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Large Deviation Principles and Generalized Sherrington-Kirkpatrick Models

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RÉSUMÉ. — Nous étudions des versions du modèle de Sherrington-Kirkpatrick où les spins sont à valeurs dans la sphère de $\mathbb{R}^d$ de rayon $\sqrt{d}$. Nous montrons qu'à haute température la solution "réplique-symétrique" est correcte. Nous en déduisons, toujours à haute température, des principes de grandes déviations pour le recouvrement de deux configurations dans la version usuelle du modèle SK. Dans le cas où les spins sont uniformes sur la sphère de rayon $\sqrt{d}$, nous montrons que la solution réplique-symétrique est valable au-dessus d'une température bornée indépendamment de d.

ABSTRACT. — We show how to prove large deviation principles for the overlaps of the usual Sherrington-Kirkpatrick model (at high enough temperature) by proving that some higher dimensional versions of this model are "solved by the replica-symmetric solution". In the version where the spins are uniform over the sphere of radius $\sqrt{d}$ of $\mathbb{R}^d$, we prove that the critical temperature is bounded independently of d.

1. Introduction

The Sherrington Kirkpatrick (SK) model for spin glasses associates to a sequence $\boldsymbol{\sigma} = (\sigma_i)_{i\leqslant N} \in \Sigma_N = \{-1,1\}^N$ the Hamiltonian

$$H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}}\sum_{i,j} g_{ij}\sigma_i\sigma_j - h\sum_{i\leqslant N}\sigma_i \tag{1.1}$$

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where g_{ij} is an i.i.d. standard normal sequence, and h a parameter (that represents an external field). The object of study is, for a typical realization of the sequence (g_{ij}) (that will be called the disorder) to understand the structure of the Gibbs measure at inverse temperature β given by

$$G(\{\boldsymbol{\sigma}\}) = Z^{-1}\exp(-\beta H_N(\boldsymbol{\sigma}))$$

where $Z = \sum_{\boldsymbol{\sigma}} \exp(-\beta H_N(\boldsymbol{\sigma}))$ is the normalization factor. While the physicists believe they understand the structure of the SK model for all values of β, rigorous results are currently known essentially only for "small β". In the case $h = 0$, precise results are obtained for $\beta < 1$ in [C-N] (following [A-L-R]). These results include central limit theorems for the overlaps. The overlap of two configurations $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ is defined as $N^{-1} \sum_{i \leqslant N} \sigma_i \sigma'_i$, and it is best viewed as a function on $\Sigma_N \times \Sigma_N$. Overlaps are of fundamental importance, as discovered in physics. The central limits theorems on the overlaps of [C-N] are extended to the case $h > 0, \beta < \beta_0 (\beta_0 > 0)$ in [T2], a case that is apparently much more difficult. The starting point of the present investigation is the following natural question: what are large deviation principles for the overlaps? In the study of the SK model (as well as in the study of disordered systems) there are two rather distinct questions about large derivations, that one can roughly state as follows.

Question 1. — Understand how rare are the exceptional realizations of the disorder for which Gibbs' measure is rather different from its typical realization.

Question 2. — For the typical realization of the disorder, understand how rare are, for Gibbs' measure, the exceptional configurations for which the overlaps are rather different from their typical (= average) value.

It is question 2 that will be addressed here. (The author is not aware of any result in the direction of question 1.) Let us denote by $\langle \ \cdot \ \rangle$ averages on Σ_N (or its products) with respect to Gibb's measure. Then question 2 essentially amounts, given $t > 0$, to estimate

$$\begin{aligned} &\log\langle \exp t \sum_{i \leqslant N} \sigma_i \sigma'_i \rangle \\ &= \log \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \exp(-\beta H_N(\boldsymbol{\sigma}) - \beta H_N(\boldsymbol{\sigma}') + t \sum_{i \leqslant N} \sigma_i \sigma'_i) - 2 \log Z_N. \end{aligned} \tag{1.2}$$

This is a quantity of order N. It is explained in [T1] that the fluctuations due to the disorder of this quantity are of order $\sqrt{N}$, so that all the information we need about the left-hand side of (1.2) is in fact contained in the number

$$E \log \langle \exp t \sum_{i \leqslant N} \sigma_i \sigma_i' \rangle \tag{1.3}$$

where E denotes expectation in the variables g_{ij}.

How can one compute this quantity? For example, its derivative with respect to t is

$$E \frac{\langle \sum_{i \leqslant N} \sigma_i \sigma_i' \exp t \sum_{i \leqslant N} \sigma_i \sigma_i' \rangle}{\langle \exp t \sum_{i \leqslant N} \sigma_i \sigma_i' \rangle} = E \langle \sum_{i \leqslant N} \sigma_i \sigma_i' \rangle_t \tag{1.4}$$

where $\langle \cdot \rangle_t$ denotes average with respect to Gibbs' measure, at inverse temperature β on $\Sigma_N \times \Sigma_N$, relative to the Hamiltonian

$$\begin{aligned} & -H_N(\boldsymbol{\sigma}) - H_N(\boldsymbol{\sigma}') - \frac{t}{\beta} \sum_{i \leqslant N} \sigma_i \sigma_i', \\ & = -\frac{1}{\sqrt{N}} \sum_{i<j} g_{ij} (\sigma_i \sigma_j + \sigma_i' \sigma_j') - \sum_{i \leqslant N} (\frac{t}{\beta} \sigma_i \sigma_i' + h \sigma_i + h \sigma_i'). \end{aligned} \tag{1.5}$$

There seems to be no other way to compute the quantity (1.4) than to gain understanding of this Gibbs' measure. Consider now the Hamiltonian

$$H(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = -\frac{1}{\sqrt{N}} \sum_{i<j} g_{ij} (\sigma_i \sigma_j + \sigma_i' \sigma_j') \tag{1.6}$$

on $\Sigma_N \times \Sigma_N = (\{-1, 1\}^2)^N$. A simple, but crucial observation is that the study of the Hamiltonian (1.5), when Σ_N is provided with uniform measure, is the same as the study of the Hamiltonian (1.6), when $\Sigma_N \times \Sigma_N$ is provided with the probability ν^N, where ν is the probability on $\{-1, 1\}^2$ given by

$$\nu((\epsilon, \epsilon')) = a \exp(t \epsilon \epsilon' + \beta h (\epsilon + \epsilon')), \tag{1.7}$$

where a is the normalization factor. Indeed, for a function f on $\Sigma_N \times \Sigma_N$,

$$\begin{aligned} & a^N \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} f(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \exp(-\beta H_N(\boldsymbol{\sigma}) - \beta H_N(\boldsymbol{\sigma}') + t \sum_{i \leqslant N} \epsilon_i \epsilon_i') \\ & = 2^{2N} \int f(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \exp -\beta H(\boldsymbol{\sigma}, \boldsymbol{\sigma}') d\nu^{\otimes N}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \end{aligned}$$

where $(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$ is identified with an element of $(\{-1, 1\}^2)^N$.

Besides the overlaps, there are other quantities for which one might want to establish large deviation principles, such as the "symmetrized overlaps" considered in [T1]. For these, the analysis performed above carries out, but one has to study a certain Gibbs measure on Σ_N^4. To treat this different cases in one stroke, we will introduce a general setting, the *generalized SK model.* In this model, the individual spins take values in the ball B of $\mathbb{R}^d$ (where d is an integer)

$$B = \{x \in \mathbb{R}^d; \quad \|x\| \leqslant \sqrt{d}\}.$$

The choice of the normalization is to ensure that $\{-1,1\}^d \subset B$, as seems natural from the previous motivating examples. A configuration $\boldsymbol{\sigma}$ is then a point of $\Sigma_N = B^N$. Denoting by $(\cdot,\cdot)$ the dot product in $\mathbb{R}^N$, we consider the Hamiltonian

$$H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j). \tag{1.8}$$

Given a probability μ on B, we define Gibbs' measure on B^N as the probability that has density proportional to $\exp -\beta H_N(\boldsymbol{\sigma})$ with respect to μ^N. The parameters of the system are then β and the probability μ. A particularly natural example is when μ is uniform on the boundary of B, an example physically interesting if $d = 3$.

THEOREM 1.1 (Informal version). — *There is a number L such that if $L\beta d \leqslant 1$, the replica symmetric (RS) solution holds for the generalized SK model.*

What is meant by the RS solution will be explained in detail in Section 2; but this means in particular that we can compute the limit as $N \to \infty$ of the quantity (1.4) (and of many others). Thus, as a consequence of Theorem 1.1, there is a number β_0 such that if $\beta < \beta_0$, we understand (at least in principle, since the solutions are given in terms of implicit functions) the large deviations of the overlaps for the usual SK model.

If no hypothesis is made upon μ, the requirement $L\beta d \leqslant 1$ is reasonable. For example, if for a certain x in $\mathbb{R}^d$, with $\|x\| = \sqrt{d}$, we have $\mu(\{x\}) = \mu(\{-x\}) = 1/2$, then the corresponding generalized SK model is isomorphic to the SK model for $h = 0$, at inverse temperature βd, so that the RS solution will not hold unless $\beta d \leqslant 1$. Thus, if we define the critical value of β as the supremum of the values for which the RS solution holds, we can reformulate Theorem 1.1 by saying that the critical temperature is of order at most d. In this example that showed that this order is optimal, the measure μ was actually "one dimensional". We feel that if μ is really d-dimensional (in a sense yet to be discovered) then the critical β should be of order 1 independently of the value of d. Here is a result in this direction, concerning the natural example.

THEOREM 1.2.— *There exists a number $\beta_0 > 0$ such that if $\beta \leqslant \beta_0$, and μ is uniform over the boundary of B, the RS solution holds for the corresponding generalized SK model, whatever the value of d.*

In contrast with the case $d = 1$, it should be pointed out that for large d the free energy density is much smaller that the anealed free energy density (by a factor d), so that Theorem 1.2 is not as easy as one might have hoped.

We will generalize Theorem 1.2 as follows.

THEOREM 1.3.— *There exists numbers $\beta_0 > 0, L > 0$ with the following property. If we assume that μ has a density $1+m$ with respect to the uniform measure on S, where $\|m\|_\infty \leqslant 1/L$, then the RS solution holds for the corresponding generalized SK model, whenever $\beta \leqslant \beta_0$, whatever the value of d.*

We would like now to explain why the interest of the generalized SK model possibly goes well beyond the application we gave to large deviation principles. We were brought to that model by our attempts to prove the validity of the RS solution for the usual SK model in the entire "high temperature region" predicted by the physicists. This problem appears very much harder than expected. At present it appears very difficult to get even close to the critical value of β, unless h is small, where one can take advantage of special features. The generalized SK model makes our shortcomings more obvious. When d is large we are currently very far from being able to get the proper *order* of the critical β. We can prove only that β is at least of order $1/d$, in cases where it is likely to be of order 1, unless we can take advantage of special features, as in the case of Theorem 1.2. Of course the reader might think that it is weird to attempt to solve a hard problem (proving the validity of the RS solution in the entire high temperature region) by working on a much harder one (studying the generalized SK model). This is not necessarily the case. The attack of [T2], that seems to follow the most natural approach, requires to estimate $E\langle f^n\rangle$ for each n, where f is a certain function on Σ_N^4. It does not appear possible, when n is of order N, to make these estimates by understanding Gibbs' measure only. Rather, it seems necessary to understand the measure of density $\exp tf$ with respect to Gibbs' measure. The form of f (that resembles an overlap) is such that this amounts to understanding a generalized SK model for $d = 4$. It is most likely that in the previous sentence the work "understand" must mean "prove that the RS solution holds". So, to prove that the RS solution holds for the ordinary SK model, at a given value of the parameters, one is naturally lead to study a generalized SK model for $d = 4$. The hope is that for this new model, one is a little bit further from the critical temperature so that the problem is a bit easier. One could then iterate the procedure until one reaches a problem easy enough to solve it directly. But the main obstacle in this program is that the dimension d doubles at each iteration, and the project has a chance to succeed only if one can develop estimates to study the generalized SK models that are independent of the value of d (at least for a sufficiently rich class of measures μ). This, by itself, appears to be a very difficult program, of which Theorems 1.2 and 1.3 are small steps.

We now describe the organization of the paper. In Section 2, we set our notation, we explain the cavity method, and describe the RS solution. In Section 3 we prove the key step toward Theorem 1.1, that is, we show that "the system is in a pure state". This follows the basic ideas of [T1], but, since the situation is more complicated, some explicit computations are no longer possible, and have to be replaced by general principles (which of course results in great simplification). We have tried to give the simplest proof we could, even though this means that some arguments will have to be repeated later in a more elaborate form to prove Theorem 1.3. Proceeding otherwise could have shortened the paper by a few pages; it would also have guaranteed that the proofs would forever be impenetrable to others. The proof of Theorem 1.1 is then completed in Section 4. In Section 5, we prove Theorem 1.2. In Section 6, using the fact that μ is close to uniform, we prove a priori estimates on Gibbs' measure. Using these, we then revisit the methods of Section 3 and 4 to prove Theorem 1.3.

2. Description of the RS solution and preliminaries

The fundamental property of the RS solution will be that "the system is in a pure state"; we refer the reader to [T4] for a detailed discussion of this idea. In the present case, the way we will define this notion is by the fact, that, given any x, y in $\mathbb{R}^d$, the function

$$\frac{1}{N}\sum_{i\leqslant N}(x,\sigma_i)(y,\sigma_i') \tag{2.1}$$

of the two configurations $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ is essentially a constant function on (Σ_N^2, G^2). It is convenient to symmetrize, and to say instead that the function

$$\frac{1}{N}\sum_{i\leqslant N}(x,\sigma_i^1-\sigma_i^2)(y,\sigma_i^3) \tag{2.2}$$

of the three configurations $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3$ is essentially zero.

To simplify notation, we will write $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2$; we will denote $x(\tilde{\boldsymbol{\sigma}})$ the sequence $(x, \tilde{\sigma}_i)_{i \leqslant N}$; and we will denote by $\cdot$ the dot product in $\mathbb{R}^N$, so that (2.2) will be written as

$$\frac{1}{N} x(\tilde{\boldsymbol{\sigma}}) \cdot y(\boldsymbol{\sigma}^3). \tag{2.3}$$

To quantify the fact that is function is nearly zero, we will consider the number

$$C_N = \sup_{\|x\|, \|y\| \leqslant 1} \langle (\frac{1}{N} x(\tilde{\boldsymbol{\sigma}}) \cdot y(\boldsymbol{\sigma}^3))^2 \rangle. \tag{2.4}$$

In this notation, $\langle \ \cdot \ \rangle$ means that $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3$ are integrated for Gibbs' measure; E denotes expectation in the r.v. $(g_{ij})_{i<j}$ (the disorder); and $\|x\|$ is the norm of x in $\mathbb{R}^d$.

In Section 3 we will prove that $C_N \to 0$ under the conditions of Theorem 1.1.

The structure of the RS solution of the standard SK model is determined by a number q. This number q has the property that for essentially all realizations of the randomness, and essentially all choices of $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ according to Gibbs' measure, we have

$$q \simeq \frac{1}{N} \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' = \frac{1}{N} \sum_{i \leqslant N} \sigma_i \sigma_i' \tag{2.5}$$

The value of q is given by

$$q = E \operatorname{th}^2(\beta g \sqrt{q} + \beta h) \tag{2.6}$$

where g is standard normal.

The situation is more complicated for the generalized SK model. We have to consider two quadratic forms R, Q on $\mathbb{R}^d$. These will have the property that for x, y in $\mathbb{R}^d$, essentially all realizations of randomness, and essentially all choices of $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ in Σ_N (according to Gibbs' measure) we have

$$Q(x, y) \simeq \frac{1}{N} x(\boldsymbol{\sigma}) \cdot y(\boldsymbol{\sigma}') (= \frac{1}{N} \sum_{i \leqslant N} (x, \sigma_i)(y, \sigma_i')) \tag{2.7}$$

$$R(x, y) \simeq \frac{1}{N} x(\boldsymbol{\sigma}) \cdot y(\boldsymbol{\sigma}). \tag{2.8}$$

Given a positive semi definite form Q on $\mathbb{R}^d$, there exists a jointly Gaussian family $(g_Q(x))_{x \in \mathbb{R}^d}$ such that $E g_Q(x) g_Q(y) = Q(x, y)$ for all x, y in

$\mathbb{R}^d$. Then in the limit the quadratic forms Q, R will by determined by the relations

$$Q(x,y) = E\frac{Av(x,\theta)(y,\theta')\mathcal{E}(\theta)\mathcal{E}(\theta')}{(Av\mathcal{E}(\theta))^2} \tag{2.9}$$

$$R(x,y) = E\frac{Av(x,\theta)(y,\theta)\mathcal{E}(\theta)}{Av\mathcal{E}(\theta)} \tag{2.10}$$

where

$$\mathcal{E}(\theta) = \exp\beta g_Q(\theta)\exp\frac{\beta^2}{2}(R(\theta,\theta) - Q(\theta,\theta)). \tag{2.11}$$

In line with our previous work, we denote above by Av integration of θ, θ' with respect to μ. The reader is reminded that while certain authors use the notation Av to mean average over the disorder, we denote average over the disorder by E. Of course in (2.9), (2.10) E stands for integration in the Gaussian variables $g_Q(\theta)$.

In Section 4 we will prove these statements under the conditions of Theorem 1.1. These already contain the information we need to evaluate quantities such as (1.4) (that is asymptotically $ER(e_1, e_2)$ where e_1, e_2 are the unit vectors of $\mathbb{R}^2$). In fact it will be clear at that point that we can compute many other quantities. It seems almost certain that in the range of values of β considered in Theorem 1.1, the generalized SK model can be understood with the accuracy that is achieved in [T2] for the standard SK model; but we did not see motivations to undertake such a large scale project.

The fundamental tool for the present paper is the cavity method, and we explain it now. Consider an independent sequence $(g_i)_{i\leqslant N}$ of i.i.d. $N(0,1)$ variables, that is independent of the variables $(g_{ij})_{i<j\leqslant N}$. If we write $g_{i,N+1} = g_i$, the collection $(g_{i,j})_{i<j\leqslant N+1}$ is independent i.i.d. We will denote by $\langle\ \cdot\ \rangle'$ Gibbs' measure for the $(N+1)$ spins system, where β is replaced by $\beta' = \beta\sqrt{1+1/N}$. Consider a function f on Σ_{N+1}. We make the convention to write a configuration in Σ_{N+1} as $(\boldsymbol{\sigma}, \theta)$, where $\boldsymbol{\sigma} \in \Sigma_N, \theta \in B$. We then have the algebraic identity

$$\langle f\rangle' = \frac{Av\langle f(\boldsymbol{\sigma},\theta)\mathcal{E}(\theta,\boldsymbol{\sigma})\rangle}{Z} \tag{2.12}$$

where $Z = Av\langle\mathcal{E}(\theta,\boldsymbol{\sigma})\rangle$ and

$$\mathcal{E}(\theta,\boldsymbol{\sigma}) = \exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\theta(\boldsymbol{\sigma}) =: \exp\frac{\beta}{\sqrt{N}}\sum_{i\leqslant N} g_i(\theta,\sigma_i) \tag{2.13}$$

In (2.12), Av means that θ is integrated with respect to μ (and of course $\boldsymbol{\sigma}$ is integrated for Gibbs measure). Once one understands the notation, (2.12) is obvious. Its purpose is to relate $\langle \cdot \rangle$ and $\langle \cdot \rangle'$ so that the information on $\langle \cdot \rangle$ can be transferred to $\langle \cdot \rangle'$, allowing induction upon N. In the previous papers where we have used formulas such as (2.12), after stating the formula, we say "the reader will check extensions of this formulas to the case where one replaces Σ_N by a power and G_N by its power (replicas)". In the present case, there is no need to appeal to the reader's good will, because replicas of a generalized SK model are themselves generalized SK models with a larger d, replacing the space and μ by a power of itself.

3. System in a pure state

The aim of this section is to prove that if $L\beta d \leqslant 1$, then $\lim_{N\to\infty} C_N = 0$. It is possible that this could be proved by the method af [F-Z], Theorem A (2), but since not all the details are provided there, it is not immediately clear whether these authors obtain the correct dependence $\beta d \leqslant 1$. In the different normalization they use this is equvalent to the fact that in their Theorem A (2) "β small enough" means small enough independently of d. In any case, it should be useful to the reader to see an argument in the spirit of the rest of our approach, argument that we present now.

What makes the problem challenging is that before we start we do not know anything about Gibbs' measure. Yet we will be able to make (with foresight) a estimate of the left hand side of (2.12). We set

$$\boldsymbol{b} = (\langle\sigma_i\rangle)_{i\leqslant N}; \dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{b} \tag{3.1}$$

$$\mathcal{E}_0(\theta,\boldsymbol{\sigma}) = \exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot\theta(\boldsymbol{b})\exp\frac{\beta^2}{2N}(\|\theta(\boldsymbol{\sigma})\|^2 - \|\theta(\boldsymbol{b})\|^2) \tag{3.2}$$

There of course $\|\theta(\boldsymbol{\sigma})\|^2 = \theta(\boldsymbol{\sigma})\cdot\theta(\boldsymbol{\sigma}) = \sum_{i\leqslant N}(\theta,\sigma_i)^2$. Even though, as mentioned at the end of the previous section, replicas of a generalized SK model can themselves be viewed as generalized SK models, it will be convenient to consider them directly. We will consider a function $f = f(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2,\boldsymbol{\sigma}^3,\theta^1,\theta^2,\theta^3)$, where $\boldsymbol{\sigma}^\ell \in \Sigma_N, \theta^\ell \in B$, and of course a quantity such as

$$Av\langle f\rangle$$

means that we integrate $\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2,\boldsymbol{\sigma}^3$ for Gibbs' measure and $\theta^1,\theta^2,\theta^3$ for μ.

PROPOSITION 3.1.— *If $\beta d \leqslant 1$ we have*

$$\begin{aligned}\Big|E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\boldsymbol{\sigma}^\ell)\rangle}{Z^3} &- E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}_0(\theta^\ell,\boldsymbol{\sigma}^\ell)\rangle}{Z^3}\Big| \\ &\leqslant Ld\beta^2(EAv\langle f^2\rangle)^{1/2}C_N^{1/2}.\end{aligned} \tag{3.3}$$

There, as well as in the rest of the paper, L denotes a number, not necessarily the same at each occurrence. The meaning of this result is that modulo a reasonable error term, we can replace in the numerator $\mathcal{E}(\theta, \boldsymbol{\sigma})$ by $\mathcal{E}_0(\theta, \boldsymbol{\sigma})$ that has a much simpler dependence upon $\boldsymbol{g}$.

Proof. — For $0 \leqslant u \leqslant 1$, we consider

$$\begin{aligned}\mathcal{E}(u) &= \mathcal{E}(u, \theta^1, \theta^2, \theta^3, \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}^3) \\ &= \exp \sum_{\ell \leqslant 3} \Big[\frac{\beta}{\sqrt{N}}(\boldsymbol{g} \cdot \theta^\ell(\boldsymbol{b}) + u\boldsymbol{g} \cdot \theta^\ell(\dot{\boldsymbol{\sigma}}^\ell)) \\ &\quad + \frac{\beta^2}{2N}(\|\theta^\ell(\boldsymbol{\sigma}^\ell)\|^2 - \|\theta^\ell(\boldsymbol{b}) + u\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell)\|^2)\Big]\end{aligned} \tag{3.4}$$

and

$$\varphi(u) = E\frac{Av\langle f\mathcal{E}(u)\rangle}{Z^3}.$$

It is good to observe that $E_g\mathcal{E}(u)$ does not depend upon u, where E_g denotes integration in $\boldsymbol{g}$ only. Also, the left-hand side of (3.3) is $|\varphi(1) - \varphi(0)|$ so that to prove (3.3) it suffices to prove that for each $0 \leqslant u < 1$ we have

$$|\varphi'(u)| \leqslant Ld\beta^2(EAv\langle f^2\rangle)^{1/2}C_N^{1/2}. \tag{3.5}$$

We write

$$\varphi'(u) = E\frac{Av\langle(W_1 - W_2)f\mathcal{E}(u)\rangle}{Z^3}$$

where

$$W_1 = \frac{\beta}{\sqrt{N}}\sum_{\ell \leqslant 3} \boldsymbol{g} \cdot \theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \tag{3.6}$$

$$W_2 = \frac{\beta^2}{N}\sum_{\ell \leqslant 3}(\theta^\ell(\boldsymbol{b}) \cdot \theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) + u\|\theta^\ell(\dot{\boldsymbol{\sigma}})\|^2). \tag{3.7}$$

To study

$$E\frac{Av\langle W_1 f\mathcal{E}(u)\rangle}{Z^3} \tag{3.8}$$

we will integrate by parts. If g is $N(0,1)$ and h is a smooth function, then

$$E(gh(g)) = Eh'(g). \tag{3.9}$$

Thus, if $(g_i)_{i \leqslant N}$ are i.i.d. $N(0,1)$ and F is a smooth function on $\mathbb{R}^N$, then

$$E(g_i F(\boldsymbol{g})) = E\frac{\partial F}{\partial g_i}(\boldsymbol{g}) \tag{3.10}$$

where $\partial F/\partial g_i$ denotes the partial derivative of F with respect to the i^{th} variable and where $\boldsymbol{g} = (g_i)_{i \leqslant N}$. We will use the functions

$$F_i^\ell(\boldsymbol{g}) = \frac{Av\langle(\theta^\ell, \dot{\sigma}_i^\ell) f \mathcal{E}(u)\rangle}{Z^3}$$

so that

$$\frac{\partial}{\partial g_i} F_i^\ell(\boldsymbol{g}) = \mathrm{I} + \mathrm{II} \tag{3.11}$$

where

$$\mathrm{I} = \frac{\beta}{\sqrt{N}} Av \frac{\langle((\theta^\ell, b_i) + u(\theta^\ell, \dot{\sigma}_i^\ell))(\theta^\ell, \dot{\sigma}_i^\ell) f \mathcal{E}(u)\rangle}{Z^3} \tag{3.12}$$

$$\mathrm{II} = -\frac{3\beta}{\sqrt{N}} \frac{Av\langle(\theta^\ell, \dot{\sigma}_i^\ell) f \mathcal{E}(u)\rangle Av\langle(\theta, \sigma_i)\mathcal{E}(\theta, \boldsymbol{\sigma})\rangle}{Z^4}. \tag{3.13}$$

When computing

$$\begin{aligned} E\frac{Av\langle W_1 f \mathcal{E}(u)\rangle}{Z^3} &= \frac{\beta}{\sqrt{N}} \sum_{\ell \leqslant 3} \sum_{i \leqslant N} E g_i F_i^\ell(\boldsymbol{g}) \\ &= \frac{\beta}{\sqrt{N}} \sum_{\ell \leqslant 3} \sum_{i \leqslant N} E \frac{\partial}{\partial g_i} F_i^\ell(\boldsymbol{g}) \end{aligned}$$

one sees that the contributions of the terms I cancel out with the contributions of W_2. This is because $E_g Av\langle f\mathcal{E}(u)\rangle$ is independent of u. Using replicas to write the contributions of the terms II, we then get that

$$\varphi'(u) = -\frac{3\beta^2}{N} E Av \frac{\langle \sum_{\ell \leqslant 3} \theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \cdot \theta^4(\boldsymbol{\sigma}^4) f \mathcal{E}(u) \mathcal{E}(\theta^4, \boldsymbol{\sigma}^4)\rangle}{Z^4}. \tag{3.14}$$

We recall that, by Jensen's inequality, we have $Z \geqslant \exp \frac{\beta}{\sqrt{N}} \boldsymbol{g} \cdot a(\boldsymbol{b})$ where a is the barycenter of μ. Thus

$$\begin{aligned} |\varphi'(u)| \leqslant \frac{3\beta^2}{N} E Av \exp \frac{-4\beta}{\sqrt{N}} \boldsymbol{g} \cdot a(\boldsymbol{b}) \\ \langle \sum_{\ell \leqslant 3} |\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \cdot \theta^4(\boldsymbol{\sigma}^4)| |f| \mathcal{E}(u) \mathcal{E}(\theta^4, \boldsymbol{\sigma}^4)\rangle. \end{aligned} \tag{3.15}$$

We first take expectation in g, using that $\|\theta(\boldsymbol{\sigma})\|^2 \leqslant dN$ for $\theta \in S, \boldsymbol{\sigma} \in S^N$, to get

$$|\varphi'(u)| \leqslant \frac{3\beta^2}{N} \exp L\beta^2 d^2 E Av \langle \sum_{\ell \leqslant 3} |\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \cdot \theta^4(\boldsymbol{\sigma}^4)| |f| \rangle \tag{3.16}$$

We now use Cauchy-Schwarz,

$$|\varphi'(u)| \leqslant \frac{L\beta^2}{N} \exp L\beta^2 d^2 (EAv\langle f^2\rangle)^{1/2} \qquad (3.17)$$
$$(EAv\langle(\theta^1(\dot{\sigma}^1)\cdot\theta^4(\sigma^4))^2\rangle)^{1/2}$$

and we observe that

$$\begin{aligned} &EAv\langle(\theta^1(\dot{\sigma}^1)\cdot\theta^4(\sigma^4))^2\rangle \\ &\leqslant d \sup_{\|x\|\leqslant 1, \|y\|\leqslant 1} E\langle(x(\dot{\sigma}^1)\cdot y(\sigma^4))^2\rangle \leqslant N^2 C_N \end{aligned}$$

where the factor d arises from the fact that θ^1, θ^4 are of norm $\sqrt{d}$. □

To prove that $\lim_{N\to\infty} C_N = 0$, it seems necessary to consider another quantity

$$D_N = \sup_{\|x\|=1} \langle(\frac{1}{N}\|x(\sigma^1)\|^2 - \frac{1}{N}\|x(\sigma^2)\|^2)^2\rangle. \qquad (3.18)$$

This quantity is of a different nature than C_N. Saying that D_N is small implies that $\|x(\sigma)\|^2$ is essentially independent of σ, a very precious information in itself, since it allows to go one step beyond Proposition 3.1, as the following shows.

PROPOSITION 3.2.— *If $\beta d \leqslant 1$, we have*

$$\begin{aligned} &|E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\sigma^\ell)\rangle}{Z^3} - E\frac{Av(\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_1(\theta^\ell))}{Z^3}| \\ &= L\beta^2 d(EAv\langle f^2\rangle)^{1/2}(C_N^{1/2} + D_N^{1/2}) \end{aligned} \qquad (3.19)$$

where $\mathcal{E}_1(\theta^\ell) = \exp(\frac{\beta}{\sqrt{N}}g\cdot\theta^\ell(b) + \frac{\beta^2}{2N}(\langle\|\theta^\ell(\sigma)\|^2\rangle - \|\theta^\ell(b)\|^2))$.

Proof. — Using Proposition 3.1, it suffices to bound

$$|E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}_0(\theta^\ell,\sigma^\ell)\rangle}{Z^3} - E\frac{Av(\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_1(\theta^\ell))}{Z^3}|. \qquad (3.20)$$

We consider the function

$$\varphi(u) = E\frac{Av\langle f\mathcal{E}(u)\rangle}{Z^3}$$

where

$$\mathcal{E}(u) = \exp\sum_{\ell\leqslant 3}[\frac{\beta}{\sqrt{N}}g\cdot\theta^\ell(b) - \frac{\beta^2}{2N}\|\theta^\ell(b)\|^2$$

$$+ u\|\theta^\ell(\boldsymbol{\sigma}^\ell)\|^2 + (1-u)\langle\|\theta^\ell(\boldsymbol{\sigma})\|^2\rangle]$$

so that (3.20) is $|\varphi(1)-\varphi(0)|$, and to prove (3.19) we bound $\varphi'(u)$ for each u. We have

$$\varphi'(u) = E\frac{Av\langle f(\sum_{\ell<3}\|\theta^\ell(\boldsymbol{\sigma}^\ell)\|^2 - \langle\|\theta^\ell(\boldsymbol{\sigma})\|^2\rangle)\mathcal{E}(u)\rangle}{Z^3}.$$

This is then bound by simple estimates, such as those previously used to bound the right-hand side of (3.14). □

Comment. — Of course one can merge the proofs of Propositions 3.1 and 3.2, but we found it more clear not to do so.

PROPOSITION 3.3. — *If $\beta d \leqslant 1$, we have*

$$C_{N+1} \leqslant L\beta^2 d^2(C_N + D_N) + L\frac{d^4}{N} \tag{3.21}$$

$$D_{N+1} \leqslant L\beta^2 d^2(C_N + D_N) + L\frac{d^4}{N}. \tag{3.22}$$

In that statement, C_{N+1} and D_{N+1} are computed for the $(N+1)$ spin system at inverse temperature $\beta' = \beta\sqrt{1+1/N}$. Iteration of the relations (3.21) to (3.22) yields that if $L\beta d < 1$, then $\lim\limits_{N\to\infty} C_N = 0 = \lim\limits_{N\to\infty} D$ (which was the main objective of this section).

Proof. — Setting $C_N(x,y) = E\langle(\frac{1}{N}x(\tilde{\boldsymbol{\sigma}})\cdot y(\boldsymbol{\sigma}^3))^2\rangle$, and using symmetry between coordinates, we see that

$$\begin{aligned} C_N(x,y) &= E\langle(x,\tilde{\sigma}_N)(y,\sigma_N^3)\frac{x(\tilde{\boldsymbol{\sigma}})\cdot y(\boldsymbol{\sigma}^3)}{N}\rangle \\ &\leqslant \frac{4d^4}{N} + E\langle(x,\tilde{\sigma}_N)(y,\sigma_N^3)\sum_{i\leqslant N-1}\frac{(x,\tilde{\sigma}_i)(y,\sigma_i^3)}{N}\rangle. \end{aligned} \tag{3.23}$$

Using this for $N+1$ rather than N, and appealing to (2.13) we get

$$C_{N+1}(x,y) \leqslant \frac{4d^4}{N} + E\frac{1}{Z^3}Av\langle(x,\tilde{\theta})(y,\theta^3)(\frac{x(\tilde{\boldsymbol{\sigma}})\cdot y(\boldsymbol{\sigma}^3)}{N+1})\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\boldsymbol{\sigma}^\ell)\rangle. \tag{3.24}$$

There, $\tilde{\theta} = \theta^1 - \theta^2$, and Av means integration over $\theta^1, \theta^2, \theta^3$. Consider

$$f = (x,\tilde{\theta})(y,\theta^3)\frac{x(\tilde{\boldsymbol{\sigma}})\cdot y(\boldsymbol{\sigma}^3)}{N+1}. \tag{3.25}$$

We observe that

$$(EAv(f^2))^{1/2} \leqslant 2d(EAv\langle(\frac{x(\tilde{\sigma})\cdot y(\sigma^3)}{N})^2\rangle)^{1/2}. \tag{3.26}$$

Using (3.19) and the essential fact that $\langle f\rangle = 0$ we get (using Cauchy-Schwarz) that

$$C_N(x,y) \leqslant L\beta^2 d^2 C_N^{1/2}(C_N^{1/2} + D_N^{1/2}) \leqslant L\beta^2 d^2 (C_N + D_N).$$

Taking the supremum over x, y yields (3.21).

The proof of (3.22) is similar; the function corresponding to (3.25) is

$$\frac{1}{N+1}((x,\theta^1)^2 - (x,\theta^2)^2)(\|x(\sigma^1)\|^2 - \|x(\sigma^2)\|^2). \qquad \square$$

4. Replica-Symmetric solution

In this section we complete the proof of Theorem 1.1. To simplify notation we will denote by $o(1)$ quantities that go to zero as $N \to \infty$, and we will not attempt to prove rates of convergence.

Assuming that $EAv\langle f^2\rangle$ remain bounded as $N \to \infty$, we see from (3.3) that

$$E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\sigma^\ell)\rangle}{Z^3} = E\frac{Av\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_1(\theta^\ell)}{Z^3} + o(1) \tag{4.1}$$

and of course the right hand side is more manageable than the left-hand side. Still, it involves Z in the denominator. We first show that we can simplify (4.1) into

$$E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\sigma^\ell)\rangle}{Z^3} = E\frac{Av\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_1(\theta^\ell)}{(Av\mathcal{E}_1(\theta))^3} + o(1). \tag{4.2}$$

Using the relation

$$E|\frac{U}{Z^3} - \frac{U}{\hat{Z}^3}| \leqslant (EU^2)^{1/2}E(\frac{(Z-\hat{Z})^2(Z^2+Z\hat{Z}+\hat{Z}^2)^2}{Z^6\hat{Z}^6})$$

for $\hat{Z} = Av\mathcal{E}_1(\theta)$, and the bounds $Z, \hat{Z} \geqslant \exp\frac{\beta}{\sqrt{N}}\boldsymbol{g}\cdot a(\boldsymbol{b})$ of last section, it suffices to show that

$$E\left(\exp\Big(-\frac{8\beta}{\sqrt{N}}\boldsymbol{g}\cdot a(\boldsymbol{b})\Big)(Z-\hat{Z})^2\right) = o(1),$$

a straight computation based upon the fact that $C_N, D_N \to 0$, and using replicas to transform products of brackets into single brackets. Of course in (4.1) we can replace 3 by any other number.

We will now show that for x, y in $\mathbb{R}^d$, the quantities

$$\frac{\langle x(\boldsymbol{b}) \cdot y(\boldsymbol{b})\rangle}{N} = \frac{\langle x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)\rangle}{N}$$

and

$$\frac{\langle x(\boldsymbol{\sigma}) \cdot y(\boldsymbol{\sigma})\rangle}{N}$$

are essentially independent of the disorder. In a second stage we will prove the relations (2.9), (2.10), and this will complete the proof of Theorem 1.1.

We will consider the two quantities

$$A_N = \sup_{\|x\|,\|y\| \leqslant 1} \operatorname{Var}\langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \rangle \tag{4.3}$$

$$B_N = \sup_{\|x\|,\|y\| \leqslant 1} \operatorname{Var}\langle \frac{x(\boldsymbol{\sigma}) \cdot y(\boldsymbol{\sigma})}{N} \rangle. \tag{4.4}$$

Of course Var refers to the variance relative to the variables (g_{ij}). We will prove that $A_N, B_N \to 0$ by arguments of the same spirit than those of Section 3. We will prove the following.

PROPOSITION 4.1.— *If $\beta d \leqslant 1$, then*

$$A_{N+1} \leqslant L\beta^2 d^2 (A_N + B_N) + o(1) \tag{4.5}$$

$$B_{N+1} \leqslant L\beta^2 d^2 (A_N + B_N) + o(1) \tag{4.6}$$

This will prove that $A_N, B_N \to 0$ if $L\beta d \leqslant 1$. We write

$$\begin{aligned} A_N(x,y) &= \operatorname{Var}\langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \rangle \\ &= E\langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \rangle^2 - (E\langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \rangle)^2 \\ &=: A'_N(x,y) - (A''_N(x,y))^2 \end{aligned} \tag{4.7}$$

We have

$$\langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \rangle^2 = \langle \frac{x(\boldsymbol{\sigma}^1) \cdot y(\boldsymbol{\sigma}^2)}{N} \frac{x(\boldsymbol{\sigma}^3) \cdot y(\boldsymbol{\sigma}^4)}{N} \rangle$$

so that using this for $N+1$ rather than N and using symmetry between the spins and (2.12), (4.2)

$$\begin{aligned} A'_{N+1}(x,y) &= E\langle x(\theta^1)y(\theta^2)(\frac{x(\boldsymbol{\sigma}^3)\cdot y(\boldsymbol{\sigma}^4)}{N+1}+\frac{x(\theta^3)y(\theta^4)}{N+1})\rangle' \qquad (4.8)\\ &= E\frac{Av(\langle x(\theta^1)y(\theta^2)x(\boldsymbol{\sigma}^3)\cdot y(\boldsymbol{\sigma}^4)\rangle \prod\limits_{\ell\leqslant 4}\mathcal{E}_1(\theta^\ell))}{N(Av\mathcal{E}_1(\theta))^4}+o(1)\\ &= E\frac{\langle x(\boldsymbol{\sigma}^3)\cdot y(\boldsymbol{\sigma}^4)\rangle Av(x(\theta^1)y(\theta^2)\prod_{\ell\leqslant 2}\mathcal{E}_1(\theta^\ell))}{N(Av\mathcal{E}_1(\theta))^2}+o(1)\end{aligned}$$

because Av means integration of $(\theta^\ell)_{\ell\leqslant 4}$ with respect to $\mu^{\otimes 4}$. In a similar (but easier) manner we have

$$A''_{N+1}(x,y) = E\frac{Avx(\theta^1)y(\theta^2)\prod_{\ell\leqslant 2}\mathcal{E}_1(\theta^\ell)}{(Av\mathcal{E}_1(\theta))^2}+o(1) \qquad (4.9)$$

and

$$A''_{N+1}(x,y) = E\frac{\langle x(\boldsymbol{\sigma}^3)\cdot y(\boldsymbol{\sigma}^4)\rangle}{N}+o(1) \qquad (4.10)$$

so that

$$A_{N+1}(x,y) = EUV - EUEV + o(1) \qquad (4.11)$$

for

$$\begin{aligned} U &= \frac{\langle x(\boldsymbol{\sigma}^3)\cdot y(\boldsymbol{\sigma}^4)\rangle}{N}\\ V &= E_g\frac{Avx(\theta^1)y(\theta^2)\prod\limits_{\ell\leqslant 2}\mathcal{E}_1(\theta^\ell)}{(Av\mathcal{E}_1(\theta))^2}\end{aligned}$$

where E_g denotes integration in g only.

Thus, from (4.10) we get

$$A_{N+1}(x,y)\leqslant \mathrm{Var}U\mathrm{Var}V+o(1)\leqslant (A_N(x,y))^{1/2}(\mathrm{Var}V)^{1/2}+o(1). \qquad (4.12)$$

To prove (4.5) we are reduced to prove that

$$\mathrm{Var}V\leqslant L\beta^4d^4(A_N+B_N). \qquad (4.13)$$

To prove this, we will show that

$$E(V-\tilde{V})^2\leqslant L\beta^4d^4(A_N+B_N) \qquad (4.14)$$

where $\tilde{V}$ is an independent copy of V, that is corresponding to an independent choice $(\tilde{g}_{ij})_{i<j\leqslant N+1}$ of the disorder. Objects related to the disorder $(\tilde{g}_{ij})$ will be denoted with a $\sim$. For $0 \leqslant u \leqslant 1$, we consider

$$\begin{aligned}\mathcal{E}_1(u,\theta) = \exp\frac{\beta}{\sqrt{N}}(\sqrt{1-u}\tilde{\boldsymbol{g}}\cdot\theta(\tilde{\boldsymbol{b}}) + \sqrt{u}\boldsymbol{g}\cdot\theta(\boldsymbol{b})) \\ \exp\frac{\beta^2}{2N}((1-u)(\langle\|\theta(\boldsymbol{\sigma})\|^2\rangle^{\sim} - \|\theta(\tilde{\boldsymbol{b}})\|^2) + \\ +u(\langle\|\theta(\boldsymbol{\sigma})\|^2\rangle - \|\theta(\boldsymbol{b})\|^2))\end{aligned} \tag{4.15}$$

and we consider

$$\varphi(u) = \frac{E_{g,\tilde{g}}Avx(\theta^1)y(\theta^2)\prod_{\ell\leqslant 3}\mathcal{E}_1(u,\theta^1)}{(Av\mathcal{E}_1(u,\theta))^2}$$

so that the left hand side (4.14) is

$$E(\varphi(1)-\varphi(0))^2 \leqslant \int_0^1 E\varphi'(u)^2du$$

and it suffices to prove that

$$\forall u, 0<u<1 \quad E\varphi'(u)^2 \leqslant L\beta^4 d^4(A_N+B_N). \tag{4.16}$$

The proof of (4.16) is a bit tedious; the computation of φ' requires integration by parts. Only crude bounds are needed, such as

$$\begin{aligned}&EAv(|x(\theta^1)y(\theta^2)|^2(\|\theta^3(\tilde{\boldsymbol{b}})\|^2 - \|\theta^3(\boldsymbol{b})\|^2)^2) \\ &\leqslant d^4 \sup_{\|x\|=1} E((\|x(\tilde{\boldsymbol{b}})\|^2 - \|x(\boldsymbol{b})\|^2)^2) \\ &\leqslant d^4 \sup_{\|x\|=1} E((\langle x(\boldsymbol{\sigma})\cdot x(\boldsymbol{\sigma}')\rangle^{\sim} - \langle x(\boldsymbol{\sigma})\cdot x(\boldsymbol{\sigma}')\rangle)^2) \\ &\leqslant d^4N^2A_N.\end{aligned}$$

The proof of (4.6) is similar. □

Combining (4.2) and Proposition 4.1, we see that we can improve (4.2) into

$$E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\boldsymbol{\sigma}^\ell)\rangle}{Z^3} = E\frac{Av\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_N(\theta^\ell)}{(Av\mathcal{E}_N(\theta))^3} + o(1) \tag{4.17}$$

for

$$\mathcal{E}_N(\theta) = \exp\beta g_{N,Q}(\theta)\exp\frac{\beta^2}{2}(R_N(\theta,\theta) - Q_N(\theta,\theta))$$

where

$$E g_{N,Q}(\theta) g_{N,Q}(\theta') = Q_N(\theta, \theta'),$$
$$Q_N(x,y) = E\langle \frac{x(\sigma^1)\cdot y(\sigma^2)}{N}\rangle$$
$$R_N(x,y) = E\langle \frac{x(\sigma)\cdot y(\sigma)}{N}\rangle.$$

We will now prove that

$$Q_N(x,y) = E\frac{Av(x,\theta^1)(y,\theta^2)\mathcal{E}_N(\theta^1)\mathcal{E}_N(\theta^2)}{(Av\mathcal{E}_N(\theta))^2} + o(1) \tag{4.18}$$

$$R_N(x,y) = E\frac{Av(x,\theta)(y,\theta)\mathcal{E}_N(\theta)}{Av\mathcal{E}_N(\theta)} + o(1). \tag{4.19}$$

To prove (4.18), we compare (4.10) and (4.9), in which using (4.17) rather than (4.2), we can now replace $\mathcal{E}_1$ by $\mathcal{E}_N$. The proof of (4.19) is similar.

To prove that (Q_N, R_N) converges to the solution (Q,R) of (2.9), (2.10) it suffices to show that for $L\beta d < 1$, these equations have a unique solution. To do this, if we define a distance on the quadratic forms on $\mathbb{R}^d$ by $\|Q - Q'\| = \sup_{\|x\|,\|y\|=1} |Q(x,y) - Q'(x,y)|$ one shows that if $T(Q,R), U(Q,R)$ denote the left hand sides of (2.9), (2.10) respectively, we have

$$\begin{aligned} &\|T(Q,R) - T(Q',R')\| + \|U(Q,R) - U(Q',R')\| \\ &\leqslant L\beta^2 d^2(\|Q-Q'\| + \|R-R'\|). \end{aligned} \tag{4.20}$$

The proof of (4.20), that can be done "moving along a path from (Q,R) to (Q',R')" as in the previous arguments require no new ideas so is better left to the reader.

5. Uniform measure on S

We set $\overline{\mu} = \mu^{\otimes N}$. We will prove the following.

THEOREM 5.1.— *There exists $L < \infty$ such that, if $L\beta \leqslant 1$ then for any value of d,*

$$\lim_{N\to\infty} \frac{1}{N} E\log Z_N = \frac{\beta^2}{4} d \tag{5.1}$$

where Z is the partition function given by

$$Z_N = \int \exp\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i,\sigma_j) d\overline{\mu}(\sigma). \tag{5.2}$$

Moreover, if $0 \leqslant u \leqslant 1$, *there is a number* $K(d)$ *independent of* N *such that, if* N *is large enough*

$$EG(\{\boldsymbol{\sigma}; \sum_{s,t}(\frac{1}{N}\sum_{i\leqslant N}\sigma_i(s)\sigma_i(t) - \delta_{s,t})^2 \geqslant u\}) \\ \leqslant K(d)\exp(-Nu/L) \tag{5.3}$$

where $(\sigma_i(s))_{s\leqslant d}$ *are the components of* σ_i *in* $\mathbb{R}^d$, $\delta_{s,t} = 1$ *if* $s = t$, $\delta_{s,t} = 0$ *if* $s \neq t$, *and*

$$EG^2\{\boldsymbol{\sigma}; \sum_{s,t}(\frac{1}{N}\sum_{i\leqslant N}\sigma_i(s)\sigma_i'(t))^2 \geqslant u\} \\ \leqslant K(d)\exp(-Nu/L) \tag{5.4}$$

It is a simple matter to deduce (5.1) from (5.3), (5.4) (by integration by parts of $\frac{\partial}{\partial\beta}E\log Z_N$) but it is useful to state (5.1) right away, because this makes apparent the main difficulty: The quantity (5.1) is very much smaller than

$$\lim_{N\to\infty}\frac{1}{N}\log EZ_N \tag{5.5}$$

that is of order $\beta^2 d^2$ (rather than $\beta^2 d$) for $\beta d \gg 1$. This difficulty does not exist in the case of $d = 1$. It will be a highly non trivial task to find the correct upper bound for $E\log Z_N$. The first step of the proof will consist in finding lower bounds for Z_N, which is much easier.

LEMME 5.2.— *Consider numbers* $(r_i)_{i\leqslant N}$. *If* $\sum_{i\leqslant N} r_i^4 \leqslant N/2$, *then for* $u \geqslant 0$,

$$P(\log 2^{-N}\sum_{\epsilon}\exp\frac{1}{\sqrt{N}}\sum_{i<j}g_{ij}\epsilon_i\epsilon_j r_i r_j \leqslant \frac{1}{4N}(\sum_{i\leqslant N}r_i^2)^2 - u) \\ \leqslant L\exp(-u^2/2), \tag{5.6}$$

where $\sum_{\epsilon}$ *means summation over all choices of* $\epsilon = (\epsilon_i), \epsilon_i = \pm 1$.

Comment.— 1) In this lemma, the condition $\sum_{i\leqslant N} r_i^4 \leqslant N/2$ can be replaced by $\sum_{i\leqslant N} r_i^4 \leqslant \gamma N$ for any $\gamma < 1$ (with a different L in (5.6)).

2) The lemma *does not* say that with high probability we have

$$\log 2^{-N}\sum_{\epsilon}\exp\frac{1}{\sqrt{N}}\sum_{i<j}g_{ij}\epsilon_i\epsilon_j r_i r_j \simeq \frac{1}{4N}(\sum_{i\leqslant N}r_i^2)^2$$

for every choice of r_i with $\sum_{i\leqslant n} r_i^4 \leqslant N/2$. If this property was true (I suspect this is not the case) (5.1) would be much easier to prove.

Proof. — The proof follows from a straightforward adaptation of the case where all r_i are equal, that is done in [T1], (1.8). The key point is that

$$2^{-2N}\sum_{\epsilon,\epsilon'} \exp\frac{1}{2}\Big(\frac{\sum \epsilon_i\epsilon_i' r_i^2}{\sqrt{N}}\Big)^2$$

remains bounded (by $\sqrt{2}$) as $\sum_{i\leqslant N} r_i^4 \leqslant N/2$, an elementary fact shown e.g. in [T2].

PROPOSITION 5.3.— *If $L\beta \leqslant 1$ and $t > 0$ we have for all $u > 0$ that*

$$P(E\log Z_N \leqslant N\frac{\beta^2 d}{4} - u) \leqslant dL\exp(-\frac{u^2}{Ld^2}). \tag{5.7}$$

Comment. — It could in fact be shown that the factor $1/d^2$ can be removed in the exponent, but we will not need this.

Proof. — Throughout the rest of the paper, we denote by $(\sigma_i(s))_{s\leqslant d}$ the components of $\sigma_i \in \mathbb{R}^d$. Using the symmetries of μ, we have

$$Z_N = \int F_N(\boldsymbol{\sigma})d\overline{\mu}(\boldsymbol{\sigma}) \tag{5.8}$$

where

$$\begin{aligned} F_N(\boldsymbol{\sigma}) &= 2^{-dN}\sum_{\epsilon^1,\cdots,\epsilon^d} \exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}\sum_{s\leqslant d}\sigma_i(s)\sigma_j(s)\epsilon_i^s\epsilon_j^s \\ &= \prod_{s\leqslant d}(2^{-N}\sum_{\epsilon}\exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}\sigma_i(s)\sigma_j(s)\epsilon_i\epsilon_j). \end{aligned} \tag{5.9}$$

There, the summations are over ϵ^s, ϵ ranging over $\{-1,1\}^N$. The reason why lower bounds on Z_N are much easier than upper bounds is that to get a lower bound, it suffices to show that for enough choices of $\boldsymbol{\sigma}, F_N(\boldsymbol{\sigma})$ is not too small. (On the other hand, showing that for many values of $\boldsymbol{\sigma}, F_N(\boldsymbol{\sigma})$ is not too large does not provide an upper bound for Z_N.) Given $\boldsymbol{\sigma}$, we know, with probability close to one, how to estimate the product in (5.9), so that by Fubini theorem, with probability close to one, we know how to estimate this product for most (but not all) values of $\boldsymbol{\sigma}$ (see the comment after Lemma 5.1). More specifically, using Jensen's inequality, and since (also by Jensen's inequality) we have $F_N(\boldsymbol{\sigma}) \geqslant 1$, we get from (5.8) that

$$\log Z_N \geqslant \int_A \log F_N(\boldsymbol{\sigma}) d\overline{\mu}(\boldsymbol{\sigma}) \tag{5.10}$$

where $A = \{\boldsymbol{\sigma}; \forall s \leqslant d, \beta^2 \sum\limits_{i \leqslant N} \sigma_i^4(s) \leqslant N/2\}$.

The law of large numbers shows (since $\int \sigma^4(s) d\mu(\sigma) \leqslant L$) that for $\beta \leqslant \beta_0$, we have $\overline{\mu}(A) \geqslant 1/2$. By (5.9) we have

$$\log F_N(\boldsymbol{\sigma}) \geqslant \sum_{s \leqslant d} \log(2^{-N} \sum_{\epsilon} \exp \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \epsilon_i \epsilon_j \sigma_i(s) \sigma_j(s)). \tag{5.11}$$

Consider the event $\Omega(\boldsymbol{\sigma}, t)$ given by

$$\begin{aligned} &\forall s \leqslant d, \log(2^{-N} \sum_{\epsilon} \exp \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \epsilon_i \epsilon_j \sigma_i(s) \sigma_j(s)) \\ &\geqslant \frac{\beta^2}{4N} (\sum_{i \leqslant N} \sigma_i^2(s))^2 - u. \end{aligned} \tag{5.12}$$

Then by Lemma 5.2 we have $P(\Omega(\boldsymbol{\sigma}, u)) \geqslant 1 - Ld \exp(-u^2/2)$ whenever $\boldsymbol{\sigma} \in A$.

By Fubini Theorem the event $\overline{\mu}(\{\boldsymbol{\sigma} \in A; \Omega(\boldsymbol{\sigma}, u) \text{ occurs}\}) \geqslant \frac{1}{4}$ has probability $\geqslant 1 - 2d \exp(-u^2/2)$. Now, if $\Omega(\boldsymbol{\sigma}, u)$ occurs, by (5.12),

$$\log F_N(\boldsymbol{\sigma}) \geqslant \sum_{s \leqslant d} (\frac{\beta^2}{4N} (\sum_{i \leqslant N} \sigma_i^2(s))^2 - u) \geqslant \frac{N\beta^2 d}{4} - du$$

using the inequality $x^2 \geqslant 2Nx - N^2$ for $x = x(s) = \sum_{i \leqslant N} \sigma_i^2(s)$, and since $\sum\limits_{s \leqslant d} x(s) = Nd$. The result follows. $\square$

We now turn to the hard part of the proof, the search of upper bounds for $\log Z_N$. The main idea is that the discrepancy between $E \log Z_N$ and $\log EZ_N$ come from the big influence of a few configurations $\boldsymbol{\sigma}$ on EZ_N. Once these are removed (and bounded by other means) we can control Z_N by EZ_N.

Even though this cannot be apparent at this stage, the central fact seems to be the following.

PROPOSITION 5.4.— *If M is large enough the following occurs. Consider for $i \leqslant N$ a number $r_i, 0 \leqslant r_i \leqslant d$, the set $S_i = r_i S$, and denote by μ_i the uniform probability measure on S_i. Assume that $L\beta \leqslant 1$. Consider the set*

$$A = \{\boldsymbol{\sigma}; \forall i \leqslant N, \sigma_i \in S_i, \forall s \leqslant d, \frac{1}{N}\sum_{i\leqslant N} \sigma_i^2(s) \leqslant 2; \tag{5.13}$$

$$\sum_{1\leqslant s<t\leqslant d} (\frac{1}{N}\sum_{i\leqslant N} \sigma_i(s)\sigma_i(t))^2 \leqslant 1\}.$$

Then for any subset B of A,

$$E(\int_B \exp \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j) d\hat{\mu}(\boldsymbol{\sigma})) \tag{5.14}$$

$$\leqslant K(d)\hat{\mu}(B)^{1/2} \exp \frac{\beta^2}{4Nd}(\sum_{i\leqslant N} r_i^2)^2$$

where $\hat{\mu} = \mu_1 \otimes \cdots \otimes \mu_N$.

Comment.— This statement is better understood when $r_i = d$ for each $i \leqslant N$, in which case the last term of (5.14) is $\exp N d\beta^2/4$. (This situation is the only one where we will use a subset $B \neq A$).

At some point in the proof of Proposition 5.4, we will have to use the fact that the uniform measure μ on S has some special properties. These will be used crucially in several occasions, so we spell out and prove them first.

PROPOSITION 5.5.— *There exists a number L (independent of d), with the following properties.*

If the numbers $(a_s)_{s\leqslant d}$ satisfy $|a_s| \leqslant 1/8$, then

$$\int \exp \sum_{s\leqslant d} a_s(\sigma^2(s) - 1) d\mu(\sigma) \leqslant \exp L \sum_{s\leqslant d} a_s^2 \tag{5.15}$$

$$\int \exp \sum_{s\leqslant d} a_s \sigma(s)\sigma'(s) d\mu(\sigma) d\mu(\sigma') \leqslant \exp L \sum_{s\leqslant d} a_s^2 \tag{5.16}$$

If the numbers $(a_{s,t})_{s,t\leqslant d}$ satisfy $\sum_{s,t\leqslant d} a_{s,t}^2 \leqslant 1/4$ and $a_{s,t} = a_{t,s}$, then

$$\int \exp \sum_{s,t\leqslant d} a_{s,t}\sigma(s)\sigma'(t) d\mu(\sigma) d\mu(\sigma') \leqslant \exp L \sum_{s,t\leqslant d} a_{s,t}^2 \tag{5.17}$$

If moreover $\sum\limits_{s\leqslant d} a_{s,s}=0$ *then*

$$\int \exp \sum_{s,t\leqslant d} a_{s,t}\sigma(s)\sigma(t)d\mu(\sigma) \leqslant \exp L \sum_{s,t\leqslant d} a_{s,t}^2. \tag{5.18}$$

Proof. — We first observe that (5.17) follows from (5.16) by diagonalization of the symmetric matrix $(a_{s,t})$ in an orthonormal basis; and (5.18) follows from (5.15), observing that $\sum\limits_{s\leqslant d} a_s(\sigma^2(s)-1) = \sum\limits_{s\leqslant d} a_s\sigma^2(s)$ if $\sum\limits_{s\leqslant d} a_s = 0$.

To prove (5.15), we will prove as an intermediate step that

$$\int \exp \sum_{s\leqslant d} a_s(\sigma^2(s)-1)d\mu(\sigma) \leqslant L \exp L \sum_{s\leqslant d} a_s^2 \tag{5.19}$$

provided $|a_s| \leqslant 1/4$. We show first that this implies (5.15), that is, that the first factor L on the right of (5.19) can be removed. If $a_s \leqslant 1/8$ for each $s \leqslant d$, then $X = \sum\limits_{s\leqslant d} a_s(\sigma^2(s)-1)$ satisfies

$$\int \exp 2X d\mu \leqslant L \exp L \sum_{s\leqslant d} a_s^2. \tag{5.20}$$

Since $2^4X^4/4! \leqslant \exp 2X$, this implies $\int X^4 d\mu \leqslant L \exp L \sum\limits_{s\leqslant d} a_s^2$. This holds under the condition $|a_s| \leqslant 1/8$; so homogeneity shows that $(\int X^4 d\mu)^{1/2} \leqslant L \sum\limits_{s\leqslant d} a_s^2$ by reducing this to the case $\sum\limits_{s\leqslant d} a_s^2 = 1/64$. Finally, since $\int X d\mu = 0$, and $e^x - 1 - x \leqslant x^2 e^x$ we have

$$\begin{aligned} \int \exp X d\mu &\leqslant 1 + \int X^2 \exp X d\mu \\ &\leqslant 1 + (\int X^4 d\mu)^{1/2} (\int \exp 2X d\mu)^{1/2} \\ &\leqslant 1 + L \sum_{s\leqslant d} a_s^2 \leqslant \exp L \sum_{s\leqslant d} a_s^2 \end{aligned} \tag{5.21}$$

provided $\sum\limits_{s\leqslant d} a_s^2 \leqslant 1/64$. Thus (5.21) proves (5.15) in that case, while if $\sum\limits_{s\leqslant d} a_s^2 \geqslant 1/64$, (5.15) follows from (5.19).

To prove (5.19), we will proceed by comparison with Gaussian r.v. If $(h(s))_{s\leqslant d}$ are i.i.d. $N(0,1)$, we have

$$E \exp \sum_{s\leqslant d} a_s(h^2(s)-1) = \prod_{s\leqslant d} \left(\frac{1}{\sqrt{1-2a_s}} \exp -a_s\right) \leqslant \exp L \sum_{s\leqslant d} a_s^2 \tag{5.22}$$

since $(1-2x)^{-1/2} \leqslant \exp x + Lx^2$ for $x \leqslant 1/4$. To relate this to (5.19), we recall that $(h(1), \cdots, h(d))$ is distributed like $(d^{-1}\sum_{s\leqslant d} h^2(s))^{1/2}\sigma$, where σ is uniform over S and independent of $h(1), \cdots, h(d)$; moreover,

$$\int \exp r \sum_{s\leqslant d} a_s(\sigma^2(s)-1)d\mu(\sigma)$$

increases with r by Hölder's inequality; and thus

$$\begin{aligned} P(\sum_{s\leqslant d} h^2(s) \geqslant d) \int \exp \sum_{s\leqslant d} a_s(\sigma^2(s)-1)d\mu(\sigma) \\ \leqslant E \exp \sum_{s\leqslant d} a_s(h^2(s)-1)d\mu(\sigma) \end{aligned}$$

which, together with (5.22) and the fact $P(\sum_{s\leqslant d} h^2(s) \geqslant d) \geqslant 1/L$ finish the proof of (5.19) and hence of (5.15). The proof of (5.16) is entirely similar.

□

Proof of Proposition 5.4. — We have to bound

$$\int_B \exp \frac{\beta^2}{2N} \sum_{i<j} (\sigma_i, \sigma_j)^2 d\hat{\mu}(\boldsymbol{\sigma}). \tag{5.23}$$

Now,

$$\begin{aligned} \sum_{i<j} (\sigma_i, \sigma_j)^2 &= \sum_{i<j} (\sum_{s\leqslant d} \sigma_i(s)\sigma_j(s))^2 \\ &= \sum_{s,t\leqslant d} \sum_{i<j} \sigma_i(s)\sigma_i(t)\sigma_j(s)\sigma_j(t) \\ &\leqslant \frac{1}{2} \sum_{s,t\leqslant d} (\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t))^2 \end{aligned}$$

where to lighten notation we write $\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t) = \sum_{i\leqslant N} \sigma_i(s)\sigma_i(t)$. Thus it suffices to bound the quantity

$$U = \int_B \exp \frac{\beta^2}{4} \sum_{s,t\leqslant d} (\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{\sqrt{N}})^2 d\hat{\mu}(\boldsymbol{\sigma}). \tag{5.24}$$

We use the identity $x^2 = 2ax - a^2 + (x-a)^2$ for $a = \frac{1}{d\sqrt{N}} \sum_{i\leqslant N} r_i^2, x =$

$x_s = \frac{1}{\sqrt{N}}\|\sigma(s)\|^2 = \frac{1}{\sqrt{N}} \sum_{i \leqslant N} \sigma_i(s)^2$ to obtain, since $\sum_{s \leqslant d} x_s = da$ that

$$\sum_{s \leqslant d} (\frac{\|\sigma(s)\|^2}{\sqrt{N}})^2 = \frac{1}{dN}(\sum_{i \leqslant N} r_i^2)^2 + \sum_{s \leqslant d}(\frac{1}{\sqrt{N}}\|\sigma(s)\|^2 - a)^2.$$

Thus we have to bound

$$V = \int_B \exp \frac{\beta^2}{4}[\sum_{s \leqslant d}(\frac{1}{\sqrt{N}}\|\sigma(s)\|^2 - a)^2 + 2 \sum_{s<t \leqslant d}(\frac{\sigma(s)\cdot\sigma(t)}{\sqrt{N}})^2]\, d\hat{\mu}(\sigma). \tag{5.25}$$

Using Cauchy-Schwarz, we have

$$V \leqslant \hat{\mu}(B)^{1/2} W^{1/2}$$

where

$$W = \int_A \exp \frac{\beta^2}{2}[\sum_{s \leqslant d}(\frac{1}{\sqrt{N}}\|\sigma(s)\|^2 - a)^2 + 2 \sum_{s<t \leqslant d}(\frac{\sigma(s)\cdot\sigma(t)}{\sqrt{N}})^2]\, d\hat{\mu}(\sigma) \tag{5.26}$$

and to finish the proof we have to show that

$$W \leqslant K(d). \tag{5.27}$$

Considering i.i.d $N(0,1)$ r.v $(g_{s,t})_{s \leqslant t \leqslant d}$, we have

$$W = \int_A E \exp \beta[\sum_{s \leqslant d} g_{s,s}(\frac{1}{\sqrt{N}}\|\sigma(s)\|^2 - a) + \sqrt{2} \sum_{s<t \leqslant d} g_{s,t}\frac{\sigma(s)\cdot\sigma(t)}{\sqrt{N}}]\, d\hat{\mu}(\sigma). \tag{5.28}$$

An essential point is that we will not need to take expectation over all the values of $g_{s,t}$, but only those that realize the following event

$$C = \{\forall s \leqslant d, |g_{s,s}| \leqslant L\sqrt{N}; \sum_{s<t} g_{s,t}^2 \leqslant LN\}.$$

We are going to show that

$$W \leqslant LE \int_A 1_C \exp [\beta \sum_{s \leqslant d} g_{s,s}(\frac{1}{\sqrt{N}}\|\sigma(s)\|^2 - a) \tag{5.29}$$

$$+ \sqrt{2}\beta \sum_{s<t \leqslant d} g_{s,t}(\frac{1}{\sqrt{N}}\sigma(s)\cdot\sigma(t))]\, d\hat{\mu}(\sigma).$$

The key fact is that the definition (5.13) of A implies that over the domain of integration in (5.28) we have

$$\sum_{s<t\leqslant d} 2\beta^2\Big(\frac{1}{\sqrt{N}}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\Big)^2 \leqslant 2\beta^2 N$$
$$\forall s\leqslant d,\ \beta\Big|\frac{1}{\sqrt{N}}\|\boldsymbol{\sigma}(s)\|^2-a\Big|\leqslant 2\beta N.$$

The tool to prove (5.29) is the following elementary fact. If $(g_\ell)_{\ell\leqslant m}$ are i.i.d. $N(0,1)$, then, for

$$y\geqslant 16\Big(m+\sum_{\ell\leqslant m}\alpha_\ell^2\Big) \tag{5.30}$$

we have

$$E\Big(1_{\{\sum_{\ell\leqslant m} g_\ell^2\geqslant y\}}\exp\sum_{\ell\leqslant m}\alpha_\ell g_\ell\Big)\leqslant \exp-\frac{y}{16}.$$

([T1], Lemma 7.3). We apply this lemma for $m=1$ to each $g_{s,s}$, and then to the family $(g_{s,t})_{s<t}$, with $y=L\beta^2N$.

The contributions to the right-hand side of (5.28) of the event C^c are then bounded by $(d+1)(\exp-\frac{y}{16})$, from which (5.29) follows.

Consider

$$J=\int \exp\Big[\beta\sum_{s\leqslant d} g_{s,s}\Big(\frac{1}{\sqrt{N}}\|\boldsymbol{\sigma}(s)\|^2-a\Big)+\sqrt{2}\beta\sum_{s\leqslant t\leqslant d} g_{s,t}\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{\sqrt{N}}\Big]d\hat{\mu}(\boldsymbol{\sigma}). \tag{5.31}$$

(Note that the integral is now over the whole space.) We will show that

$$1_C J\leqslant \exp L\beta^2\sum_{s\leqslant t\leqslant d} g_{s,t}^2 \tag{5.32}$$

so that for $\beta\leqslant\beta_0$

$$W\leqslant 2E1_C J\leqslant L^{d^2}$$

which proves (5.27). Since in (5.31) the integral is over all the values of $\boldsymbol{\sigma}$, we have

$$J=\prod_{i\leqslant N} J_i$$

where

$$J_i=\int_{S_i}\exp\Big[\beta\sum_{s\leqslant d}\frac{g_{s,s}}{\sqrt{N}}\Big(\sigma^2(s)-\frac{r_i^2}{d}\Big)+\sqrt{2}\beta\sum_{s<t\leqslant d}\frac{g_{s,t}}{\sqrt{N}}\sigma(s)\sigma(t)\Big]d\mu_i(\sigma).$$

Going back to S with the change of variable $\sigma \to r_i\sigma/\sqrt{d}$, we see that we have to bound

$$\int_S \exp\Big[\sum_{s\leqslant d} a_s(\sigma^2(s)-1)+\sum_{s<t\leqslant d} a_{s,t}\sigma(s)\sigma(t)\Big]d\mu(\sigma)$$

where the coefficients $a_s, a_{s,t}$ satisfy

$$\forall s\leqslant d, |a_s|\leqslant L\beta \tag{5.33}$$

$$\Big(\sum_{s<t} a_{s,t}^2\Big)^{1/2}\leqslant L\beta \tag{5.34}$$

so it suffices to use Cauchy-Schwarz and Proposition 5.5 to conclude. □

LEMME 5.6.— *If $r_i = d$ for each i, A is given by (5.13), and*

$$B(u)=\{\boldsymbol{\sigma}\in S^N; \sum_{s\leqslant d}\Big(\frac{1}{N}\|\boldsymbol{\sigma}(s)\|^2-1\Big)^2+\sum_{s\leqslant t\leqslant d}\Big(\frac{1}{N}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\Big)^2\geqslant u\} \tag{5.35}$$

then

$$\overline{\mu}(A\cap B(u))\leqslant \exp(-Nu/L). \tag{5.36}$$

Proof. — Inspection of the proof of Proposition 5.4 shows that we have proved that for a certain β_0, we have

$$\int_A \exp\frac{N\beta_0^2}{4}\Big[\sum_{s\leqslant d}\Big(\frac{1}{N}\|\boldsymbol{\sigma}(s)\|^2-1\Big)^2+\sum_{s\leqslant t\leqslant d}\Big(\frac{1}{N}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\Big)^2\Big]d\overline{\mu}(\boldsymbol{\sigma})\leqslant L \tag{5.37}$$

from which (5.36) follows by Chebishev inequality. □

Comparison of (5.36), (5.14) shows that we have succeeded in controlling the configurations in A. It remains to control the others.

LEMME 5.7.— *If $L\beta\leqslant 1$, consider*

$$A'=\{\boldsymbol{\sigma}\in S^N; \exists s\leqslant d, \frac{1}{N}\|\boldsymbol{\sigma}(s)\|^2\geqslant 2, \sum_{s<t\leqslant d}\Big(\frac{1}{N}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\Big)^2\leqslant 1\}. \tag{5.38}$$

Then, with probability $\geqslant 1-\exp(-N/L)$ we have

$$\int_{A'}\exp\frac{1}{\sqrt{N}}\sum_{i<j} g_{ij}(\sigma_i,\sigma_j)d\overline{\mu}(\boldsymbol{\sigma})\leqslant K(d)\exp\Big(\frac{N\beta^2 d}{4}-\frac{N}{L}\Big). \tag{5.39}$$

Proof. — It suffices to prove (5.39) when one replaces A' by

$$A_k = \{\boldsymbol{\sigma} \in S^N; \forall s \leqslant k, \frac{1}{N}\|\boldsymbol{\sigma}(s)\|^2 \geqslant 2, \forall s, k < s \leqslant d, \frac{1}{N}\|\boldsymbol{\sigma}(s)\|^2 \leqslant 2$$
$$\sum_{1 \leqslant s < t \leqslant d} (\frac{1}{N}\boldsymbol{\sigma}(s) \cdot \boldsymbol{\sigma}(t))^2 \leqslant 1\}.$$

We observe that

$$\sum_{i<j} g_{ij} x_i x_j \leqslant w \sum_{i \leqslant N} x_i^2$$

where u is the largest eigenvalue of the matrix $(\frac{1}{2}g_{ij})_{ij}$, and where we define $g_{ij} = g_{ji}$ if $i > j$. Moreover there is a set C with $P(C) \geqslant 1 - \exp -N/L$ and $w \leqslant L_0\sqrt{N}$ on C. Thus

$$\begin{aligned} &1_C \exp \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j) \\ &\leqslant \exp(L\beta(\sum_{s \leqslant k} \sum_{i \leqslant N} \sigma_i^2(s)) + \frac{\beta}{\sqrt{N}} \sum_{s>k} \sum_{i<j} g_{ij}\sigma_i(s)\sigma_j(s)). \end{aligned}$$

We can now apply (5.14) (with $d - k$ rather than d and $r_i^2 = d - \sum_{s \leqslant k} \sigma_i^2(s)$) to obtain

$$\begin{aligned} E(1_C \int_{A_k} &\exp \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j) d\overline{\mu}(\boldsymbol{\sigma})) \\ &\leqslant \int_{A_k} \exp\ [L_0\beta(\sum_{s \leqslant k} \|\boldsymbol{\sigma}(s)\|^2) \\ &+ \frac{\beta^2}{4N(d-k)}(Nd - \sum_{s \leqslant k} \|\boldsymbol{\sigma}(s)\|^2)^2] d\overline{\mu}(\boldsymbol{\sigma}). \end{aligned}$$

For $\boldsymbol{\sigma}$ in A_k, we have $\sum_{s \leqslant k} \|\boldsymbol{\sigma}(s)\|^2 \geqslant kN$. Now, if $x \geqslant kN$, we have $Nd - x \leqslant N(d-k)$ so that

$$\frac{\beta^2}{4N(d-k)}(Nd - x)^2 \leqslant \frac{N\beta^2}{4}(d-k).$$

Thus, all we have to prove is that, for $L\beta \leqslant 1$ we have

$$\int_{A_k} \exp L_0\beta(\sum_{s \leqslant k} \|\boldsymbol{\sigma}(s)\|^2) d\overline{\mu}(\boldsymbol{\sigma}) \leqslant K(d) \tag{5.40}$$

To do this, we observe that by (5.15) we have (for $|t| \leqslant 1/8$)

$$\int \exp t(\sum_{s\leqslant k} \|\boldsymbol{\sigma}(s)\|^2)d\overline{\mu}(\boldsymbol{\sigma}) \leqslant \exp kN(t+Lt^2) \tag{5.41}$$

so that by Chebishev inequality,

$$\overline{\mu}(A_k) \leqslant \exp(-kNt + kLNt^2) \leqslant \exp -\frac{kN}{L}$$

by optimization over t; from which (5.40) follows by Hölder's inequality using (5.41) again. □

Proof of Theorem 5.1. — We consider the set

$$A_1 = \{\boldsymbol{\sigma}; \sum_{s<t}(\frac{1}{N}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t))^2 \leqslant 1\}$$

and, for $0 \leqslant u \leqslant 1$, the set $B(u)$ given by (5.35).

It follows from Proposition 5.4 that

$$E\int_{A\cap B(u)} \exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}(\sigma_i,\sigma_j)d\overline{\mu}(\boldsymbol{\sigma}) \leqslant K(d)\hat{\mu}(B(u))^{1/2}\exp\frac{\beta^2 dN}{4}. \tag{5.42}$$

We use Lemma 5.6 to control $\hat{\mu}(B(u))$, and Lemma 5.7 to show that $A' = A_1\backslash A$ can be added to the domain of integration; we then see that by Chebishev inequality we have

$$\begin{aligned} P(\int_{A_1\cap B(u)} \exp\frac{\beta}{\sqrt{N}}\sum_{i<j} g_{ij}(\sigma_i,\sigma_j)d\overline{\mu}(\boldsymbol{\sigma}) &\geqslant K(d)\exp(\frac{\beta^2 Nd}{4} - \frac{uN}{L})) \\ &\leqslant K(d)\exp(-\frac{Nu}{L}). \end{aligned} \tag{5.43}$$

A crucial ingredient of the proof is that we can improve (5.43) by replacing $A_1 \cap B(u)$ by $B(u)$. To see this, given an orthonormal basis W of $\mathbb{R}^d$, let us consider the sets $A_{1,W}, B_W(u)$ obtained as A_1 and $B(u)$ when the components $\sigma_i(s)$ of σ_i in the canonical basis are replaced by their components in the basis W. Then (5.43) holds when one replaces A_1 by $A_{1,W}, B(u)$ by $B_W(u)$. The matrices $D(\boldsymbol{\sigma}) = (\frac{1}{N}\sum_{i\leqslant N}\sigma_i(s)\sigma_i(t))_{s,t} = (\frac{1}{N}\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t))_{s,t}$ relate by the same change of basis, so that $B_W(u) = B(u)$. Moreover, for any $\boldsymbol{\sigma}$ we can find a basis that diagonalizes the symmetric matrix $D(\boldsymbol{\sigma})$. It follows that we can find a finite set of orthonormal bases of $\mathbb{R}^d$, (with a cardinality independent of N) such that for each $\boldsymbol{\sigma}$, we can find one basis W in this collection with $\boldsymbol{\sigma} \in A_{1,W}$. Thus we have proved that

$$P(\int_{B(u)} \exp \frac{\beta}{\sqrt{N}} g_{ij}(\sigma_i, \sigma_j) d\overline{\mu}(\boldsymbol{\sigma}) \geqslant K(d) \exp(\frac{\beta^2 Nd}{4} - \frac{uN}{L_1})) \leqslant K(d) \exp(-\frac{Nu}{L}), \tag{5.44}$$

the improvement of (5.43) that was promised earlier.

Now, by (5.7),

$$P(Z_N \leqslant \exp(\frac{\beta^2 Nd}{4} - \frac{Nu}{2L_1})) \leqslant dL \exp(-\frac{N^2u^2}{d^2L})$$

and comparing with (5.44) we get

$$P(G(B(u) \geqslant K(d) \exp -\frac{Nu}{2L_1})) \leqslant K(d) \exp(-\frac{Nu}{L}) + dL \exp(-\frac{N^2u^2}{d^2L})$$

from which (5.3) follows.

Using (5.42), (5.43), we see that for $u \leqslant 1$ we have

$$P(\log Z_N \geqslant K(d) + \frac{\beta^2 dN}{4} + uN) \leqslant K(d) \exp(-\frac{Nu}{L}) \tag{5.45}$$

from which (5.1) follows easily.

It remains to prove (5.4). The proof is rather similar to that of (5.3). Using a suitable coordinate system in $\mathbb{R}^d$, we reduce the proof of (5.4) to the case where $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ moreover satisfy

$$\sum_{s<t} (\frac{1}{N} \boldsymbol{\sigma}(s) \cdot \boldsymbol{\sigma}'(t))^2 \leqslant 1.$$

Use of Lemma 5.8 shows that we can moreover assume

$$\frac{1}{N} \|\boldsymbol{\sigma}(s)\|^2 \leqslant 2, \frac{1}{N} \|\boldsymbol{\sigma}'(t)\|^2 \leqslant 2.$$

Following the pattern of Proposition 3.5, we have to show that if $|a(s)| \leqslant 1/L$ for $s \leqslant d$, then

$$E \int \exp \sum_{s \leqslant d} a(s) \sigma(s) \sigma'(s) d\mu(\sigma) d\mu(\sigma') \leqslant \exp L \sum_{s \leqslant d} a(s)^2,$$

which is proved in Proposition 5.6.

6. Perturbation of uniform measure on S_1

The proof of Theorem 1.3 faces serious obstacles. When using the cavity method, we face quantities such as $\beta^2\|\theta(\sigma)\|^2/N$ in exponents, that can be as large as $\beta^2 d^2 >> 1$, and these are very difficult to control. Also, Z will now be of order $\exp\beta^2 d^2$, and this is very large. The lower bounds on Z obtained using Jensen's inequality are now ineffective, and much more work will be required to control Z from below. The proof of Theorem 1.3 is much easier if, rather than asking only that $\|m\|_\infty \leqslant 1/L$, where L does not depend upon d, we require, say, $\|m\|_\infty \leqslant 1/Ld^2$. On the other hand, as the prime objective of the present paper is to gain a better understanding of the cavity method, we have made the effort to reach the strongest statement we could.

We consider a large constant M, that will be determined later, and is currently better considered as a parameter. We consider a function m on S, and the measure ν on S of density $1+m$ with respect to μ. We now denote by G Gibbs' measure with Hamiltonian (1.8), when S^N is provided with the measure $\overline{\nu} = \nu^{\otimes N}$, while G_0 denotes Gibbs' measure when S^N is provided with $\overline{\mu}$ (so it corresponds to $m = 0$).

PROPOSITION 6.1.— *If M is large enough and $\|m\|_\infty \leqslant 1/M$, then*

$$EG\{\boldsymbol{\sigma};\sum_{s,t\leqslant d}(\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N}-\delta_{s,t})^2 \geqslant 1\} \leqslant K(d)\exp -\frac{N}{L} \tag{6.1}$$

$$EG^2\{(\boldsymbol{\sigma},\boldsymbol{\sigma}');\sum_{s,t\leqslant d}(\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}'(t)}{N})^2 \geqslant 1\} \leqslant K(d)\exp -\frac{N}{L}. \tag{6.2}$$

Comment. — The point is that a quantity such as $\sum_{s,t\leqslant d}(\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}'(t)/N)^2$ can be as large as d^2; The content of (6.2) is that it is in practice not larger than 1.

Proof. — If f is any function on S^N, we have

$$(1-\frac{1}{M})^N\int f d\overline{\mu} \leqslant \int f d\overline{\nu} \leqslant (1+\frac{1}{M})^N\int f d\overline{\mu}$$

and thus for any set A in S^N,

$$G(A) \leqslant (\frac{M+1}{M-1})^N G_0(A)$$

so that (6.1) follows from (5.3) for $u = 1$ if M is large enough. Similarly, (6.2) follows from (5.4). □

We fix M such that (6.1), (6.2) hold, and we now turn towards the problem of finding lower bounds for

$$Z = Av\langle \exp \frac{\beta}{\sqrt{N}} g \cdot \theta(\boldsymbol{\sigma}) \rangle$$

where now average means integration of θ with respect to ν.

We will think to Z as a function of g, the disorder relative to $\langle\cdot\rangle$ (that is the r.v. $g_{ij}, i,j \leqslant N$) being fixed. As should be obvious later, events of exponentially small probability are irrelevant, so using Proposition 6.1 we will assume in our estimates that

$$G\{\boldsymbol{\sigma};\ \sum_{s,t\leqslant d} (\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N} - \delta_{s,t})^2 \leqslant 1\} \geqslant 1 - \exp -\frac{N}{L} \tag{6.3}$$

$$G^2\{(\boldsymbol{\sigma},\boldsymbol{\sigma}');\ \sum_{s,t\leqslant d} (\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}'(t)}{N})^2 \leqslant 1\} \geqslant 1 - \exp -\frac{N}{L}. \tag{6.4}$$

PROPOSITION 6.2.— *Under (6.3), (6.4) we have*

$$E_g \frac{1}{Z^8} \leqslant L \exp -4\beta^2 d. \tag{6.5}$$

Proof. — We will prove that, if $u > 0$, then

$$P_g(\log Z \leqslant \frac{\beta^2 d}{2} - L(1+u)) \leqslant L \exp -u^2 \tag{6.6}$$

of which (6.5) is an immediate consequence. The proof of (6.6) builds upon the ideas of [T1, Section 2], and the reader might like to read first this simpler version of the same argument.

We view Z as a function $Z(\boldsymbol{g})$ of $\boldsymbol{g} = (g_i)_{i\leqslant N} \in \mathbb{R}^N$, where $\mathbb{R}^N$ is endowed with the canonical gaussian measure γ^N. The key to (6.6) is the following.

LEMME 6.3.— *There exists a subset A of $\mathbb{R}^N$ with $\gamma^N(A) \geqslant 1/L$ satisfying the following properties*

$$\boldsymbol{g} \in A \Rightarrow Z(\boldsymbol{g}) \geqslant \frac{1}{L} \exp \frac{\beta^2 d}{2} \tag{6.7}$$

$$\boldsymbol{g} \in A \Rightarrow \frac{1}{N} \sum_{i\leqslant N} \langle (\theta, \sigma_i) \rangle'^2 \leqslant L \tag{6.8}$$

where in (6.8) $\langle\cdot\rangle'$ is the Gibbs' measure on S^{N+1} as in (2.12), and where $\langle(\theta,\sigma_i)\rangle'$ is the integral for this measure of the map $(\sigma_1, \cdots, \sigma_N, \theta) \to (\theta, \sigma_i)$, the dot product of θ and σ_i in $\mathbb{R}^N$.

We will prove Lemma 6.3 later, and we prove (6.6) now. For $\boldsymbol{g}$ in A, $\boldsymbol{u}$ in $\mathbb{R}^N$, we write

$$\begin{aligned} Z(\boldsymbol{g}+\boldsymbol{u}) &= Av\langle \exp \frac{\beta}{\sqrt{N}} \boldsymbol{g}\cdot\theta(\boldsymbol{\sigma}) \exp \frac{\beta}{\sqrt{N}} \boldsymbol{u}\cdot\theta(\boldsymbol{\sigma})\rangle \\ &= Z(\boldsymbol{g})\langle \exp \frac{\beta}{\sqrt{N}} \boldsymbol{u}\cdot\theta(\boldsymbol{\sigma})\rangle' \end{aligned} \tag{6.9}$$

by definition of $\langle\cdot\rangle'$. Now, by Jensen's inequality

$$\begin{aligned} \langle \exp \frac{\beta}{\sqrt{N}} \boldsymbol{u}\cdot\theta(\boldsymbol{\sigma})\rangle' &\geqslant \exp \frac{\beta}{\sqrt{N}} \sum_{i\leqslant N} u_i\langle(\theta,\sigma_i)\rangle' \\ &\geqslant \exp -\beta L(\sum_{i\leqslant N} u_i^2)^{1/2} \end{aligned}$$

where we use Cauchy-Schwarz and (6.8) in the last inequality. Thus, using (6.7), (6.9) we get, for $\boldsymbol{g}\in A$ that

$$\log Z(\boldsymbol{g}+\boldsymbol{u}) \geqslant \frac{\beta^2 d}{2} - L(1+\|\boldsymbol{u}\|).$$

It follows that

$$\log Z(\boldsymbol{g}') < \frac{\beta^2 d}{2} - L(1+u) \Rightarrow \boldsymbol{g}' \notin A + uB$$

where B is the euclidean unit ball of $\mathbb{R}^N$. Thus (6.6) follows from the Gaussian isoperimetric inequality (as in [T1]).

Proof of Lemma 6.3. — We compute first

$$E_g Z = \langle Av \exp \frac{\beta^2}{2N} \sum_{i\leqslant N} (\theta,\sigma_i)^2\rangle.$$

First,

$$\begin{aligned} \frac{1}{N}\sum_{i\leqslant N}(\theta,\sigma_i)^2 &= \frac{1}{N}\sum_{i\leqslant N}(\sum_{s\leqslant d}\theta(s)\sigma_i(s))^2 \\ &= \sum_{s,t\leqslant d}\theta(s)\theta(t)\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N} \\ &= d + \sum_{s,t\leqslant d}\theta(s)\theta(t)(\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N} - \delta_{s,t}) \end{aligned} \tag{6.9}$$

because $\sum_{s\leqslant d}\theta^2(s) = d$ on the support of ν.

Thus

$$Av\frac{1}{N}\sum_{i\leqslant w}(\theta,\sigma_i)^2 \geqslant d-(Av(\sum_{s,t\leqslant d}\theta(s)\theta(t)a_{s,t})^2)^{1/2} \tag{6.10}$$

where $a_{s,t}=\dfrac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N}-\delta_{s,t}$. We note that $\sum a_{s,s}=0$; whenever $\sum a_{s,t}^2\leqslant 1$, it is proved in Proposition 5.6 that $Av_0(\sum_{s,t}\theta(s)\theta(t)a_{s,t})^2\leqslant L$, where Av_0 refers to integration in μ. This implies $Av(\sum_{s,t}\theta(s)\theta(t)a_{s,t})^2\leqslant L$, because for any positive function on S we have $Avf\leqslant 2Av_0 f$ if $M\leqslant 1/4$. We thus have shown that (provided $\beta\leqslant 1$)

$$E_g Z\geqslant\frac{1}{L}\exp\frac{\beta^2 d}{2}. \tag{6.11}$$

The next step is to find an upper bound for $E_g Z^2$. We have, using replicas

$$E_g Z^2=\langle Av\exp\frac{\beta^2}{2N}\Big[\sum_{i\leqslant N}(\theta,\sigma_i)^2+(\theta',\sigma_i')^2+2(\theta,\sigma_i)(\theta',\sigma_i')\Big]\rangle$$

where Av means integration of θ,θ' for ν and $\langle\ \cdot\ \rangle$ means integration of $\boldsymbol{\sigma},\boldsymbol{\sigma}'$ for G. As in (6.9), we see that the exponent is

$$\beta^2 d+\frac{\beta^2}{2}\sum_{s,t\leqslant d}(\theta(s)\theta(t)a_{s,t}+\theta'(s)\theta'(t)a'_{s,t}+2\theta(s)\theta'(t)a''_{s,t})$$

where

$$a_{s,t}=\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N}-\delta_{s,t},\ a'_{s,t}=\frac{\boldsymbol{\sigma}'(s)\cdot\boldsymbol{\sigma}'(t)}{N}-\delta_{s,t},\ a''_{s,t}=\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}'(t)}{N}. \tag{6.12}$$

Thus, by Hölder's inequality applied to Av,

$$E_g Z^2\leqslant\exp\beta^2 d\langle(UU'U'')^{1/3}\rangle$$

for

$$U=Av\exp\frac{3}{2}\beta^2\sum_{s,t\leqslant d}\theta(s)\theta(t)a_{s,t}$$

$$U'=Av\exp\frac{3}{2}\beta^2\sum_{s,t\leqslant d}\theta'(s)\theta'(t)a'_{s,t}$$

$$U''=Av\exp\frac{3}{2}\beta^2\sum_{s,t\leqslant d}\theta(s)\theta'(t)a''_{s,t}.$$

We appeal to (5.15), (5.16) (that remains true when averaging for ν rather than for μ) and we get, using (6.3) and (6.4),

$$E_g Z^2 \leqslant L \exp \beta^2 d. \tag{6.13}$$

Now, combining (6.11) and (6.13), we obtain

$$P_g(Z \geqslant \frac{1}{2} E_g Z) \geqslant \frac{1}{4} \frac{(E_g Z)^2}{E_g Z^2} \geqslant \frac{1}{L}. \tag{6.14}$$

This means that we have found a subset A_1 of $\mathbb{R}^N$ such that (6.7) holds for A_1, and $\gamma^N(A_1) \geqslant 1/L$. The set A of Lemma 6.3 will be a subset of A_1.

We observe that, using replicas, we have

$$\begin{aligned} \frac{1}{N} \sum_{i \leqslant N} \langle (\theta, \sigma_i) \rangle'^2 &= \langle \frac{1}{N} \sum_{i \leqslant N} (\theta, \sigma_i)(\theta', \sigma_i') \rangle' \\ &= U/Z^2 \end{aligned} \tag{6.15}$$

where

$$U = Av \langle \frac{1}{N} \sum_{i \leqslant N} (\theta, \sigma_i)(\theta', \sigma_i') \exp \frac{\beta}{\sqrt{N}} (\boldsymbol{g} \cdot \theta(\boldsymbol{\sigma}) + \boldsymbol{g} \cdot \theta'(\boldsymbol{\sigma}')) \rangle.$$

To complete the proof of Lemma 6.3 it suffices to prove that

$$E_g U \leqslant L \exp \beta^2 d. \tag{6.16}$$

Indeed, the set

$$A_2 = \{U \leqslant L' \exp \beta^2 d\}$$

satisfies $\gamma^N(A_2) \geqslant 1 - L/L'$ by Chebychev inequality, so $\gamma^N(A_1 \cap A_2) \geqslant 1/L$ for L' large enough; and, on $A = A_1 \cap A_2$, we bound U from above and Z from below, so that (6.15) implies (6.8). To prove (6.16), we compute $E_g U$ as

$$\exp \beta^2 d Av \langle \textstyle\sum_{s,t \leqslant d} \theta(s)\theta'(t) a''_{s,t}$$
$$\exp \tfrac{\beta^2}{2} (\textstyle\sum_{s,t \leqslant d} \theta(s)\theta(t) a_{s,t} + \theta'(s)\theta'(t) a'_{s,t} + 2\theta(s)\theta'(t) a''_{s,t}) \rangle$$

where $a_{s,t}, a'_{s,t}, a''_{s,t}$ are given by (6.12). To obtain (6.16), we simply use Hölder's inequality on Av as in the proof (6.13). This proves Lemma 6.3. □

We now start a series of lemmas that will culminate in the proof "that the system is in a pure state". These lemmas perform the necessary adaptation of the methods of Section 3.

The central quantity will be

$$C_N = E\langle \sum_{s \leqslant t} \Big(\frac{\tilde{\boldsymbol{\sigma}}(s) \cdot \boldsymbol{\sigma}^3(t)}{N}\Big)^2 \rangle \tag{6.17}$$

where $\tilde{\boldsymbol{\sigma}}(s) = \boldsymbol{\sigma}^1(s) - \boldsymbol{\sigma}^2(s)$. The vanishing of this quantity will express that the system is in a pure state.

Our next result is an adaptation of Proposition 3.1, of which we keep the notation (except for C_N).

PROPOSITION 6.4.— *If $\beta \leqslant 1/L$, then we have*

$$\begin{aligned} &\Big|E\frac{Av\langle f \prod_{\ell \leqslant 3} \mathcal{E}(\theta^\ell, \boldsymbol{\sigma}^\ell)\rangle}{Z^3} - E\frac{Av\langle f \prod_{\ell \leqslant 3} \mathcal{E}_0(\theta^\ell, \boldsymbol{\sigma}^\ell)\rangle}{Z^3}\Big| \\ &\leqslant L\beta^2 (E\langle (Avf^4)^{1/2}\rangle)^{1/2} C_N^{1/2}. \end{aligned} \tag{6.18}$$

Comment.— 1) We have gained a crucial factor d compared to (3.3).

2) It is essential to have $\langle (Avf^4)^{1/2}\rangle$ rather than $(Av\langle f^4\rangle)^{1/2}$ in the bounds.

Proof.— We consider the function $\varphi(u)$ as in the proof of Proposition 3.1, and we will show that for $0 \leqslant u \leqslant 1, |\varphi'(u)|$ is bounded by the right-hand side of (6.18); $\varphi'(u)$ is given by (3.14). Thus, using Cauchy-Schwarz in E_g

$$|\varphi'(u)| \leqslant \frac{3\beta^2}{N} E\big[(E_g \frac{1}{Z^8})^{1/2} (E_g T^2)^{1/2}\big] \tag{6.19}$$

where

$$T = Av\langle \sum_{\ell \leqslant 3} \frac{\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \cdot \theta^4(\boldsymbol{\sigma}^4)}{N} f \mathcal{E}(u) \mathcal{E}(\theta^4, \boldsymbol{\sigma}^4)\rangle.$$

To prove (6.18), using Proposition 6.2, it is enough to prove that if

$$T_\ell = Av\langle \frac{\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \cdot \theta^4(\boldsymbol{\sigma}^4)}{N} f \mathcal{E}(u) \mathcal{E}(\theta^4, \boldsymbol{\sigma}^4)\rangle$$

then

$$E((E_g T_\ell^2)^{1/2}) \leqslant L(E\langle (Avf^4)^{1/2}\rangle)^{1/2} C_N^{1/2} \exp 2d\beta^2. \tag{6.20}$$

We will use replicas to write T_ℓ^2 as a single average. Since in T_ℓ we have an order 4 replica, we need an order 8 replica. To simplify notation, we use the symbol $\wedge$ to mean that replicas of order $\ell \leqslant 4$ are replaced by replicas of order $\ell + 4$. Thus

$$T_\ell^2 = Av\langle \frac{\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell)\cdot\theta^4(\boldsymbol{\sigma}^4)}{N}\frac{\theta^{\ell+4}(\dot{\boldsymbol{\sigma}}^{\ell+4})\cdot\theta^8(\boldsymbol{\sigma}^8)}{N} f\hat{f}\mathcal{E}(u)\hat{\mathcal{E}}(u)\mathcal{E}(\theta^4,\boldsymbol{\sigma}^4)\mathcal{E}(\theta^8,\boldsymbol{\sigma}^8)\rangle$$

To prove (6.20) we will prove the following

$$(Av(E_g\mathcal{E}(u)\hat{\mathcal{E}}(u)\mathcal{E}(\theta^4,\boldsymbol{\sigma}^4)\mathcal{E}(\theta^8,\boldsymbol{\sigma}^8))^6)^{1/6} \leqslant L\exp 4d\beta^2 \tag{6.21}$$

$$(Av(\frac{\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell)\cdot\theta^4(\boldsymbol{\sigma}^4)}{N})^6)^{1/6} \leqslant L(\sum_{s,t\leqslant d}(\frac{\dot{\boldsymbol{\sigma}}^\ell(s)\cdot\boldsymbol{\sigma}^4(t)}{N})^2)^{1/2}. \tag{6.22}$$

Using (6.21), (6.22) and Hölder's inequality for Av, we get that

$$E_gT_\ell^2 \leqslant L\exp 4d\beta^2\langle U\hat{U}(Avf^4)^{1/4}(Av\hat{f}^4)^{1/4}\rangle \tag{6.23}$$

where

$$U = (\sum_{s,t\leqslant d}(\frac{\dot{\boldsymbol{\sigma}}^\ell(s)\cdot\boldsymbol{\sigma}^4(t)}{N})^2)^{1/2}.$$

Thus, by (6.23) and independence

$$\begin{aligned} E_gT_\ell^2 &\leqslant L\exp Ld\beta^2\langle U(Avf^4)^{1/4}\rangle^2 \\ &\leqslant L\exp 4d\beta^2\langle U^2\rangle\langle (Avf^4)^{1/2}\rangle^2 \end{aligned}$$

using Cauchy Schwarz. To prove (6.20), it suffices that $E\langle U^2\rangle^{1/2} \leqslant C_N^{1/2}$; but in fact using Jensen's inequality we have $E\langle U^2\rangle \leqslant C_N$.

We now prove (6.21). We have

$$\begin{aligned} &E_g\mathcal{E}(u)\hat{\mathcal{E}}(u)\mathcal{E}(\theta^4,\boldsymbol{\sigma}^4)\mathcal{E}(\theta^8,\boldsymbol{\sigma}^8) \\ &= \exp\frac{\beta^2}{2N}[\sum_{\ell\leqslant 8}\|\theta^\ell(\boldsymbol{\sigma}^\ell)\|^2 + 2\sum_{\ell,\ell'\leqslant 8}\boldsymbol{A}_\ell\cdot\boldsymbol{A}_{\ell'}] \end{aligned} \tag{6.24}$$

where

$$\begin{aligned} \boldsymbol{A}_\ell = \theta^\ell(\boldsymbol{b}) + u\theta^\ell(\dot{\boldsymbol{\sigma}}^\ell) \quad &\text{for} \quad \ell \neq 4,8 \\ \boldsymbol{A}_\ell = \theta^\ell(\boldsymbol{\sigma}^\ell) \quad &\text{for} \quad \ell = 4,8. \end{aligned}$$

We use as before that

$$\|\theta^\ell(\boldsymbol{\sigma}^\ell)\|^2 = d + \sum_{s,t\leqslant d}\theta^\ell(s)\theta^\ell(t)(\frac{\boldsymbol{\sigma}^\ell(s)\cdot\boldsymbol{\sigma}^\ell(t)}{N} - \delta_{s,t}).$$

To prove (6.21), we simply use Hölder's inequality for Av, and we use Proposition 5.5, and (6.3), (6.4) to control quantities such as $\sum_{s,t}(N^{-1}\boldsymbol{b}(s)\cdot\boldsymbol{\sigma}^{\ell}(t))^2$, etc. It remains only to prove (6.22); but this is weaker than Proposition 5.5. □

We now consider the quantity

$$D_N = E\langle\sum_{s,t\leqslant d}(\frac{\boldsymbol{\sigma}^1(s)\cdot\boldsymbol{\sigma}^1(t)}{N}-\frac{\boldsymbol{\sigma}^2(s)\cdot\boldsymbol{\sigma}^2(t)}{N})^2\rangle. \tag{6.25}$$

PROPOSITION 6.5.— *If $\beta\leqslant 1/L$, we have*

$$\begin{aligned}&|E\frac{Av\langle f\prod_{\ell\leqslant 3}\mathcal{E}(\theta^\ell,\sigma^\ell)\rangle}{Z^3}-E\frac{Av\langle f\rangle\prod_{\ell\leqslant 3}\mathcal{E}_1(\theta^\ell)}{Z^3}| \\ &\leqslant L\beta^2E\langle(Avf^4)^{1/2}\rangle(C_N^{1/2}+D_N^{1/2}).\end{aligned} \tag{6.26}$$

Proof. — The method to gain a factor d comparing to (3.19) is similar (but simpler) to what we did in Proposition 6.4, so we will simply explain the new feature, the occurrence of D_N. It occurs through the following inequality (that corresponds to (6.22))

$$\begin{aligned}&Av((\frac{\|\theta(\boldsymbol{\sigma})\|^2}{N}-\frac{\langle\|\theta(\boldsymbol{\sigma})\|^2\rangle}{N})^8)^{1/8} \\ &\leqslant L(\sum_{s,t\leqslant d}(\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N}-\frac{\langle\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\rangle}{N})^2)^{1/2}\end{aligned}$$

and the fact that

$$E\langle(\sum_{s,t\leqslant d}(\frac{\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)}{N}-\frac{\langle\boldsymbol{\sigma}(s)\cdot\boldsymbol{\sigma}(t)\rangle}{N})^2)^{1/2}\rangle\leqslant D_N^{1/2}. \qquad \square$$

PROPOSITION 6.6.— *If $\beta\leqslant 1/L$, we have*

$$\lim_{N\to\infty} C_N = 0 \tag{6.27}$$

$$\lim_{N\to\infty} D_N = 0. \tag{6.28}$$

Proof. — We will prove that

$$C_{N+1}\leqslant L\beta^2(C_N+D_N)+\frac{Ld^4}{N} \tag{6.29}$$

$$D_{N+1}\leqslant L\beta^2(C_N+D_N)+\frac{Ld^4}{N}. \tag{6.30}$$

To prove (6.29), the formula that corresponds to (3.24) is now

$$C_{N+1} \leqslant \frac{Ld^4}{N} + E\frac{1}{Z^3}Av\langle f \prod_{\ell \leqslant 3} \mathcal{E}(\theta^\ell, \boldsymbol{\sigma}^\ell)\rangle \tag{6.31}$$

where

$$f = \sum_{s,t \leqslant d} \tilde{\theta}(s)\theta^3(t)\frac{\tilde{\boldsymbol{\sigma}}(s)\cdot \boldsymbol{\sigma}^3(t)}{N} = \frac{\tilde{\theta}(\tilde{\boldsymbol{\sigma}})\cdot \theta^3(\boldsymbol{\sigma}^3)}{N}. \tag{6.32}$$

Since $\langle f \rangle = 0$, (6.26) shows that

$$C_{N+1} \leqslant \frac{Ld^4}{N} + L\beta^2 E\langle (Avf^4)^{1/2}\rangle (C_N^{1/2} + D_N^{1/2})$$

so that (6.29) follows from the fact that

$$(Avf^4)^{1/4} \leqslant L(\sum_{s,t \leqslant d}(\frac{\tilde{\boldsymbol{\sigma}}(s)\cdot \boldsymbol{\sigma}^3(t)}{N})^2)^{1/2}$$

(of the same nature than (6.22)). To prove (6.30), one proceeds similarly, except that now

$$f = \sum_{s,t \leqslant d} (\theta^1(s)\theta^1(t) - \theta^2(s)\theta^2(t))(\frac{\boldsymbol{\sigma}^1(s)\cdot \boldsymbol{\sigma}^1(t)}{N} - \frac{\boldsymbol{\sigma}^2(s)\cdot \boldsymbol{\sigma}^2(t)}{N}). \qquad \square$$

As in Section 4, one shows that (4.2) holds.

We consider the parameters

$$A_N = \sum_{s,t \leqslant d} \mathrm{Var}\langle \frac{\boldsymbol{\sigma}^1(s)\cdot \boldsymbol{\sigma}^2(t)}{N}\rangle$$

$$B_N = \sum_{s,t \leqslant d} \mathrm{Var}\langle \frac{\boldsymbol{\sigma}(s)\cdot \boldsymbol{\sigma}(t)}{N}\rangle$$

and we will prove the following.

PROPOSITION 6.7.— *If $\beta \leqslant 1/L$, then we have*

$$\lim_{N\to\infty} A_N = 0 \tag{6.33}$$

$$\lim_{N\to\infty} B_N = 0. \tag{6.34}$$

Proof. — We will prove that

$$A_{N+1} \leqslant L\beta^2(A_N + B_N) + o(1) \qquad (6.35)$$
$$B_{N+1} \leqslant L\beta^2(A_N + B_N) + o(1).$$

To prove (6.35), we write

$$A_N = A'_N - A''_N$$

where

$$A'_N = \sum_{s,t \leqslant d} E\langle \frac{\boldsymbol{\sigma}^1(s) \cdot \boldsymbol{\sigma}^2(t)}{N} \rangle^2$$
$$A''_N = \sum_{s,t \leqslant d} (E\langle \frac{\boldsymbol{\sigma}^1(s) \cdot \boldsymbol{\sigma}^2(t)}{N} \rangle)^2.$$

The argument is a bit more delicate than the one of Proposition 4.1. We write

$$A'_N = E \sum_{s,t \leqslant d} \langle \frac{\boldsymbol{\sigma}^1(s) \cdot \boldsymbol{\sigma}^2(t)}{N} \frac{\boldsymbol{\sigma}^3(s) \cdot \boldsymbol{\sigma}^4(t)}{N} \rangle$$

so that, by symmetry among the sites,

$$\begin{aligned} A'_{N+1} &= E \sum_{s,t \leqslant d} \langle \theta^1(s)\theta^2(t) \frac{\boldsymbol{\sigma}^3(s) \cdot \boldsymbol{\sigma}^4(t)}{N} \rangle' + o(1) \qquad (6.36) \\ &= E \frac{Av\langle \theta^1(\boldsymbol{\sigma}^3) \cdot \theta^2(\boldsymbol{\sigma}^4)\rangle \mathcal{E}_1(\theta^1)\mathcal{E}_1(\theta^2)}{N Av \mathcal{E}_1(\theta^1)\mathcal{E}_1(\theta^2)} + o(1) \end{aligned}$$

using (4.2). Next, we write

$$\begin{aligned} A''_{N+1} &= \sum_{s,t \leqslant d} E \frac{\langle \boldsymbol{\sigma}^3(s) \cdot \boldsymbol{\sigma}^4(t) \rangle}{N} E \frac{Av \theta^1(s)\theta^2(t)\mathcal{E}_1(\theta^1)\mathcal{E}_1(\theta^2)}{(Av\mathcal{E}_1(\theta))^2} + o(1) \\ &= E \frac{Av\langle \theta^1(\boldsymbol{\sigma}^3) \cdot \theta^2(\boldsymbol{\sigma}^4)\rangle \tilde{\mathcal{E}}_1(\theta^1)\tilde{\mathcal{E}}_1(\theta^2)}{N(Av\tilde{\mathcal{E}}_1(\theta))^2} + o(1) \end{aligned}$$

where

$$\tilde{\mathcal{E}}_1(\theta^1) = \exp \frac{\beta}{\sqrt{N}} g \cdot \theta^1(\tilde{\boldsymbol{b}}) + \frac{\beta^2}{2N}((\langle \|\theta^1(\boldsymbol{\sigma})\|^2 \rangle^{\sim} - \|\theta^1(\tilde{\boldsymbol{b}})\|^2)$$

and where the ˜ indicates an independent realization of the disorder. Thus

$$A_{N+1} = E(AvU(\theta^1, \theta^2)V(\theta^1, \theta^2)) + o(1) \qquad (6.37)$$

where

$$U(\theta^1,\theta^2) = \langle \theta^1(\boldsymbol{\sigma}^3)\cdot\theta^2(\boldsymbol{\sigma}^4)\rangle - E\langle \theta^1(\boldsymbol{\sigma}^3)\cdot\theta^2(\boldsymbol{\sigma}^4)\rangle$$
$$V(\theta^1,\theta^2) = E_g\frac{\mathcal{E}_1(\theta^1)\mathcal{E}_1(\theta^2)}{(Av\mathcal{E}_1(\theta))^2} - E_g\frac{\tilde{\mathcal{E}}_1(\theta^1)\tilde{\mathcal{E}}_1(\theta^2)}{(Av\tilde{\mathcal{E}}_1(\theta))^2}$$

(we use here that $EV(\theta^1,\theta^2)=0$). Thus from (6.37) we have

$$A_{N+1} \leqslant (EAvU(\theta^1,\theta^2)^2)^{1/2}(E(AvV(\theta^1,\theta^2)^2)^{1/2} + o(1).$$

To prove (6.35), we show that

$$EAvU(\theta^1,\theta^2)^2 \leqslant LA_N \tag{6.38}$$
$$EAvV(\theta^1,\theta^2)^2 \leqslant L(A_N+B_N). \tag{6.39}$$

To prove (6.38) we write

$$U(\theta^1,\theta^2) = \sum_{s,t}\theta^1(s)\theta^2(t)\Big(\frac{\langle\boldsymbol{\sigma}^3(s)\cdot\boldsymbol{\sigma}^4(t)\rangle}{N} - E\frac{\langle\boldsymbol{\sigma}^3(s)\cdot\boldsymbol{\sigma}^4(t)\rangle}{N}\Big)$$

and we use the fact that $Av(\sum\limits_{s,t}\theta^1(s)\theta^2(t)a_{s,t})^2 \leqslant L\sum\limits_{s,t}a_{s,t}^2$.

To prove (6.39), we consider $\mathcal{E}_0(u,\theta)$ as in (4.15), and

$$\varphi(u,\theta^1,\theta^2) = E_{g,g'}\frac{\mathcal{E}_0(u,\theta^1)\mathcal{E}_0(u,\theta^2)}{(Av\mathcal{E}_0(u,\theta))^2}.$$

To prove (6.39), one has to show that for $0\leqslant u\leqslant 1$,

$$EAv\varphi'(u,\theta^1,\theta^2)^2 \leqslant L(A_N+B_N)$$

that requires a few lines of computation, but no imagination. The proof of (6.36) is entirely similar.

To prove that the equations (2.9), (2.10) have a unique solution, we prove a result similar to (4.20) without the factor d^2, using now the distance

$$\Delta(Q,Q') = \Big(\sum_{s,t}(Q(e_s,e_t)-Q'(e_s,e_t))^2\Big)^{1/2}$$

where $(e_s)_{s\leqslant d}$ is an orthonormal basis of $\mathbb{R}^d$.

Bibliography

[A-L-R] AIZENMAN (M.), LEBOWITZ (J.L.), RUELLE (D.). — *Some rigorous results on the Sherrington-Kirkpatrick model*, Commun Math. Phys., 112 (1987), 3–20.

[C-N] COMETS (F.), NEVEU (J.). — *Sherrington-Kirkpatrick model of spin glasses and stochastic calculus: the high temperature case*, Commun Math. Phys., 166 (1995), 549–564.

[F-Z] FRÖHLICH (J.), ZEGARLINSKI (B.). — *Some comments on the Sherrington-Kirkpatrick model for spin glasses*, Commun Math. Phys., 112 (1987), 553–556.

[T1] TALAGRAND (M.). — *The Sherrington-Kirkpatrick model: a challenge to mathematicians*, Prob. Th. Relat. Fields, 110 (1998), 109–176.

[T2] TALAGRAND (M.). — *Exponential inequalities and replica-symmetry breaking for the Sherrington-Kirkpatrick model*, Ann. Probab., to appear.

[T3] TALAGRAND (M.). — *Exponential inequalities and convergence of moments in the replica-symmetric regime of the Hopfield model*, Manuscript, 1998.

[T4] TALAGRAND (M.). — *Huge random structures and mean field models for spin glasses*, Proceeding of the international congress of Mathematicians, Berlin, 1998. Doc. Math., Extra Vol. I, 1998, 507–536.