLARY 3.3. — *Let* $(d\alpha_j)_{1 \leqslant j \leqslant r}$ *be a familly of bounded measures such that* $T_i(\rho) d\alpha_j$ *verifies* $(H7)$ *and* $(H8)$. *Suppose that* $(H9)$, $(H11)$ – $(H13)$ *are satisfied and that*

$$\det\Big( \int_0^1 \frac{\sigma(\rho)}{\rho^2} T_i(\rho) d\alpha_j(\rho) \Big)_{1 \leqslant i,j \leqslant r} \neq 0 \,. \qquad (H14)$$

*Then* $\{S_i \,;\ 1 \leqslant i \leqslant r\}$ *are recovered explicitly, if we know the moments of the solution with respect to the measures* $d\alpha_j(\rho) \otimes dS(w)$ *and* $d\beta_j(\rho) \otimes dS(w)$ $(1 \leqslant j \leqslant r)$, *where* $d\beta_j$ *is the measure defined by (3.25).* $\square$

*Proof.* — Let

$$\varphi_1^j(x) = \int_0^1 d\alpha_j(\rho) \int_{S^{N-1}} \psi(x, \rho w) dS(w)$$

and

$$\varphi_2^j(x) = \int_0^1 d\beta_j(\rho) \int_{S^{N-1}} \psi(x, \rho w) dS(w), \ (1 \leqslant j \leqslant r).$$

According to Corollary 3.2

$$\Delta \varphi_1^j(x) - \varphi_2^j(x) = \sum_{i=1}^{r} C_{N,i}^j S_i(x),$$

where $C_{N,i}^j = (N-2)|S^{N-1}| \int_0^1 \sigma(\rho) T_i(\rho) \frac{d\alpha_j(\rho)}{\rho^2}$. Thanks to $(H14)$, the matrix $\left(C_{N,i}^j\right)_{1 \leqslant i,j \leqslant r}$ is invertible and this ends the proof. □

*Remark* 3.5. — As in the previous section, it is easy to extend the results to certain transport equations with collision operators. We do not elaborate on this point. □

*Concluding remark.* — The treatment of time dependent problems is possible by converting them into stationnary ones by means of Laplace transform. Thus, recovering internal sources or even initial datum from suitable time-velocity moments follows the ideas developped here (see [23]). □

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