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On singular perturbations for quasilinear IBV problems (*)

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RÉSUMÉ. — On montre que les perturbations singulières hyperboliques (1.1) des problèmes paraboliques quasilinéaires (1.2) admettent des solutions locales dans des espaces du type Kato-Sobolev, définies sur un intervalle de temps commun, sur lequel on peut étudier leur convergence singulière vers une solution de (1.2). On en déduit aussi que si (1.2) admet des solutions Kato-Sobolev globales, le temps de vie des solutions de (1.1) croît jusqu'à l'infini quand ε tends vers zéro.

ABSTRACT. — We show that the hyperbolic singular perturbations (1.1) of the quasilinear parabolic problems (1.2) admit local Kato-Sobolev solutions defined on a common time interval, on which their singular convergence to a solution of (1.2) can be studied. We also find that if this limit problem admits a global Kato-Sobolev solution, the life span of the solutions of (1.1) grows to $+\infty$ as $\varepsilon \to 0$.

1. Introduction.

1.1. Many physical problems can be modelled by second order quasilinear dissipative equations of the form

$$\begin{cases} \varepsilon u_{tt} + u_t - a_{ij}(\nabla u)\partial_i\partial_j u = f(x,t) & \text{in} \quad \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in} \quad \Omega \times \{t=0\}, \\ u(\cdot,t) = 0 & \text{in} \quad \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

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where $\Omega \subset R^n$ is a bounded domain, $T > 0$, and $\varepsilon > 0$. For example, in Maxwell's equations for the electromagnetic potentials ε represents the displacement currents; for the heat equation with delay, ε is a measure of the thermal relaxation. In applications and numerical simulations, it is usual to neglect the term εu_{tt} in (1.1), and to consider instead the reduced parabolic problem

$$\begin{cases} v_t - a_{ij}(\nabla v)\partial_i\partial_j v = g(x,t) & \text{in} \quad \Omega \times (0,T), \\ v(x,0) = v_0(x) & \text{in} \quad \Omega \times \{t=0\}, \\ v(\cdot,t) = 0 & \text{in} \quad \partial\Omega \times (0,T). \end{cases} \tag{1.2}$$

This simplification is often motivated by the observation that either the value of ε is very small, or that the long time-behavior of solutions to (1.1) and (1.2) (when these exist globally) is similar, or both. This brings forth the natural question of the comparison of solutions to these problems, both on compact time intervals, and on all of $[0,+\infty)$. In the latter case, one typical result is the so-called diffusion phenomenon of hyperbolic waves, whereby solutions to (1.1) converge (for fixed ε) to those of (1.2), as $t \to +\infty$. For example, a result of this type was established in [11] for equations in divergence form in the whole space R^n, with no source terms. Of course, this requires the preliminary knowledge that solutions of both problems exist globally, which is usually obtained by showing that local solutions can be extended to all later times. When comparison of solutions is sought on compact intervals, the main question is instead to determine whether, and in what sense, solutions of (1.1) converge, as $\varepsilon \to 0$, to a solution of (1.2). Because of the loss of the initial condition on u_t, which gives rise to an initial layer at $t = 0$, this convergence is in general singular; therefore, it becomes essential, in applications, to obtain reasonable estimates on the rate of convergence, which are interpreted as an information on the error caused by considering v as an approximation of u^ε. In [7], we have given some results on the singular perturbation problem in the case of the Cauchy problem for (1.1) and (1.2), that is, when $\Omega = R^n$. When boundary conditions are present, the same type of results are much more difficult to obtain because, in addition to the initial layer, we also have to deal with the boundary layer due to the different type of compatibility conditions for the data on $\partial\Omega$ at $t = 0$, which must necessarily be satisfied by local or global solutions of either problem. Thus, one expects convergence on compact intervals non including $t = 0$, i.e. on intervals $[\tau, T]$, $\tau \in (0,T)$. Physically, this means that after a (presumably short) transient, the observable evolution of a system governed by (1.1) can be well approximated by model (1.2).

1.2. In either case, i.e. global or local point of view, the first necessary step is to establish at least a uniformly local existence result for problem

(1.1). This means that if the data $\{f, u_0, u_1\}$ of (1.1) are independent of ε, there should exist an interval $[0, T_0] \subseteq [0, T]$ on which all solutions of (1.1) are defined, independently of ε (or, at least, for all ε sufficiently small). In many situations, local solvability of (1.1) for each fixed ε is known, but one is in general only able to determine existence on intervals $[0, \tau_\varepsilon]$, with $\tau_\varepsilon \to 0$ as $\varepsilon \to 0$ (the situation is similar to the 3-dimensional Navier-Stokes equations, where the size of the interval on which smooth solutions are known to exist vanishes with the viscosity). If this were the case here, the singular perturbation problem would of course no longer make sense; it turns out, however, that uniformly local existence for (1.1) is assured by the presence in the equation of the dissipation term u_t. This result is much easier to establish for the Cauchy problem (see [7]): this is because, since the space variables are not affected by the parameter ε, it is sufficient to establish a priori estimates for the space derivatives of u and u_t and, when $\Omega = R^n$, we can achieve this by simply differentiating the equation with respect to the space variables. Clearly, this procedure does not carry over to initial-boundary value problems, because the space derivatives need not satify the boundary conditions. Thus, we are forced to consider time derivatives only, and the corresponding estimates are not uniform in ε (space regularity is obtained, at least for fixed ε, by ellipticity). For the same reason, we mention that, contrary to the initial value problem, we cannot recover a global existence result for small data, uniform in the sense that the smallness required of the data $\{f, u_0, u_1\}$ is independent of ε. This question remains open; nevertheless, we do obtain a sufficient condition for the almost global existence of solutions of (1.1), independently of the size of the data $\{f, u_0, u_1\}$ (see claim (1.3) below on the behavior of the life-span of solutions of (1.1) as $\varepsilon \to 0$). In contrast, when $\Omega = R^n$ the uniform global existence result for small data can be obtained as in [7], following the method of Matsumura, [6]; here, we emphasize that we consider data $\{f, u_0, u_1\}$ of arbitrary size (but independent of ε).

1.3. Our first goal is thus to establish the mentioned uniformly local existence result for (1.1): in Theorem 2.2 we show that, if ε is sufficiently small, the corresponding problems (1.1) are all solvable in a Kato-Sobolev class $X_m(0, T_0)$, with T_0 independent of ε. These classes are defined in (2.2) below; if m is sufficiently large, these solutions are, by embedding, also Hölder continuous. This allows us to study next the question of the convergence, on the common interval $[0, T_0]$, of the solutions $u = u^\varepsilon$ to a function v which should be recognizable as the solution of (1.2). There are three main questions related to this problem: first, to recognize which equation v solves, i.e. for which data $\{g, v_0\}$ (1.2) holds. This is perhaps the physically more realistic point of view, when the hyperbolic data $\{f, u_0, u_1\}$ are those actually given, and problem (1.2) is taken as a simplification of the "real" one (1.1). Conversely, if the given data are the parabolic ones $\{g, v_0\}$, the question is to construct suitable hyperbolic data $\{f, u_0, u_1\}$, which permit a reasonable control of the convergence of u^ε to v. In either case, the solutions u^ε and v should be comparable, in the sense that they should belong to a common space, in which it makes sense to estimate the difference $u^\varepsilon - v$ in terms of ε. Our second goal is to give some results at least on the second and third of these questions, in preparation for the first: in §4 we recall from [12] how to construct, from compatible parabolic data $\{g, v_0\}$ of (1.2), a set of compatible hyperbolic data $\{f, u_0, u_1\}$, and then show and estimate the singular convergence of the corresponding solutions u^ε to a limit v, which is the solution of (1.2) corresponding to $\{g, v_0\}$. In contrast, the first question is much more difficult; indeed, the same technique produces a limit v which, while formally a solution of (1.2) with data $\{f, u_0\}$, cannot inherit the smoothness of the u^ε at the "corner" $\partial\Omega \times \{t = 0\}$, because the data $\{f, u_0\}$ do not satisfy the necessary parabolic compatibility conditions (this is precisely the initial-boundary layer). However, we recover the same type of smoothness and estimates on compact intervals not containing $t = 0$. Note that the inherent difficulties are not specifically related to the nonlinear structure of the equations; indeed, they already appear in the linear case (e.g., the telegraph vs. the heat equation). We will investigate these questions in a future paper.

1.4. The final motivation for this work, and specifically for the uniformly local existence result, is that in [12] we were also able to treat the "opposite" situation in which, starting from given hyperbolic data $\{f, u_0, u_1\}$, we construct suitable parabolic data $\{g, v_0\}$, so that the difference $u^\varepsilon - v$ can be estimated in the (same) Kato-Sobolev class. This result, however, hinges precisely on the availability of a uniformly local existence result like the one established here. Furthermore, as a consequence of the essential equivalence of the solvability of problems (1.1) and (1.2) in the same Kato-Sobolev class, it follows that if the reduced problem (1.2) can be solved on arbitrary time intervals $[0, T]$ for any choice of compatible data $\{g, v_0\}$, of arbitrary size, then problem (1.1) is also solvable on the same interval $[0, T]$ for any choice of compatible data $\{f, u_0, u_1\}$, again of arbitrary size, at least if ε is sufficiently small (as determined by T and $\{f, u_0, u_1\}$). The emphasis here is on the arbitrarity of the data, because for small data both problems can be globally solvable by the methods of Matsumura, [6] (albeit with the above mentioned caveat for (1.1)). As a consequence, in those cases when (1.2) is globally solvable (see e.g. [9]), we deduce an almost global result for (1.1): that is, if T_ε denotes the life span of solutions of (1.1),

$$\lim_{\varepsilon \to 0} T_\varepsilon = +\infty. \tag{1.3}$$

We refer to [10] for a more detailed discussion of some questions associated to the initial-boundary layers, and to [8] for further illustration of our motivations and applications of this type of results.

2. Notations and Results.

2.1. Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$. For integer $m \geqslant 1$ we denote by $||.||_m$ the norm in the Sobolev space $H^m(\Omega)$, and set $H^m_*(\Omega) \doteq H^m(\Omega) \cap H^1_0(\Omega)$; $\|.\|$ denotes the norm in $H^0(\Omega) = L^2(\Omega)$ and, finally, $H^\infty(\Omega) \doteq \cap_{m \geqslant 0} H^m(\Omega)$. Given $T > 0$, we set $Q_T \doteq \Omega \times (0, T)$; if $u = u(x,t)$ is defined in Q_T, we denote space derivatives by $\partial_i u = \partial u / \partial x_i$, and set $\nabla u = \{\partial_1, \ldots, \partial_n\}$, $D^2 u = \{\partial_i \partial_j u \mid 1 \leqslant i,\, j \leqslant n\}$. Time differentiation is denoted by $\partial_t^k u = \partial^k u / \partial t^k$, and we write u_t and u_{tt} instead of $\partial_t u$ and $\partial_t^2 u$. We assume that the coefficients a_{ij} in (1.1), (1.2), are smooth and symmetric (i.e. $a_{ij} = a_{ji}$), and satisfy the uniformly strong ellipticity condition

$$\exists\, \nu > 0 \,\big|\quad \forall p,\, q \in R^n, \qquad a_{ij}(p) q^i q^j \geqslant \nu |q|^2. \tag{2.1}$$

Following Kato, [2], for integer m we introduce the spaces

$$\begin{aligned} X_m(0,T) &\doteq \cap_{j=0}^m C^j([0,T]; H^{m-j}(\Omega)), & (2.2)\\ X_m^{(1)}(0,T) &\doteq \left\{ u \in X_m(0,T) \mid \partial_t^{m+1} u \in L^2(0,T; L^2(\Omega)), \right\}, & (2.3)\\ X_m^{(2)}(0,T) &\doteq \left\{ u \in X_m^{(1)}(0,T) \mid \nabla u \in X_m(0,T) \right\}, & (2.4) \end{aligned}$$

and consider solutions of (1.1) in these spaces for sufficiently large m; more precisely, we fix integer $s \geqslant \left[\frac{n}{2}\right] + 2$, $[y]$ denoting the integer part of y, and assume that

$$f \in X_{s-1}^{(1)}(0,T), \qquad u_0 \in H_*^{s+1}(\Omega), \quad u_1 \in H_*^s(\Omega), \tag{2.5}$$

and that $\{f, u_0, u_1\}$ satisfy the hyperbolic compatibility conditions of order s at $\partial\Omega$ for $t = 0$, defined as follows: setting

$$u_k(x) \doteq (\partial_t^k u)(x,0), \qquad 0 \leqslant k \leqslant s+1, \tag{2.6}$$

as formally computed from (1.1) by means of an explicit formula of the type

$$\varepsilon u_{k+2} = (\partial_t^k f)(\cdot, 0) - u_{k+1} + A_k[u_0, \ldots, u_k], \qquad 0 \leqslant k \leqslant s-1 \tag{2.7}$$

(for example,

$$\begin{aligned} \varepsilon u_2 &= f(\cdot,0) - u_1 + a_{ij}(\nabla u_0) \partial_i \partial_j u_0,\\ \varepsilon u_3 &= f_t(\cdot,0) - u_2 + a_{ij}(\nabla u_0) \partial_i \partial_j u_1 + (\nabla a_{ij}(\nabla u_0) \cdot \nabla u_1) \partial_i \partial_j u_0, \end{aligned}$$

etc.), we require that the $s+1$ conditions $u_k \in H_*^{s+1-k}(\Omega)$ for $0 \leqslant k \leqslant s$ are satisfied. These conditions make sense, since our assumptions on the data guarantee that $u_k \in H^{s+1-k}(\Omega)$ for $0 \leqslant k \leqslant s+1$, and are necessary for the solvability of (1.1) in $X_{s+1}(0,T)$, for if $u \in X_{s+1}(0,T)$ solves (1.1), then (2.6) does hold in $H^{s+1-k}(\Omega)$, so the traces of u_k on $\partial\Omega$ are defined at least for $0 \leqslant k \leqslant s$, and must therefore vanish.

2.2. Local in time solvability of (1.1) is established e.g. in Kato, [2] (Theorem 14.3):

THEOREM 2.1.— *Assume $\{f, u_0, u_1\}$ satisfy (2.5) and the HCC of order s. There exist then $\tau \in (0,T]$, and a unique $u \in X_{s+1}(0,\tau)$, solution of (1.1).*

We note that u is also a classical solution on Q_τ, by virtue of

PROPOSITION 2.1.— *Assume that $\frac{n}{2} < m < \frac{n}{2}+1$, so that $\alpha \doteq m-\frac{n}{2} \in (0,1)$. Then*

$$\left\{u \in C([0,T]; H^m(\Omega)) \;\middle|\; u_t \in L^2(0,T; L^2(\Omega))\right\} \hookrightarrow C^{\alpha,\alpha/2}(\overline{Q}_T),$$

with continuous injection.

We omit the routine proof of this result, and refer to Friedman, [1], for the definition and the main properties of the Hölder spaces $C^{\alpha,\alpha/2}(\overline{Q}_T)$. In particular, Proposition 2.1 implies that $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\tau)$, with $\alpha \doteq s-1-n/2$.

A direct application of Theorem 2.1 to problem (1.1) would yield that $\tau = \tau(\varepsilon) \to 0$ as $\varepsilon \to 0$, since the functions u_k defined in (2.7) are such that

$$\|u_k\|_{s+1-k} = O(\varepsilon^{1-k}), \qquad 1 \leqslant k \leqslant s+1; \tag{2.8}$$

our first goal is precisely to show that these local solutions of (1.1) are in fact all defined on a common interval independent of ε. To this end, in §3 we prove

THEOREM 2.2.— *Let $\tau = \tau(\varepsilon)$ be the local existence time given by Theorem 2.1: then, $T_0 \doteq \inf_{\varepsilon>0} \tau(\varepsilon) > 0$. Moreover, there are ε_0, $M > 0$ such that, for all $\varepsilon \leqslant \varepsilon_0$ and $t \in [0, T_0]$,*

$$\|u(\cdot,t)\|_{s+1} + \|u_t(\cdot,t)\|_s \leqslant M. \tag{2.9}$$

The same result also holds if f is allowed to depend on ε, provided that, as $\varepsilon \to 0$,

$$\sup_{0\leqslant t\leqslant T} \|f(\cdot,t)\|_{s-1} \;\leqslant\; O(1), \tag{2.10}$$

$$\int_0^T \|\partial_t^s f(\cdot,\theta)\|^2 d\theta \;\leqslant\; O(\varepsilon^{1-2s}). \tag{2.11}$$

2.3. The uniform estimate (2.9) shows that if $\varepsilon \leqslant \varepsilon_0$, the set $\{u^\varepsilon\}$ of the solutions of (1.1) is in a bounded set of the space

$$W(0,T_0) \doteq \{u \in L^2(0,T;H^{s+1}(\Omega)) \mid u_t \in L^2(0,T;H^s(\Omega))\}. \tag{2.12}$$

Consequently, there are a function $v \in W(0,T_0)$ and a subsequence, still noted $\{u^\varepsilon\}$, such that as $\varepsilon \to 0$

$$u^\varepsilon \to v \qquad \text{weakly in} \quad L^2(0,T_0;H^{s+1}(\Omega)), \tag{2.13}$$

$$u_t^\varepsilon \to v_t \qquad \text{weakly in} \quad L^2(0,T_0;H^s(\Omega)). \tag{2.14}$$

The natural question is now whether, and in which sense, v is a solution of the reduced problem (1.2) on $[0,T_0]$, with, presumably, data $\{f,u_0\}$. The difficulty in this resides in the fact that these data in general satisfy only the parabolic compatibility conditions (PCC in short) of order 1. We recall that the PCC of order m (a positive integer) are defined as in the hyperbolic case: namely, we require that the functions $v_k \doteq (\partial_t^k v)(\cdot,0)$, understood in the same sense as in (2.6), be in $H_*^{m+1-2k}(\Omega)$ for $0 \leqslant k \leqslant [m/2]$. We will come back to this problem in §4.5; in preparation for this, we address instead a somewhat more favorable situation: namely, we assume that we are given smooth parabolic data $\{g,v_0\}$ satisfying the PCC of order s, and construct a set of smooth hyperbolic data $\{f,u_0,u_1\}$, such that:

1. $\{f,u_0,u_1\}$ satisfy the HCC of order s;
2. Theorem 2.2 is applicable to the corresponding problem (1.1);
3. The weak limit v defined in (2.13) is a local Sobolev solution of (1.2), and the norm of v in the space $H_{s+2}(0,T_0)$ defined in (2.15) below can be estimated in terms of the norm of g in $X_{s-1}^{(1)}(0,T)$ and of v_0 in $H^{s+1}(\Omega)$ (that is, the spaces where $\{f,u_0\}$ belong);
4. v is more regular for $t>0$, in the sense that for $\rho \in (0,T_0)$, similar estimates hold for the norm of v in $X_s^{(2)}(\rho,T_0)$;
5. $u^\varepsilon \to v$ in $X_s^{(2)}(\rho,T_0)$, and the convergence rate can be explicitly estimated in terms of ε.

These results are described by the Theorems of §4; the Sobolev solutions of (1.2) are in the spaces

$$H_m(0,T) \doteq \left\{u \in H^{m,m/2}(Q_T) \mid \partial_t^{m'} u \in C([0,T];H^{m-2m'-1}(\Omega))\right\}, \tag{2.15}$$

where m is a positive integer, $m' \doteq \left[\frac{m-1}{2}\right]$, and $H^{m,m/2}(Q_T)$ is the space introduced by Lions and Magenes in [5] (Chapter 4).

2.4. Our last result concerns claim (1.3) on the life span of solutions of (1.1). Thus, we assume that the source term f of (1.1) is defined on all of $[0,+\infty)$, and satisfies the corresponding part of (2.5) for all $T>0$. We define the life span T_ε of solutions of (1.1) as the supremum of those $T>0$ such that the local solution of (1.1) can be extended to a solution $u \in X_{s+1}(0,T)$. It may well happen that, for a fixed ε, $T_\varepsilon < +\infty$, i.e., that the corresponding solution blows up in finite time; however, by proving (1.3) we show that the smaller ε is, the closer the long time behavior of the corresponding solution of the hyperbolic problem (1.1) is to that of the parabolic problem (1.2) of which (1.1) is a perturbation. In [12] we have shown that, if the uniformly local existence result of Theorem 2.2 holds, then the hyperbolic problem (1.1) is globally solvable if so is the parabolic problem (1.2) in the Sobolev spaces $H_m(Q_T)$, and ε is sufficiently small; more precisely, setting

$$Y_\alpha^m(Q_T) \doteq H_m(0,T) \cap C^{\alpha,\alpha/2}(\overline{Q}_T), \tag{2.16}$$

we have proved

THEOREM 2.3.— *Let $T>0$ and $\{f,u_0,u_1\}$ be given, satisfying the same assumptions of Theorem 2.1. Assume that problem (1.2) has, for all $m \in N$ and all choices of compatible data $g \in Y_\alpha^m(Q_T)$ and $v_0 \in H^{m+1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$, a solution $v \in Y_{\alpha+2}^{m+2}(Q_T)$. There exists $\varepsilon_0>0$, depending on T and $\{f,u_0,u_1\}$, such that if $\varepsilon \leqslant \varepsilon_0$, the local solution $u \in X_{s+1}(0,\tau)$ of (1.1) given by Theorem 2.1 can be extended to a global solution $u \in X_{s+1}(0,T)$.*

As a consequence, we can deduce an almost global result for solutions of (1.1):

THEOREM 2.4.— *Under the same assumptions of Theorem 2.3, (1.3) holds; that is,*

$$\lim_{\varepsilon \to 0} T_\varepsilon = +\infty. \tag{2.17}$$

Again, this means that solutions of (1.1) can be extended to arbitrary intervals $[0,T]$, for any (compatible) data $\{f,u_0,u_1\}$ independent of ε, provided ε is sufficiently small (depending on T). Theorem 2.4 is immediately proven: Given arbitrary $T>0$, by assumption we can determine $\varepsilon_0 = \varepsilon_0(T,\{f,u_0,u_1\})$ such that we can solve (1.1) on $[0,T]$ for all $\varepsilon \leqslant \varepsilon_0$. Thus, $T_\varepsilon > T$ for all such ε, which is exactly (2.17). □

Of course, Theorem 2.4 is meaningful only when the parabolic problem (1.2) is known to be globally solvable in the spaces $Y_{\alpha+2}^{m+2}(Q_T)$; for a result on this direction, established under additional assumptions of boundedness and decay of the coefficients a_{ij}, we refer to [9].

2.5. We conclude by reporting two technical results that we shall need in the sequel. First, we recall the generalized Gagliardo-Nirenberg inequalities, whose proof can be found e.g. in [9]:

PROPOSITION 2.2.— *Let $I \subset R$ be an interval, $U = \Omega \times I$, and denote by $|\,.\,|_p$ the norm in $L^p(U)$, $1 \leqslant p \leqslant +\infty$. Let u be a smooth function on $\overline{U}$, m a positive integer, and p, $r \in [1, +\infty]$. Given $j = 0, \dots, m$, define $q \in [1, +\infty]$ by $\frac{1}{q} = \frac{j}{pm} + \frac{1}{r}\left(1 - \frac{j}{m}\right)$. Then, if $s = \max\{p, r\}$,*

$$|\partial_t^j u|_q \leqslant C\, |\partial_t^m u|_p^{j/m}\, |u|_r^{1-j/m} + C\, |u|_s, \tag{2.18}$$

with C independent of u. Inequality (2.18) also holds for those functions u, whose corresponding norms are finite, and for all $s \geqslant q$ such that $u \in L^s(U)$; its last term can be omitted if $I = R$.

As a consequence, the following composition estimate can be proven as in Racke, [14] (Lemma 4.7):

PROPOSITION 2.3.— *Let U be as in Proposition 2.2, $m \in N$, $a \in C^m(R^n; R^n)$, and $u \in L^\infty(U)$ be such that $\partial_t^m u \in L^p(U)$. Then $\partial_t^m a(u) \in L^p(U)$, and there is a continuous, positive increasing function h such that*

$$|\partial_t^m a(u)|_p \leqslant h(|u|_\infty)(1 + |\partial_t^m u|_p). \tag{2.19}$$

3. Proof of Theorem 2.2

3.1. We show that if T_ε is the life-span of the solution of (1.1), there is $t_0 > 0$ such that $T_\varepsilon > t_0$ for all $\varepsilon \in (0, 1]$ (which we assume without loss of generality). In the estimates that follow, we denote by C a generic positive constant, possibly different from estimate to estimate, or even within the same estimate; we may allow C to depend on T, $T > 0$ being fixed, sufficiently large and independent of ε. Also, a notation like $M = O(1)$ means that $M > 0$ and, if M depends on ε, there are constants C_1, C_2 such that, as $\varepsilon \to 0$, $0 < C_1 \leqslant M = M(\varepsilon) \leqslant C_2$.

Let $u \in X_{s+1}(0, \tau)$ be a solution of (1.1): by continuity, there is $t_1 \in (0, T_\varepsilon)$ such that

$$\forall t \in [0, t_1], \;\; \|u(\cdot, t)\|_{s+1} + \|u_t(\cdot, t)\|_s \leqslant 2\left(\|u_0\|_{s+1} + \|u_1\|_s\right) \equiv \Delta_0 \tag{3.1}$$

(note that $\Delta_0 = O(1)$, but t_1 could vanish as $\varepsilon \to 0$). In the sequel, we denote by D_i, $i = 1, \dots$, a sequence of constants that depend on Δ_0, but not on ε; thus, $D_i = O(1)$. By ellipticity, we obtain from (1.1) that, for $0 \leqslant t \leqslant t_1$,

$$\|u(\cdot,t)\|_{s+1} \leqslant L\left(\|f(\cdot,t)\|_{s-1} + \varepsilon\|u_{tt}(\cdot,t)\|_{s-1} + \|u_t(\cdot,t)\|_{s-1}\right), \qquad (3.2)$$

where the ellipticity constant L depends only on $\max_{0\leqslant t\leqslant t_1}\|\nabla u(\cdot,t)\|_{s-1}$; thus, since $\nabla u \in C([0,t_1];H^s(\Omega))$, by (3.1) there is D_0 such that $L \leqslant D_0$, and from (3.2) we have

$$\int_0^t \|u\|_{s+1}^2 d\theta \leqslant CD_0 \int_0^t \left(\|f\|_{s-1}^2 + \varepsilon^2\|u_{tt}\|_{s-1}^2 + \|u_t\|_{s-1}^2\right) d\theta. \qquad (3.3)$$

Denote by $|||\cdot|||_m$ the norm in $L^2(0,t;H^m(\Omega))$: by the interpolation inequalities of Lions and Magenes, [4], for $0 \leqslant j \leqslant s+1$,

$$|||\partial_t^j u|||_{s+1-j} \leqslant C\,|||u|||_{s+1}^{1-\theta_j}|||\partial_t^{s+1}u|||_0^{\theta_j} + C|||u|||_1, \qquad (3.4)$$

$\theta_j = j/(s+1)$; using this for $j = 1,\ 2$, so that $0 < \theta_1 < \theta_2 < 1$, we deduce from (3.3) that for $\eta > 0$ and suitable D_1,

$$\int_0^t \|u\|_{s+1}^2 d\theta \leqslant \frac{1}{2}D_1 \int_0^t \left(\|f\|_{s-1}^2 + \|\partial_t^{s+1}u\|^2 + \|u\|_1^2\right) d\theta + \eta\int_0^t \|u\|_{s+1}^2 d\theta,$$

so that, for $\eta \leqslant 1/2$,

$$\int_0^t \|u\|_{s+1}^2 d\theta \leqslant D_1 \int_0^t \left(\|f\|_{s-1}^2 + \|\partial_t^{s+1}u\|^2 + \|u\|_1^2\right) d\theta. \qquad (3.5)$$

3.2. To estimate the term with $\partial_t^{s+1}u$, we differentiate the equation of (1.1) s times with respect to t (this is formal; we should in fact resort to approximations, as in [10]), and multiply in $L^2(\Omega)$ by $2\partial_t^{s+1}u$: setting $A_{ij} \doteq a_{ij}(\nabla u)$, $Q(\nabla z) \doteq (A_{ij}\partial_i z, \partial_j z)$ and

$$C_s \doteq \sum_{l=1}^{s} \partial_t^l A_{ij}\partial_t^{s-l}\partial_i\partial_j u, \qquad (3.6)$$

we obtain

$$\begin{aligned}\frac{d}{dt}\big(\varepsilon\|\partial_t^{s+1}u\|^2 \quad &+ \quad Q(\nabla\partial_t^s u)\big) + 2\|\partial_t^{s+1}u\|^2 = \\ &= \quad 2\big(\partial_t^s f + C_s + (A'_{ij}\cdot\nabla\partial_j u)\,\partial_i\partial_t^s u, \partial_t^{s+1}u\big) + \\ &\quad + \big((A'_{ij}\cdot\nabla u_t)\,\partial_i\partial_t^s u, \partial_j\partial_t^s u\big)\,.\end{aligned}$$

Integrating in $[0,t_1]$ and recalling (2.1), we have

$$\begin{aligned}\varepsilon\|\partial_t^{s+1}u(\cdot,t)\|^2 + \nu\|\nabla\partial_t^s u(\cdot,t)\|^2 + \int_0^t \|\partial_t^{s+1}u\|^2 d\theta \leqslant \\ \leqslant \left\{\varepsilon\|u_{s+1}\|^2 + Q(\nabla u_s) + C\int_0^t \|\partial_t^s f\|^2 d\theta\right\} + \\ +C\int_0^t \|C_s\|^2 d\theta \\ +h(|\nabla u|_\infty)\int_0^t \left(\|\nabla u\|_s^2 + \|u_t\|_s\right)\|\nabla\partial_t^s u\|^2 d\theta,\end{aligned} \tag{3.7}$$

where h is as in (2.19). Thus, denoting by M_ε the quantity inside the brackets in (3.7), recalling (3.1) (with, for simplicity, $\nu = 1$) we deduce that for $0 \leqslant t \leqslant t_1$,

$$\begin{aligned}\varepsilon\|\partial_t^{s+1}u(\cdot,t)\|^2 \quad &+ \quad \|\nabla\partial_t^s u(\cdot,t)\|^2 + \int_0^t \|\partial_t^{s+1}u\|^2 d\theta \leqslant \\ &\leqslant \quad M_\varepsilon + C\int_0^t \|C_s\|^2 d\theta + D_2\int_0^t \|\nabla\partial_t^s u\|^2 d\theta. \quad (3.8)\end{aligned}$$

We estimate the term with C_s as in [9]: if p, $q \in [2,+\infty]$ are such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, from (3.6) we obtain, via Proposition 2.3:

$$\begin{aligned}|C_s|_2 \quad &\leqslant \quad C\sum_{l=1}^{s} |\partial_t^l A_{ij}|_p |\partial_t^{s-l}\partial_i\partial_j u|_q \leqslant \\ &\leqslant \quad h(|\nabla u|_\infty)\sum_{l=1}^{s}\left(1+|\partial_t^l\nabla u|_p\right)|\partial_t^{s-l}\partial_i\partial_j u|_q. \qquad (3.9)\end{aligned}$$

By Proposition 2.2, for $\lambda = \frac{l-1}{s-1} \in [0,1]$ and $\frac{1}{p} = \frac{\lambda}{2}$:

$$|\partial_t^l\nabla u|_p \leqslant C\,|\partial_t^s\nabla u|_2^\lambda |\nabla u_t|_\infty^{1-\lambda} + C\,|\nabla u_t|_\infty \leqslant D_3\left(1+|\partial_t^s\nabla u|_2^\lambda\right), \tag{3.10}$$

via (3.1); similarly, for $\mu = \frac{s-l}{s-1} \in [0,1]$ and $\frac{1}{q} = \frac{\mu}{2}$:

$$\begin{aligned}|\partial_t^{s-l}\partial_i\partial_j u|_q \quad &\leqslant \quad C\,|\partial_t^{s-1}\partial_i\partial_j u|_2^\mu |\partial_i\partial_j u|_\infty^{1-\mu} + C\,|\partial_i\partial_j u|_\infty \leqslant \\ &\leqslant \quad D_4\left(1+|\partial_t^{s-1}\partial_i\partial_j u|_2^\mu\right).\end{aligned}$$

Noting that $\lambda+\mu=1$, and therefore $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, we can proceed from (3.9) with

$$\begin{aligned}|C_s|_2 \quad &\leqslant \quad D_5\sum_{l=1}^{s}\left(1+|\partial_t^s\nabla u|_2^\lambda\right)\left(1+|\partial_t^{s-1}\partial_i\partial_j u|_2^\mu\right) \leqslant \\ &\leqslant \quad D_6\left(1+|\partial_t^s\nabla u|_2 + |\partial_t^{s-1}\partial_i\partial_j u|_2\right); \qquad (3.11)\end{aligned}$$

analogously to (3.4), we also have the interpolation inequality

$$|\partial_t^{s-1}\partial_i\partial_j u|_2 \leqslant C\,|||\partial_t^{s-1}u|||_2 \leqslant C\,|||u|||_{s+1}^{1/s}|||\partial_t^s u|||_1^{1-1/s} + C\,|||u|||_1,$$

so we conclude from (3.11) that, for $\sigma > 0$,

$$C\int_0^t \|C_s\|^2 d\theta \leqslant D_7\left(1+\int_0^t \|\nabla\partial_t^s u\|^2 d\theta\right) + \sigma\int_0^t \|u\|_{s+1}^2 d\theta. \qquad (3.12)$$

Inserting this into (3.8) we obtain

$$\begin{aligned} \varepsilon\|\partial_t^{s+1}u(\cdot,t)\|^2 + \|\nabla\partial_t^s u(\cdot,t)\|^2 + \int_0^t \|\partial_t^{s+1}u\|^2 d\theta \doteq \\ \doteq \Phi(u,t) \ \leqslant (M_\varepsilon + D_7) + \sigma\int_0^t \|u\|_{s+1}^2 d\theta + \qquad (3.13) \\ +(D_2+D_7)\int_0^t \|\nabla\partial_t^s u\|^2 d\theta; \end{aligned}$$

noting that $D_7 = O(1)$ while $M_\varepsilon = O(\varepsilon^{1-2s})$ because of (2.8) (and even if (2.11) holds), by Gronwall's inequality we deduce from (3.13)

$$\Phi(u,t) \leqslant \left(M_1\varepsilon^{1-2s} + \sigma\int_0^t \|u\|_{s+1}^2 d\theta\right)e^{D_8 t}, \qquad (3.14)$$

with $M_1 = O(1)$. In particular, (3.14) yields an estimate of the term with $\partial_t^{s+1}u$ in (3.5), whose last term we estimate as in (3.7): setting

$$M_0(t) \doteq \varepsilon\|u_1\|^2 + Q(\nabla u_0) + C\int_0^t \|f\|^2 d\theta,$$

we have first that

$$\varepsilon\|u_t(\cdot,t)\|^2 + \|\nabla u(\cdot,t)\|^2 + \int_0^t \|u_t\|^2 d\theta \leqslant M_0(t) + D_2\int_0^t \|\nabla u\|^2 d\theta;$$

from this, by Gronwall's inequality and then integrating, we obtain

$$\int_0^t \|u\|_1^2 d\theta \leqslant M_0(t)\,D_2^{-1}\left(e^{D_2 t}-1\right). \qquad (3.15)$$

3.3. We can now go back to (3.5), from which we obtain, by (3.14) and (3.15),

$$\begin{aligned} \int_0^t \|u\|_{s+1}^2 d\theta \ &\leqslant \ D_1\int_0^t \|f\|_{s-1}^2 d\theta + M_0(t)D_2^{-1}\left(e^{D_2 t}-1\right) + \\ &+ \ D_1\left(M_1\varepsilon^{1-2s} + \sigma\int_0^t \|u\|_{s+1}^2 d\theta\right)e^{D_8 t}; \qquad (3.16) \end{aligned}$$

choosing σ so small that $2\sigma D_1 e^{D_8 T} \leqslant 1$, and setting $F(t) \doteq \|f\|^2_{C([0,t];H^{s-1}(\Omega))}$, we obtain from (3.16)

$$\int_0^t \|u\|^2_{s+1} d\theta \leqslant 2D_1 tF(t) + 2M_0(t)D_2^{-1}\left(e^{D_2 t} - 1\right) + + 2D_1 M_1 \varepsilon^{1-2s} e^{D_8 t} \equiv \gamma(t). \tag{3.17}$$

Since $\gamma(0) = 2D_1 M_1 \varepsilon^{1-2s}$, there is $t_2 > 0$ such that $\gamma(t) \leqslant 8D_1 M_1 \varepsilon^{1-2s}$ for $0 \leqslant t \leqslant t_2$; we note that $t_2 = O(1)$ (in particular, if (2.10) holds), so that if $t_1 \geqslant t_2$, our claim is proven. If instead $t_1 < t_2$, we deduce from (3.17) that, for $0 < t \leqslant t_1$,

$$\int_0^t \|u\|^2_{s+1} d\theta \leqslant 8D_1 M_1 \varepsilon^{1-2s}. \tag{3.18}$$

We can therefore conclude: by continuity, there is $t_3 \in (0, t_1]$ such that $\Phi(u,t) \geqslant \frac{1}{2}\Phi(u,0)$ for $0 \leqslant t \leqslant t_3$; now,

$$\Phi(u,0) = \varepsilon\|u_{s+1}\|^2 + \|\nabla u_s\|^2 = O(\varepsilon^{1-2s}),$$

so if $\Phi(u,0) = M_2\varepsilon^{1-2s}$, $M_2 = O(1)$, from (3.14) and (3.18) we have, for $0 \leqslant t \leqslant t_3$,

$$\frac{1}{2}M_2\varepsilon^{1-2s} \leqslant M_1(1 + 8\sigma D_1)\varepsilon^{1-2s}e^{D_8 t_3} \leqslant D_9\,\varepsilon^{1-2s}e^{D_8 t_3}. \tag{3.19}$$

From this, we deduce that

$$t_1 \geqslant t_3 \geqslant \frac{1}{D_8}\ln\frac{M_2}{2D_9} = O(1):$$

this allows us to conclude the proof of Theorem 2.2, with (2.9) following from (3.1). □□

We remark that since $t_1 < T_\varepsilon$, there certainly is a number A_ε, depending on t_1, such that for $0 < t \leqslant t_1$,

$$\int_0^t \|u\|^2_{s+1} d\theta \leqslant A_\varepsilon;$$

the whole point of our argument is to show that $A_\varepsilon = O(\varepsilon^{1-2s})$, as in (3.18). Indeed, if instead $A_\varepsilon = O(\varepsilon^{1-2s-\alpha})$, $\alpha > 0$, then from (3.14) we could only conclude that, instead of (3.19),

$$O(\varepsilon^\alpha) \leqslant (O(\varepsilon^\alpha) + 1)\,e^{D_8 t}:$$

but when ε is small, this inequality is satisfied for all $t \geqslant 0$, so it doesn't allow us to deduce the desired lower bound for t_1.

4. The Singular Perturbation

4.1. In this section we describe and estimate the singular convergence of solutions $u = u^\varepsilon$ of (1.1) corresponding to a special choice of data $\{f, u_0, u_1\}$, constructed from given parabolic data $\{g, v_0\}$ of (1.2). Local solvability of (1.2) in the spaces $H_s(0,T)$ defined in (2.15) can be established as in Kato, [2]; as shown in [9], global solvability in the same classes follows if there is an a priori bound on the norm of the solution in the Hölder spaces $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$. Furthermore, regularity results hold for $t > 0$ (see [13]). We summarize these results in

THEOREM 4.1.— *Let $s \geqslant \left[\frac{n}{2}\right] + 2$ be an integer, and $g \in H_s(0,T)$, $v_0 \in H_*^{s+1}(\Omega)$ satisfy the PCC of order s. There exist then $\tau \in (0,T]$, and a unique $v \in H_{s+2}(0,\tau)$, solution of (1.2). v satisfies the estimate*

$$\|v\|^2_{H_{s+2}(0,\tau)} \leqslant C\left\{\|g\|^2_{H_s(0,T)} + \sum_{2k=0}^{s+1} \|v_k\|^2_{s+1-2k}\right\}, \tag{4.1}$$

where C depends on $\|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_\tau)}$. If in addition $g \in X^{(1)}_{s-1}(0,T)$, then for $\rho \in (0,\tau/2)$, $v \in X^{(2)}_s(2\rho,\tau)$, and satisfies the estimate

$$\|v\|_{X^{(2)}_s(2\rho,\tau)} \leqslant C\,\|g\|_{X^{(1)}_{s-1}(\rho,\tau)}. \tag{4.2}$$

In this result, the additional regularity assumption on g is comparable to that of f in (2.5), and the conclusion essentially states that for $t > 0$, v is comparable to u^ε in $X^{(2)}_s(2\rho,T)$ (the only reason we cannot conclude that $v \in X_{s+1}(2\rho,\tau)$ is that we are not able to show that $\partial_t^{s+1}v \in C([2\rho,\tau];L^2(\Omega))$). Consequently, we will assume that $g \in X^{(1)}_{s-1}(0,T)$. Since this does not imply that the functions v_k (formally defined as $(\partial_t^k v)(\cdot,0)$) are in $H^{s+1-k}(\Omega)$, we regularize the data $\{g, v_0\}$ and consider, by means of the techniques described in [10], a sequence of smooth data $\{g^\delta, v_0^\delta\}$, satisfying the PCC of order $2s+2$. At this point we can construct, again as in [10], a corresponding sequence of hyperbolic data $\{f^\delta, u_0^\delta, u_1^\delta\}$, satisfying the HCC of order $s+1$. To this end, we set $u_0^\delta = v_0^\delta$, $u_1^\delta = v_1^\delta$ and

$$f^\delta(x,t) = g^\delta(x,t) + \varphi^\delta(x,t/\varepsilon) \doteq g^\delta(x,t) + \varphi^{\delta\varepsilon}(x,t), \tag{4.3}$$

where the "corrector" function φ^δ is defined in

LEMMA 4.1.— *There exists a C^∞ function $\varphi^\delta = \varphi^\delta(x,t)$ such that for $k = 0,\dots,s-1$,*

$$(\partial_t^k \varphi^\delta)(\cdot,0) = \varepsilon^{k+1} v^\delta_{k+2}. \tag{4.4}$$

Setting $K_\delta^2 \doteq \sum_{j=0}^{s-1} \|\varepsilon^{j+1} u_{j+2}^\delta\|_{s-j-1}^2$, *the function* $\varphi^{\delta\varepsilon}(\cdot,t) \doteq \varphi^\delta(\cdot,t/\varepsilon)$ *satisfies the following estimates: For all* $m \in N$, *there exists positive* C *such that for all* $\rho \in (0,T)$ *there is* $\rho_m \in (0,\rho]$ *such that for all* $\delta > 0$ *and* $\varepsilon \leqslant \rho_m$,

$$\|\partial_t^k \varphi^{\delta\varepsilon}\|_{C([\rho,T];H^{m-k}(\Omega))} \leqslant \frac{CK_\delta}{\varepsilon^k} e^{-\rho/\varepsilon}, \qquad 0 \leqslant k \leqslant m, \quad (4.5)$$

$$\|\partial_t^m \varphi^{\delta\varepsilon}\|_{L^2(\rho,T;L^2(\Omega))} \leqslant \frac{CK_\delta}{\varepsilon^{m-1/2}} e^{-\rho/\varepsilon}. \quad (4.6)$$

If $m \leqslant s-1$, *these estimates also hold for* $\rho = 0$ *and all* $\varepsilon > 0$.

This Lemma is proven in [12]; note that $K_\delta = O(1)$ as $\varepsilon \to 0$, because each u_k^δ satisfies (2.8), and that for $m = 0$, (4.6) implies that

$$\|\varphi^{\delta\varepsilon}\|_{L^2(0,T;H^m(\Omega))} \leqslant CK_\delta \sqrt{\varepsilon}. \quad (4.7)$$

In the same way, it is also possible to show that if $m \geqslant s$, estimate (4.6) can be pushed down to $\rho=0$ if we replace K_δ by $K_{1,\delta}^2 \doteq \sum_{j=0}^{s-1} \|\varepsilon^{j+1} u_{j+2}^\delta\|_{s-j-1/2}^2$: that is, the estimate

$$\|\partial_t^m \varphi^{\delta\varepsilon}\|_{L^2(\rho,T;L^2(\Omega))} \leqslant \frac{CK_{1,\delta}}{\varepsilon^{m-1/2}} e^{-\rho/\varepsilon} \quad (4.8)$$

holds for all $\rho \in [0,T)$ (and all $\varepsilon > 0$).

4.2. We now solve the hyperbolic problems (1.1) corresponding to the data $\{f^\delta, u_0^\delta, u_1^\delta\}$; because of estimates (4.5) for $m = s-1$, $k = 0$, and (4.8) for $m = s$, we see that conditions (2.10) and (2.11) of Theorem 2.2 are satisfied, so these problems have solutions $u = u^{\varepsilon\delta}$ all defined on a common interval $[0,\tau_\delta]$, with τ_δ independent of ε, and $u^{\varepsilon\delta} \in X_{s+1}(0,\tau_\delta)$. We can then establish the singular convergence of the solutions $u^{\varepsilon\delta}$ on this common interval as $\varepsilon \to 0$:

THEOREM 4.2.— *There exists a function* $v = v^\delta$ *such that as* $\varepsilon \to 0$

$$u^{\varepsilon\delta} \to v^\delta \qquad \text{in } W(0,\tau_\delta) \text{ weakly.} \quad (4.9)$$

v^δ *solves* (1.2) *with data* $\{g, v_0\}$, *and satisfies estimate* (4.1) *on* $[0,\tau_\delta]$.

We prove this Theorem in §4.3. Note that, while (4.9) implies, by compactness, that

$$u^{\varepsilon\delta} \to v^\delta \qquad \text{in } C([0,\tau_\delta];H^{s+\gamma}(\Omega)) \text{ strongly} \quad (4.10)$$

for $0 \leqslant \gamma < 1/2$ (see (4.15) below), we cannot establish analogous results on the uniform convergence of the time derivatives of $u^{\varepsilon\delta}$ in $[0,\tau_\delta]$; this is precisely because of the initial-boundary layer at $\partial\Omega \times \{t = 0\}$ (note that even if we can choose $u_1^\delta = v_1^\delta$, the effect of this layer is still recorded by the different behavior, as $\varepsilon \to 0$ and $t \to 0$, of the estimates on $\varphi^{\delta\varepsilon}$ given in Lemma 4.1). However, as expected, this effect disappears as soon as we keep away from $t = 0$:

THEOREM 4.3.— *Let $\rho \in (0, \tau_\delta/2)$, and $G = G(\rho, \delta) \doteq \|\varphi^{\delta\varepsilon}\|_{X^{(1)}_{s-1}(\rho,T)}$. The following estimates hold:*

$$\|\partial_t^k(u^{\varepsilon\delta} - v^\delta)\|_{C([2\rho,\tau_\delta];H^{s+1-k}(\Omega))} \leqslant C\,(G(\rho,\delta) + \varepsilon), \ 0 \leqslant k \leqslant s, \quad (4.11)$$

$$\|\partial_t^{s+1}(u^{\varepsilon\delta} - v^\delta)\|_{L^2(2\rho,\tau_\delta;L^2(\Omega))} \leqslant C\,(G(\rho,\delta) + \varepsilon), \quad (4.12)$$

with C independent of ρ and ε.

We give a sketch of the proof of this Theorem in §4.4. As a consequence, recalling (4.5) and (4.8), for $\varepsilon \leqslant \varepsilon_s$ we have the initial layer estimates

$$\|u^{\varepsilon\delta} - v^\delta\|^2_{X^{(2)}_s(2\rho,\tau_\delta)} \leqslant C\left\{\frac{K_\delta^2}{\varepsilon^{s-1/2}}e^{-2\rho/\varepsilon} + \varepsilon^2\right\}. \quad (4.13)$$

This estimate still depends "badly" on δ, because of K_δ; still, this bad dependence can be offset by taking small ε. Moreover, we remark that once we have the solution v^δ on $[0, \tau_\delta]$, we can try to extend it beyond τ_δ by means of the a priori estimates (4.1). In relation to this, note that if $g \in X^{(1)}_{s-1}(0,T)$ and $v_0 \in H^{s+1}(\Omega)$, by Proposition 2.1 and standard embedding we have that $g \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ and $v_0 \in C^{2,\alpha}(\overline{\Omega})$; consequently, the classical Hölder estimates can be used to bound the norm of v in $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, and therefore the constant C of (4.1).

4.3. We now prove Theorem 4.2. From now on, we drop for convenience the reference to δ, unless where essential; that is, we write u^ε, v, g, τ, ..., instead of $u^{\varepsilon\delta}$, v^δ, g^δ, τ_δ, etc. From estimate (3.1) (we did mention that Theorem 2.2 applies to (1.1) with data $\{f^\delta, u_0^\delta, u_1^\delta\}$), we deduce that the set $\{u^\varepsilon\}$ is uniformly bounded with respect to ε in $W(0,\tau)$. Consequently, there are a subset, still denoted $\{u^\varepsilon\}$, and a function $v \in W(0,\tau)$, such that (4.9) holds. By compactness (see e.g. Lions, [3], Theorem 5.1 of Chapter 1), also

$$u^\varepsilon \to v \qquad \text{in } L^2(0,\tau;H^{s+1-\eta}(\Omega)) \text{ strongly}, \qquad 0 < \eta < 1. \quad (4.14)$$

By trace theorems (see e.g. Lions-Magenes, [4], Theorem 3.1 of Chapter 1), we have for $\gamma = (1-\eta)/2$ the estimate

$$\|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{s+\gamma} \leqslant C\,|||u^\varepsilon - v|||^{1/2}_{s+1-\eta}\,|||u_t^\varepsilon - v_t|||_s^{1/2}, \quad (4.15)$$

from which (4.10) follows, via (4.14). Recalling that $H^{s-1}(\Omega)$ is an algebra, estimates (4.10) and (4.14) imply that $a_{ij}(\nabla u^\varepsilon)\partial_i\partial_j u^\varepsilon \to a_{ij}(\nabla v)\partial_i\partial_j v$ weakly in $L^2(0,\tau;H^{s-1}(\Omega))$. Thus, if we multiply (1.1) by an arbitrary function $\psi \in C^1([0,T];H^{s-1}(\Omega))$, integrate the term with εu_{tt} by parts and let $\varepsilon \to 0$, recalling estimate (4.7) on φ^ε we deduce that for all such ψ

$$\int_0^\tau (v_t - a_{ij}(\nabla v)\partial_i\partial_j v - g, \psi)\, dt = 0. \tag{4.16}$$

This proves that v solves (1.2) with data $\{g, v_0\}$, where the equation is interpreted in $H^{s-1}(\Omega)$, for almost every $t \in (0,\tau)$; note that (4.10) implies that $v(\cdot,0) = v_0$. Since $v_t \in L^2(0,\tau;H^{s-1}(\Omega))$, $g = g^\delta$ is smooth, and the PCC of order s are satisfied, we can differentiate the equation of (1.2), and find that $v_{tt} \in L^2(0,\tau;H^{s-2}(\Omega))$. Hence, $v_t \in C([0,\tau];H^{s-1}(\Omega))$ and therefore, by ellipticity, $v \in C([0,\tau];H^{s+1}(\Omega))$. By Proposition 2.1, this implies that $v \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_\tau)$: consequently, v satisfies (4.1). This concludes the proof of Theorem 4.2. □

4.4. The proof of Theorem 4.3 is really a consequence of the regularity estimates established in [13]. Although these are carried out for the parabolic equations, we immediately see that exactly the same technique applies to the hyperbolic equation as well; in fact, these estimates are "hyperbolic" in nature, in the sense that they establish regularity in the "hyperbolic" spaces $X_m(0,T)$ of solutions of a parabolic problem. We recall the main steps of the proof, adapted to the present situation. The function $z\,(= z^\delta) \doteq u^\varepsilon - v$ is at least in $H_{s+1}(0,\tau)$, and solves the IBV problem

$$\begin{cases} \varepsilon z_{tt} + z_t - a_{ij}(\nabla u)\partial_i\partial_j z = F \doteq \varphi^\varepsilon - \varepsilon v_{tt} + H & \text{in} \quad \Omega \times (0,T), \\ z(x,0) = 0, \quad z_t(x,0) = 0 & \text{in} \quad \Omega \times \{t=0\}, \\ z(\cdot,t) = 0 & \text{in} \quad \partial\Omega \times (0,T), \end{cases} \tag{4.17}$$

where

$$\begin{aligned} H = H(u,v) \; &\doteq \; [a_{ij}(\nabla u) - a_{ij}(\nabla v)]\partial_i\partial_j v = \\ &= \left(\int_0^1 a'_{ij}(\lambda\nabla u + (1-\lambda)\nabla v)d\lambda\right)\cdot \nabla z\, \partial_i\partial_j v. \end{aligned} \tag{4.18}$$

We fix $\rho \in (0,\tau/2)$ and for $-1 \leqslant k \leqslant s$ we set $\rho_k = \rho\left(1 + \frac{k+1}{s+1}\right)$, so that $\rho_{-1} = \rho < \rho_k < \rho_s = 2\rho$, and claim:

LEMMA 4.2. — *For each $k = 0, \dots, s$ there exists positive C_k, independent of ε, such that for all $t \in [\rho_k, \tau]$,*

$$\int_{\rho_k}^{t} \|\partial_t^{k+1} z(\cdot,\theta)\|^2 d\theta \;+\; \|\nabla \partial_t^k z(\cdot,t)\|^2 \leqslant \tag{4.19}$$
$$\leqslant\; C_k^2 \int_{\rho_k}^{t} \left(\|\partial_t^k \varphi^\varepsilon(\cdot,\theta)\|^2 + \|\partial_t^k(\varepsilon v_{tt})(\cdot,\theta)\|^2\right) d\theta.$$

Proof. — This is proved exactly as Lemma 3.1 of [13], proceeding by induction on k, with the help of cut-off functions ψ_k such that $\psi_k(t) \equiv 0$ for $t \leqslant \rho_{k-1}$ and $\psi_k(t) \equiv 1$ for $t \geqslant \rho_k$. The only differences are the proof of the case $k = 0$, which we consider now, and the extra term H at the right side of (4.18). As (4.18) shows, this term can be estimated in terms of ∇z, as for instance in (4.21) below, so its presence does not introduce any essential difficulty. When $k = 0$, we proceed as in §3.2: multiplying the equation in (4.17) in $L^2(\Omega)$ by $2z_t$, after integration we obtain

$$\varepsilon\|z_t(\cdot,t)\|^2 + \|\nabla z(\cdot,t)\|^2 + \int_{\rho_0}^{t} \|z_t(\cdot,\theta)\|^2 d\theta \leqslant \int_{\rho_0}^{t} \|F(\cdot,\theta)\|^2 d\theta. \tag{4.20}$$

Recalling (4.18), we estimate

$$\|H\| \leqslant C\,\|D^2 v\|_{s-1}\|\nabla z\|, \tag{4.21}$$

with C depending on the norms of ∇u^ε and ∇v in $C([0,\tau]; H^{s-1}(\Omega))$. Inserting (4.21) into (4.20), (4.19) for $k = 0$ follows by Gronwall's inequality. □

The proof of Lemma 4.2 hinges on an auxiliary estimate of the commutators $C_k(u^\varepsilon, v)$, defined by

$$C_k(u,w) \doteq \sum_{l=1}^{k} \binom{k}{l} \partial_t^l a_{ij}(\nabla u)\partial_t^{k-l}\partial_i\partial_j w; \tag{4.22}$$

this estimate is given by

LEMMA 4.3. — *Let m be such that $1 \leqslant m \leqslant s$, and assume (4.19) holds for $0 \leqslant k \leqslant m-1$. Then for all $\eta > 0$ there is $C_k' > 0$ independent of ε, such that if $w = \psi_k z$, with ψ_k the cut-off function mentioned above, then*

$$\int_{\rho_m}^{t} \|C_k(u^\varepsilon, z)\|^2 d\theta \;\leqslant\; \eta \int_{\rho_m}^{t} \|\partial_t^{k+1} z\|^2 d\theta + \tag{4.23}$$
$$+\; C_k' \int_{\rho_k}^{t} \left(\|\partial_t^k \varphi^\varepsilon\|^2 + \|\partial_t^k(\varepsilon v_{tt})\|^2\right) d\theta.$$

We refer to [13] (Lemma 3.2) for the rather lengthy and technical proof of this Lemma. Now, since $w = \psi_s z = z$ for $t \geqslant \rho_s = 2\rho$, we see that (4.12), and (4.11) for $k = s$, follow from (4.19) with $k = s$. Finally, (4.11) for the other values of k is obtained by ellipticity from (4.17), using (4.23) to estimate $C_k(u^\varepsilon, z)$ for $t \geqslant \rho_k$. □□

4.5. We conclude with a remark on the singular convergence in the case of the non modified data $\{f, u_0, u_1\}$. With exactly the same techniques, we see that the solutions $\{u^\varepsilon\}$ converge to a weak limit v, solution of (1.2) with data $\{f, u_0\}$. These data certainly satisfy the first parabolic compatibility condition, since

$$\begin{aligned} v_1 &\doteq f(\cdot, 0) + a_{ij}(\nabla u_0)\partial_i\partial_j u_0 = \\ &= [f(\cdot, 0) + a_{ij}(\nabla u_0)\partial_i\partial_j u_0 - u_1] + u_1 \end{aligned} \tag{4.24}$$

and both u_1 and the term in brackets in (4.24) (which was called εu_2 in (2.7)) have vanishing trace on $\partial\Omega$; however, the data $\{f, u_0\}$ will in general not satisfy the higher order PCC. With some refinements of the arguments of §4.3, we can show that $v \in C^{2+\alpha, 1+\alpha/2}(\overline{Q}_{\tau_\delta})$, so that v is a classical solution to (1.2) on $[0, \tau_\delta]$; we can then bootstrap regularity for $t > 0$ exactly as in Theorem 4.3, to obtain that $v \in X^{(2)}(2\rho, \tau_\delta)$. Further regularity at $t = 0$ is prevented, as we said, by the fact that $\{f, u_0\}$ do not satisfy the higher order PCC. In fact, there appear additional distributional corrections in the time derivatives of v, which account for the initial-boundary layer at $\partial\Omega \times \{t = 0\}$; these corrections are evidently not detected by the weak convergence process described in (2.13) and (2.14). Note that this phenomenon already appears in the simpler case of a linear equation, even with constant coefficients, and is not restricted to the particular degree of regularity (high order Sobolev spaces) that we have considered here. We shall describe the initial-boundary layer, in a general setting, in another paper.

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Bibliography

[1] FRIEDMAN (A.). — *Partial Differential Equations of Parabolic Type.* Krieger, Malabar, FL 1983.

[2] KATO (T.). — *Abstract Differential Equations and Nonlinear Mixed Problems.* Fermian Lectures, Pisa, 1985.

[3] LIONS (J. L.). — *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires.* Dunod, Paris, 1969.

[4] LIONS (J. L.), MAGENES (E.). — *Non-Homogeneous Boundary value Problems, Vol. I.* Springer Verlag, New York, 1972.

[5] LIONS (J. L.), MAGENES (E.). — *Non-Homogeneous Boundary value Problems, Vol. II.* Springer Verlag, New York, 1972.

[6] MATSUMURA (A.). — *Global Existence and Asymptotics of the Solutions of Second Order Quasilinear Hyperbolic Equations with First Order Dissipation term.* , Publ. RIMS Kyoto Univ., **13**, 349-379 (1977).

[7] MILANI (A.). — *Long Time Existence and Singular Perturbation Results for Quasilinear Hyperbolic Equations with Small Parameter and Dissipation Term.* Non Linear An. TMA, 10/11 (1986), 1237-1248.

[8] MILANI (A.). — *Global Existence via Singular Perturbations for Quasilinear Evolution Equations.* Adv. Math. Sci. Appl., 6/2 (1996), 419-444.

[9] MILANI (A.). — *A Remark on the Sobolev Regularity of Classical Solutions to Uniformly Parabolic Equations.* Math. Nachr., 199 (1999), 115-144.

[10] MILANI (A.). — *On the Construction of Compatible Data for Hyperbolic-Parabolic Initial-Boundary Value Problems.* Rend. Sem. Mat. Univ. Trieste, 29 (1997), 167-188.

[11] YANG (H.), MILANI (A.). — *On the Diffusion Phenomenon of Quasilinear Hyperbolic Flows.* To appear on Bull. Sc. Math.

[12] MILANI (A.). — *Global Existence via Singular Perturbations for Quasilinear Evolution Equations: the Initial-Boundary Value Problem.* Preprint, 1999.

[13] MILANI (A.). — *Sobolev Regularity for $t > 0$ in Quasilinear Parabolic Equations.* Preprint, 1999.

[14] RACKE (R.). — *Lectures on Nonlinear Evolution Equations.* Vieweg, Braunschweig, 1992.