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# Cauchy Problem for Hyperbolic Systems in Gevrey Class. A note on Gevrey Indices (*)

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*Dedicated to Professor Kunihiko KAJITANI*
*for his sixtieth birthday*

RÉSUMÉ. — Dans cette article nous considérons une $4 \times 4$ matrice $L$ d'opérateurs différentiels hyperboliques d'ordre 1 et nous déterminons les classes de Gevrey dans lesquelles le problème de Cauchy est bien posé pour $L$. Les résultats entraînent que la multiplicité maximale des zéros du polynôme minimal de la partie principale ne donne pas, en général, l'indice de Gevrey pour lequel le problème de Cauchy est bien posé.

ABSTRACT. — In the present paper we determine completely the Gevrey indices for the well-posedness of the Cauchy problem to a certain first order differential hyperbolic $4 \times 4$ systems. This leads us to the fact that the maximal multiplicity of zeros of the minimal polynomial of the principal part does not give, in general, the appropriate index for the Gevrey well-posedness.

## 1. Introduction

Cauchy problem for hyperbolic equations with multiple characteristic roots is not in general well-posed in the space of $C^\infty$-functions. If the multiplicities are constant, Ohya [1] and Leray-Ohya [2] found that the Cauchy

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problem is well-posed for any lower order terms in appropriate Gevrey classes. They showed that the indices of Gevrey classes are determined by the multiplicities of the characteristic roots. Following these works, many papers contributed in this field. Among them, Bronstein [3] studied hyperbolic equations without the assumption of the constant multiplicities of the characteristic roots and showed that the Cauchy problem is well-posed for any lower terms in Gevrey classes whose indices are determined by the maximum of their multiplicities. Kajitani [4], Nishitani [5] and Mizohata [6] developed the study of these problems from another view points.

In the case of hyperbolic systems, the above theorem are also true. However those results are not satisfactory for hyperbolic systems. The appropriate indices could not be determined only by the multiplicities of the characteristic roots in this case. Once the author gave a conjecture that the indices of Gevrey classes, in which the Cauchy problem is well-posed, are determined instead by the multiplicities of zeros of the minimal polynomial of the principal symbol. We know that this is true provided that the multiplicities of the characteristic roots are constant (Yamahara [7]). Here we must remark that Vaillant [8], [9] studies intensively the relations between the indices of Gevrey classes and the Levi conditions, and he explained completely the relations when the multiplicities are at most 5.

If we drop this assumption of constant multiplicities, the situation is in fact much more complicated. In this note, we will give an example of the $4 \times 4$-hyperbolic system which shows that, besides multiplicities of the characteristic roots, the degeneracy of the Jordan normal form of the principal part determine the appropriate Gevrey indices.

We study the following Cauchy problem:

$$\text{(C.P.)} \qquad \begin{cases} L[u] \equiv (I_4 D_t + A(t) D_x + B(t)) u(t,x) = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

in $\Omega = [0,T] \times \mathbf{R}^1_x$, where $I_4$ denotes the unit matrix of order 4 and

$$A(t) = \left(\begin{array}{cc|cc} \lambda(t) & 1 & 0 & 0 \\ 0 & \lambda(t) & a(t) & 0 \\ \hline 0 & 0 & \mu(t) & 1 \\ 0 & 0 & 0 & \mu(t) \end{array}\right) \tag{2}$$

Here we assume that $\lambda(t), \mu(t)$ and $a(t)$ are real and smooth functions satisfying following (A. 1) and (A. 2).

$$\begin{aligned} &\text{(A.1)} \qquad \lambda(t) - \mu(t) = \nu t^p(1 + o(1)), \quad \nu \neq 0 \\ &\text{(A.2)} \qquad a(t) = at^q(1 + o(1)), \quad a \neq 0 \end{aligned} \tag{3}$$

as $t \to 0$ , where $p$ and $q$ are positive integers.

We are concerned with the Gevrey-wellposedness in (C.P.), more precisely we shall study how $p$ and $q$ determine the index of the Gevrey class in which (C.P.) is wellposed.

As we assumed that the coefficients depend only on time variable, we study (C.P.) by Fourier transform w.r.t. space variable.

$$\text{(C.P.)}' \qquad \begin{cases} \hat{L}[\hat{u}] \equiv (I_4 D_t + A(t)\xi + B(t))\hat{u}(t,\xi) = 0 \\ \hat{u}(0,x) = \hat{u}_0(\xi) \end{cases} \tag{4}$$

We define the Gevrey well-posedness of (C.P.) in the following way which is equivalent to the classical definition.

DEFINITION. — *Let $s$ be a positive number. We say that the Cauchy problem (C.P.) is $\gamma^{(s)}$-wellposed if for any lower order term $B(t)$, there exists a constant $T > 0$ and (C.P.)' has the solution which satisfies the following inequality*

$$|\hat{u}(t,\xi)| \leqslant C exp(\delta|\xi|^{\frac{1}{s}}t)|\hat{u}_0(\xi)|$$

*for $0 \leqslant t \leqslant T$ , where $C$ and $\delta$ are positive constants independent of the initial data $\hat{u}_0(\xi)$ .*

Our result is as follows.

THEOREM 1. — *Under the assumptions (A.1) and (A.2), (C.P.) is $\gamma^{(s)}$-wellposed for any $s$ satisfying*

$$\text{(1)} \qquad s \leqslant 2 \qquad \text{(when} \quad 2p \leqslant q)$$

$$\text{(2)} \qquad s \leqslant \frac{4p-q}{3p-q} \qquad \text{(when} \quad 2p > q)$$

THEOREM 2. — *Under the assumptions (A.1) and (A.2), (C.P.) is not $\gamma^{(s)}$-wellposed for any $s(> 2)$, and moreover for any $s$ which satisfies*

$$s > \frac{4p-q}{3p-q} \qquad \text{(when} \quad 2p > q)$$

## 2. Proof of Theorem 1

In the Cauchy problem (C.P.)', set $\xi = n$ and without loss of generality we regard it as a positive large parameter. Now we start with the ordinary differential equations:

$$D_t v + nA(t)v + B(t)v = 0 \tag{5}$$

with the Cauchy data $\hat{u}(0,n) = v_0(n)$ which satisfies

$$|v_0(n)| \leqslant \text{const.}\exp(-cn^{\frac{1}{s}})\ , \tag{6}$$

where $c$ is a positive constant. At first we shall study (5) in the interval $0 \leqslant t \leqslant n^{-\sigma}$, where $\sigma$ is a positive constant which will be determined later.

Let us denote a matrix of weight:

$$W = W(n;\epsilon_1,\epsilon_2) = \begin{pmatrix} 1 & & & \\ & n^{-\epsilon_1} & & \\ & & n^{-\epsilon_2} & \\ & & & n^{-\epsilon_1-\epsilon_2} \end{pmatrix}, \tag{7}$$

where $\epsilon_1$ and $\epsilon_2$ are non-negative constants which will be also determined later.

By the change of the unknowns $v$ such that $v = Wv_1$, (5) turns out to

$$D_t v_1 + n\begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{pmatrix} v_1 + nW^{-1}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) Wv_1 + W^{-1}BWv_1 = 0 \tag{8}$$

In order to make the crutial terms in (8) smallest (with a viewpoint of the order of $n$), $\epsilon_1$ and $\epsilon_2$ will be taken in such a way that $1-\epsilon_1 = \epsilon_1+\epsilon_2 = 1 - q\sigma + \epsilon_1 - \epsilon_2$, hence

$$\begin{cases} \epsilon_1 = \dfrac{1}{4} + \dfrac{q}{4}\sigma \\[2ex] \epsilon_2 = \dfrac{1}{2} - \dfrac{q}{2}\sigma \end{cases} \tag{9}$$

We shall study our problem with two cases

(case 1) $\qquad \sigma \geqslant \dfrac{1}{q}$

(case 2) $\sigma < \frac{1}{q}$

In case 1 regardless of (9) we can take $\epsilon_1 = \frac{1}{2}$, and $\epsilon_2 = 0$, but in case 2 $\epsilon_1$ and $\epsilon_2$ are determined as in (9). Then we obtain following inequalitiy in each case.

PROPOSITION 1. — *For any solution $v(t) = v(t;n)$ of (5) with the Cauchy data satisfying (6), following inequalities hold on the interval* $0 \leqslant t \leqslant n^{-\sigma}$

$$\begin{aligned} |v(t)|^2 &\leqslant \text{const.}\, n^M \exp(\delta n^{\frac{1}{2}} t - 2cn^{\frac{1}{s}}) \quad , \quad \text{in case 1,} \\ |v(t)|^2 &\leqslant \text{const.}\, n^M \exp(\delta n^{\frac{3}{4}-\frac{q}{4}\sigma} t - 2cn^{\frac{1}{s}}) \quad , \quad \text{in case 2.} \end{aligned} \tag{10}$$

*Next we shall evaluate $v(t)$ when $t \geqslant n^{-\sigma}$. For this, at first we construct a matrix $N(t) = N(t;n)$ satisfying*

$$N(t)^{-1}A(t)N(t) = A_1(t) \equiv \left(\begin{array}{cc|cc} \lambda(t) & 1 & 0 & 0 \\ 0 & \lambda(t) & 0 & 0 \\ \hline 0 & 0 & \mu(t) & 1 \\ 0 & 0 & 0 & \mu(t) \end{array}\right)$$

*Actually $N(t)$ is obtained such that*

$$N(t) = \begin{pmatrix} I_2 & N_{12} \\ 0 & I_2 \end{pmatrix} \quad , \quad N_{12} = a(t) \begin{pmatrix} \frac{1}{(\lambda-\mu)^2} & \frac{2}{(\lambda-\mu)^3} \\ \frac{-1}{\lambda-\mu} & \frac{-1}{(\lambda-\mu)^2} \end{pmatrix}.$$

*By changing the unknowns $v(t)$ in such a way that $v(t) = N(t)v_1$ and $v_1(t) = W_1 v_2$, where $W_1$ is the same type with (7) and $\epsilon_1, \epsilon_2$ will be determined later, (5) turns out to*

$$D_t v_2 + n \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{pmatrix} v_2 + nW_1^{-1} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$W_1 v_2 + W_1^{-1}(N^{-1}BN + N^{-1}(D_t N))W_1 v_2 = 0. \tag{11}$$

We study each term carefully. First,

$$nW_1^{-1} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) W_1 = n^{1-\epsilon_1} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{12}$$

Second to evaluate $W_1^{-1}N^{-1}BNW_1$, we denote $B$ blockwisely such that

$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ , where each $B_{ij}$ is a $2 \times 2$-matrix. Then

$$N^{-1}BN = \begin{pmatrix} B_{11} - N_{12}B_{21} & B_{11}N_{12} - N_{12}B_{21}N_{12} - N_{12}B_{22} \\ B_{21} & B_{21}N_{12} + B_{22} \end{pmatrix} \tag{13}$$

Moreover when we denote $B_{ij}$ precisely by $B_{ij} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, we see that

$$N_{12}B_{ij} = a(t) \begin{pmatrix} \frac{b_{11}}{(\lambda-\mu)^2} + \frac{b_{21}}{(\lambda-\mu)^3} & \frac{b_{12}}{(\lambda-\mu)^2} + \frac{2b_{22}}{(\lambda-\mu)^3} \\ \frac{-b_{11}}{\lambda-\mu} - \frac{b_{21}}{(\lambda-\mu)^2} & \frac{-b_{12}}{\lambda-\mu} - \frac{b_{22}}{(\lambda-\mu)^2} \end{pmatrix},$$

$$B_{ij}N_{12} = a(t) \begin{pmatrix} \frac{b_{11}}{(\lambda-\mu)^2} - \frac{b_{12}}{\lambda-\mu} & \frac{2b_{11}}{(\lambda-\mu)^3} - \frac{b_{12}}{(\lambda-\mu)^2} \\ \frac{b_{21}}{(\lambda-\mu)^2} - \frac{b_{22}}{\lambda-\mu} & \frac{2b_{21}}{(\lambda-\mu)^3} - \frac{b_{22}}{(\lambda-\mu)^2} \end{pmatrix},$$

$$N_{12}B_{ij}N_{12} = a(t)^2 \begin{pmatrix} \frac{1}{(\lambda-\mu)^2} & \frac{2}{(\lambda-\mu)^3} \\ \frac{-1}{\lambda-\mu} & \frac{-1}{(\lambda-\mu)^2} \end{pmatrix} \begin{pmatrix} \frac{b_{11}}{(\lambda-\mu)^2} - \frac{b_{12}}{\lambda-\mu} & \frac{2b_{11}}{(\lambda-\mu)^3} - \frac{b_{12}}{(\lambda-\mu)^2} \\ \frac{b_{21}}{(\lambda-\mu)^2} - \frac{b_{22}}{\lambda-\mu} & \frac{2b_{21}}{(\lambda-\mu)^3} - \frac{b_{22}}{(\lambda-\mu)^2} \end{pmatrix}.$$

Third in the term of $W_1^{-1}N^{-1}(D_tN)W_1$ we remark that $N^{-1}(D_tN) = \begin{pmatrix} 0 & D_tN_{12} \\ 0 & 0 \end{pmatrix}$.

Taking account of the above structures we can determine the orders of all components w. r. t. $n$. In (12) terms are majorized by $n^{1-\epsilon_1}$ . Terms in (13) are estimated as follows:

In (1,1)-block and in (2,2)-block, they are majorized by

$$n^{\epsilon_1} \quad , \quad t^{q-2p}n^{\epsilon_1} \quad , \quad t^{q-3p} \quad ;$$

in (1,2)-block, by

$$t^{q-2p}n^{\epsilon_1-\epsilon_2} \quad , \quad t^{q-3p}n^{-\epsilon_2} \quad ,$$

$$t^{2q-4p}n^{\epsilon_1-\epsilon_2} \quad , \quad t^{2q-5p}n^{-\epsilon_2} \quad , \quad t^{2q-6p}n^{-\epsilon_1-\epsilon_2} \quad ;$$

in (2,1)-block, by

$$n^{\epsilon_1+\epsilon_2} \quad .$$

Finally, the terms of $W_1^{-1}N^{-1}(D_tN)W_1$ , by

$$t^{q-p-1}n^{\epsilon_1-\epsilon_2} \quad , \quad t^{q-2p-1}n^{-\epsilon_2} \quad , \quad t^{q-3p-1}n^{-\epsilon_1-\epsilon_2} \quad .$$

We shall take $\epsilon_1$ and $\epsilon_2$ in $W_1$ so that $\epsilon_1 > 0$ and $\epsilon_2 \geqslant 0$. Hence we expect that the terms dominated just by $n^{1-\epsilon_1}$ or $n^{\epsilon_1+\epsilon_2}$ are the top terms, which means all the other terms are majorized by these terms when $n^{-\sigma} \leqslant t$. For this purpose we must divide our argument in two cases.

In the case $2p \leqslant q$ :

We take $\epsilon_1$ and $\epsilon_2$ so that $\epsilon_1 = \frac{1}{2}, \epsilon_2 = 0$ , and in order that all the other terms are majorized by $n^{\frac{1}{2}}$, $\sigma$ has to satisfy following conditions ;

$$(3p-q)\sigma \leqslant \frac{1}{2}\,, \quad (6p-2q)\sigma - \frac{1}{2} \leqslant \frac{1}{2}\,, \quad (3p-q+1)\sigma - \frac{1}{2} \leqslant \frac{1}{2}\,.$$

In conclusion we obtain the following

PROPOSITION 2.— *Assume that* $2p \leqslant q$ *and that*

$$\sigma \leqslant \min\{\frac{1}{6p-2q}\,, \quad \frac{1}{3p-q+1}\}\,.$$

*Then for any soluton* $v(t) = v(t;n)$ *of (5), following inequality holds when* $t \geqslant n^{-\sigma}$

$$|v(t)|^2 \leqslant \text{const.}\, n^M \exp(\delta_1 n^{\frac{1}{2}}(t-n^{-\sigma}))|v(n^{-\sigma})|^2\,, \tag{14}$$

*where* $M$ *and* $\delta_1$ *are positive constants independent of* $n$.

Combining (14) with the first inequality in (10), we can see that

$$|v(t)|^2 \leqslant \text{const.}\, n^M \exp(\delta_1 n^{\frac{1}{2}}(t-n^{-\sigma}) + \delta n^{\frac{1}{2}-\sigma} - 2cn^{\frac{1}{s}})\,, \tag{15}$$

when $t \geqslant n^{-\sigma}$. Since that $\sigma > 0$ and $s \leqslant 2$ , (15) leads us to

$$|v(t)|^2 \leqslant \text{const.}\exp(\delta_1' n^{\frac{1}{2}} t - c' n^{\frac{1}{s}}) \tag{16}$$

for $t \geqslant 0$. Here we enphasize that $\sigma$ must be taken under the condition

$$\frac{1}{q} \leqslant \sigma \leqslant min\{\frac{1}{6p-2q}\,, \frac{1}{3p-q+1}\}\,, \tag{17}$$

and remark that in the above inequality $1/(6p-2q)$, $1/(3p-q+1)$ has no meaning when $6p-2q \leqslant 0$, $3p-q+1 \leqslant 0$ respectively. Thus we established the first part of Theorem 1.

Next we continue the proof of Theorem 1 in the case $2p > q$ :

We take $\epsilon_1$ and $\epsilon_2$ in $W_1$ so that

$$1-\epsilon_1 = \epsilon_1 + \epsilon_2 = (3p-q)\sigma\,.$$

Here we remark that $\epsilon_2 \geqslant 0$ means that

$$\sigma \geqslant \frac{1}{2(3p-q)} .$$

Like the previous argument, in order that all the other terms are majorized by $n^{(3p-q)\sigma}$, $\sigma$ has to satisfy following conditions:

$$(4p-q)\sigma \geqslant 1 , \quad (11p-3q-1)\sigma \geqslant 2 , \quad (7p-2q+1)\sigma \geqslant 1 .$$

In conclusion we obtain the following

PROPOSITION 3.— *Assume that $2p > q$ and that*

$$\sigma \geqslant \frac{1}{4p-q} . \tag{18}$$

*Then for any solution $v(t) = v(t;n)$ of (5), following inequality holds when $t \geqslant n^{-\sigma}$*

$$|v(t)|^2 \leqslant \text{const.}\, n^M \exp(\delta_1 n^{(3p-q)\sigma}(t-n^{-\sigma}))|v(n^{-\sigma})|^2 . \tag{19}$$

Combining with the second inequality in (10) we can see that

$$|v(t)|^2 \leqslant \text{const.}\, n^M \exp(\delta_1 n^{(3p-q)\sigma}(t-n^{-\sigma}) + \delta n^{\frac{3}{4}-\frac{q}{4}\sigma-\sigma} - 2cn^{\frac{1}{s}}) , \tag{20}$$

when $t \geqslant n^{-\sigma}$. On account of (18), (20) leads us to

$$|v(t)|^2 \leqslant \text{const.} \exp(\delta_1' n^{(3p-q)\sigma} t - c' n^{\frac{1}{s}}) \tag{21}$$

for any $t \geqslant 0$. This means that for any $s$ satisfying

$$(3p-q)\sigma \leqslant \frac{1}{s} \tag{22}$$

the Cauchy problem to (5) with the Cauchy data which satisfies (6) is $\gamma^{(s)}$-wellposed.

Here we recall that the energy inequality (21) is obtained when $2p > q$ and when $\sigma$ satisties the following inequalities

$$\frac{1}{2(3p-q)} \leqslant \sigma < \frac{1}{q} ,$$

$$\frac{1}{4p-q} \leqslant \sigma ,$$

$$\sigma \leqslant \frac{1}{s(3p-q)} .$$

Thus we established the second part of Theorem 1.

## 3. Proof of Theorem 2

The proof of the first part of Theorem 2 is similar to the second part and is rather easy. So we shall only prove the second one of the theorem. To prove this we will find the lower order term $B(t)$ and the initial data $v_0(x)$ which will cause the ill-posedness in the class $\gamma^{(s)}$ when $s > (4p-q)/(3p-q)$ and when $2p > q$.

As in §2 we also denote that $\xi = n$ and we regard it as a positive large parameter in the Cauchy problem (C.P.)'. More precisely we start with the following equations:

$$D_t v + n \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \mu & \\ & & & \mu \end{pmatrix} v + n \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) v + \begin{pmatrix} & & & \\ & & \mathbf{0} & \\ & & & \\ b & & & \end{pmatrix} v = 0 \tag{23}$$

Remark that we take $B(t)$ as above whose $(4,1)$-element $b$ is a non-zero constant which will be determined later.

Let $\sigma$ be a positive constant, then by the asymptotic transformation $t = n^{-\sigma}\tau$ , (23) changes to

$$n^{\sigma} D_{\tau}\tilde{v} + n \begin{pmatrix} \tilde{\lambda} & & & \\ & \tilde{\lambda} & & \\ & & \tilde{\mu} & \\ & & & \tilde{\mu} \end{pmatrix} \tilde{v} + n \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{a} & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \tilde{v} + \begin{pmatrix} & & & \\ & & \mathbf{0} & \\ & & & \\ b & & & \end{pmatrix} \tilde{v} = 0\,, \tag{24}$$

where $\tilde{v} = \tilde{v}(\tau, n) = v(n^{-\sigma}\tau, n)$, $\tilde{\lambda} = \lambda(n^{-\sigma}\tau)$ and so on. Next as in §2 we introduce a matrix of weght:

$$W = W(n; \epsilon_1, \epsilon_2) = \begin{pmatrix} 1 & & & \\ & n^{-\epsilon_1} & & \\ & & n^{-\epsilon_2} & \\ & & & n^{-\epsilon_1-\epsilon_2} \end{pmatrix}, \tag{25}$$

where $\epsilon_1$ and $\epsilon_2$ are non-negative constants which are determined such that

$$1 - \epsilon_1 = \epsilon_1 + \epsilon_2 = 1 + \epsilon_1 - \epsilon_2 - q\sigma$$

Here we remark that the first term is from the Jordan form of order 2, the second term is from the lower order term $B$ and the last term is from $\tilde{a}(\tau, n)$.

Thus

$$\begin{cases} \epsilon_1 = \dfrac{1}{4} + \dfrac{q}{4}\sigma \\ \\ \epsilon_2 = \dfrac{1}{2} - \dfrac{q}{2}\sigma \end{cases} \tag{26}$$

Here we assume that $\epsilon_2 > 0$, which means that

$$\sigma < \frac{1}{q} \tag{27}$$

Changing the unknown $\tilde{v}$ such that $\tilde{v} = W\tilde{v}_1$, (24) comes to

$$D_\tau \tilde{v}_1 + n^{1-\sigma} \begin{pmatrix} \tilde{\lambda} & & & \\ & \tilde{\lambda} & & \\ & & \tilde{\mu} & \\ & & & \tilde{\mu} \end{pmatrix} \tilde{v}_1 + n^{1-\epsilon_1-\sigma} \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{a}(\tau,n) & 0 \\ \hline 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \end{array}\right) \tilde{v}_1 = 0 \tag{28}$$

Remark that from the assumption of $a(t)$ we can denote that $\tilde{a}(\tau, n) = \nu\tau^q + \tilde{a}_2(\tau, n)$, where $\nu \neq 0$ and $\tilde{a}_2(\tau, n) \to 0$ when $n \to \infty$ and $\tau \in [T_1, T_2]$ for any $T_1$ and $T_2$ $(0 < T_1 < T_2)$. We rewrite this such that

$$D_\tau \tilde{v}_1 + n^{1-\sigma} A_1^{(0)} \tilde{v}_1 + n^{1-\epsilon_1-\sigma}(A_1^{(1)} + A_1^{(2)})\tilde{v}_1 = 0 \,, \tag{29}$$

where $A_1^{(1)} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & \nu\tau^q & 0 \\ \hline 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \end{array}\right)$ and $A_1^{(2)} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{a}_2 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ .

In (29) we regard the term $D_\tau \tilde{v}_1 + n^{1-\epsilon_1-\sigma} A_1^{(1)} \tilde{v}_1$ as a principal part which dominates the other terms. Now we will diagonarize $A_1^{(1)}(\tau)$ . For this we denote that

$$\det(\lambda - A_1^{(1)}) = \prod_{j=1}^{4} (\lambda - \lambda_j(\tau)) \,,$$

where $\lambda_j(\tau) = b_j \tau^{\frac{q}{4}}$. Here we take the element $b$ of the lower order term $B(t)$ as follows:

All $b_j$ are distinct each other and there exists a positive constant $\delta_1$ such that

$$\begin{cases} \mathrm{Im} b_1 = \mathrm{Im} b_2 = \delta_1 > 0 \\ \mathrm{Im} b_3 = \mathrm{Im} b_4 = -\delta_1 < 0 \end{cases} \tag{30}$$

From now on we consider our system of equations for $\tau \geqslant 1$. Then there exists a non-singular matrix $N(\tau)$ such that

$$N^{-1}A_1^{(1)}N = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix},$$

$$\det N(\tau) = \tau^{-\frac{q}{2}} \prod_{i<j} (b_i - b_j) .$$

We change again the unknowns $\tilde{v}_1$ such that $\tilde{v}_1 = N(\tau)\tilde{v}_2$, then (29) turns out to

$$D_\tau \tilde{v}_2 + n^{1-\sigma} N(\tau)^{-1} A_1^{(0)} N(\tau) \tilde{v}_2 + n^{1-\epsilon_1-\sigma} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \tilde{v}_2$$

$$+ n^{1-\epsilon_1-\sigma} N(\tau)^{-1} A_1^{(2)} N(\tau) \tilde{v}_2 + N(\tau)^{-1}(D_\tau N(\tau)) \tilde{v}_2 = 0 . \qquad (31)$$

Denoting the components of the unknowns $\tilde{v}_2$ by

$$\tilde{v}_2 = {}^t(\tilde{v}_2^{(1)}, \tilde{v}_2^{(2)}, \tilde{v}_2^{(3)}, \tilde{v}_2^{(4)}) ,$$

we set the energy form in such a way that

$$S_n(\tau) = |\tilde{v}_2^{(1)}|^2 + |\tilde{v}_2^{(2)}|^2 - |\tilde{v}_2^{(3)}|^2 - |\tilde{v}_2^{(4)}|^2 .$$

Then we see that

$$\frac{d}{d\tau} S_n(\tau) \geqslant \delta_1 n^{1-\epsilon_1-\sigma} S_n(\tau) - \text{const.} n^{-(1-\epsilon_1-\sigma)} |g_n(\tau)|^2 , \qquad (32)$$

where $g_n(\tau)$ is

$$g_n(\tau) = n^{1-\sigma}(\tilde{\lambda} - \tilde{\mu}) N(\tau)^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} N(\tau) \tilde{v}_2$$

$$+ n^{1-\epsilon_1-\sigma} N(\tau)^{-1} A_1^{(2)} N(\tau) \tilde{v}_2 + (N(\tau))^{-1}(D_\tau N) \tilde{v}_2 .$$

Evaluating $g_n(\tau)$, we obtain a following estimate.

PROPOSITION 4. — *Assume that* $\sigma < 1/q$ *and that* $1-\epsilon_1-\sigma > 1-\sigma-p\sigma$. *Then*

$$S_n(\tau) \geqslant \exp(\delta_1' n^{1-\epsilon_1-\sigma}(\tau-1))S_n(1)\,, \tag{33}$$

*for* $\tau \geqslant 1$ *and for large* $n$.

Now we shall determine the Cauchy data. We define our Cauchy data $v_0(n)$ at $t=0$ so that

$$\tilde{v}_2^{(1)}(1) = 1,\ \tilde{v}_2^{(j)}(1) = 0\ (j = 2,3,4)$$

More precisely first we determine $\tilde{v}_2$ at $\tau = 1$, that means $v(t)$ is given at $t = n^{-\sigma}$. Then we define $v_0(n)$ at $t = 0$ as the solution of (23) (backward Cauchy problem) with this Cauchy data. From this consideration we can see the following inequality for our Cauchy data $v(0,n) = v_0(n)$

$$|v(0,n)|^2 \leqslant \text{const.} n^{2(1-\epsilon_1)} \exp(\delta_2' n^{1-\epsilon_1-\sigma})\,. \tag{34}$$

We emphasize that for any $n$ the inequality (33) holds with $S_n(1) = 1$ when we define the Cauchy data $v(0,n)$ in such a way as above which satisfies the estimate (34).

We are going to prove the theorem by contradiction. So we assume that our Cauchy problem is $\gamma^{(s)}$-wellposed which means by definition that

$$|v(t,n)|^2 \leqslant \text{const.}\exp(\delta_2 n^{\frac{1}{s}} t)|v(0,n)|^2\,. \tag{35}$$

It follows from the definition of $S_n(\tau)$ that

$$S_n(\tau) \leqslant |\tilde{v}_2(\tau)|^2 = |N(\tau)^{-1}W(n)^{-1}\tilde{v}(\tau)|^2$$
$$\leqslant \text{const.} n^{2(1-\epsilon_1)}|\tilde{v}(\tau)|^2\,.$$

When we remark that $\tilde{v}(\tau) = \tilde{v}(\tau, n) = v(n^{-\sigma}\tau, n)$, we can see that

$$S_n(\tau) \leqslant \text{const.} n^{2(1-\epsilon_1)}|v(n^{-\sigma}\tau, n)|^2\,.$$

Thus from (35) following inequality holds:

$$S_n(\tau) \leqslant \text{const.} n^{2(1-\epsilon_1)} \exp(\delta_2 n^{\frac{1}{s}-\sigma}\tau)|v(0,n)|^2\,. \tag{36}$$

Moreover owing to (34) we see at last the estimte of $S_n(\tau)$ such that

$$S_n(\tau) \leqslant \text{const.} n^{4(1-\epsilon_1)} \exp(\delta_2 n^{\frac{1}{s}-\sigma}\tau + \delta_2' n^{1-\epsilon_1-\sigma})\,. \tag{37}$$

Let us compare this inequality with (33) under the condition that $S_n(1) = 1$.

$$\exp(\delta_1' n^{1-\epsilon_1-\sigma}(\tau - 1)) \leqslant \text{const.} n^{4(1-\epsilon_1)} \exp(\delta_2 n^{\frac{1}{s}-\sigma}\tau + \delta_2' n^{1-\epsilon_1-\sigma}) \ , \quad (38)$$

for $\tau \geqslant 1$ and for large $n$.

If we impose the condition that

$$1 - \epsilon_1 - \sigma > \frac{1}{s} - \sigma \ ,$$

then with fixed some $\tau(> 1)$ the above inequality leads us to a contradiction when $n$ tends to infinity.

Here we review our conditions imposed on $\sigma$ in our argument

$$\sigma < \frac{1}{q}$$

$$1 - \epsilon_1 - \sigma > 0$$

$$1 - \epsilon_1 - \sigma > 1 - \sigma - p\sigma$$

$$1 - \epsilon_1 - \sigma > \frac{1}{s} - \sigma$$

Taking account of the choice of $\epsilon_1$, it is easily shown that $\sigma$ can exist, when

$$q < 2p$$

and when

$$s > \frac{4p - q}{3p - q}$$

Thus the proof of Theorem 2 is complete.

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