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Annales de la faculté des sciences de Toulouse 6^e série, tome 8, n^o 4
(1999), p. 661-675

http://www.numdam.org/item?id=AFST_1999_6_8_4_661_0

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Regular Foliations along Curves^(*)

PAULO SAD⁽¹⁾

RÉSUMÉ. — On étudie des feuilletages réguliers le long d'une courbe holomorphe compacte invariante, avec une attention spéciale pour le cas d'une courbe elliptique.

ABSTRACT. — A special type of foliation which is regular along an invariant compact holomorphic curve is studied. When the curve is an elliptic one, the relation to a naturally associated elliptic fibration is analysed.

This paper is concerned with the existence of holomorphic foliations, in complex surfaces, without singularities along a smooth, compact, holomorphic curve. If this curve is a leaf of the foliation, it is well known that its self-intersection number is zero. It is natural to ask if, given a curve of zero self-intersection number, one can make it a leaf of a foliation on the surface. This situation will be referred to as *the foliation is regular along the curve* or *the curve admits a regular foliation*.

We study this problem in a fairly simple context. Let C be a smooth projective plane curve of degree $d \in \mathbb{N}$; one has $C.C = d^2$. We select a number of points in C and blow up d^2 times the complex projective plane $\mathbb{P}(2)$ at these points; the strict transform \widehat{C} of C has zero self-intersection number in the new surface (\widehat{C} depends on the choice of points and on the sequence of blow-ups). We show that when $d \geq 3$, a generic choice of centers of blow-ups produces a curve that does not admit a regular foliation. It should be

(*) Reçu le 5 janvier 1999, accepté le 8 février 2000

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remarked here that a smooth rational curve of zero self-intersection number in a compact surface admits a regular foliation (see [1], Proposition V.4.3).

Next, we assume $d = 3$, that is, C is a plane elliptic curve, and the points where $\mathbb{P}(2)$ is to be blown up are $d^2 = 9$ distinct points. A necessary and sufficient condition is given to assure that \widehat{C} admits a regular foliation; the proof of sufficiency presented here produces an elliptic fibration containing \widehat{C} as a fiber. This is the unique regular foliation admitted by \widehat{C} if we demand the singularities along C to be of simple type. We give also an example which has only a finite number of compact leaves: the holonomy group of the curve C is not trivial, although the local holonomy diffeomorphisms at the singularities are all equal to the identity map (compare with ([12])).

The arguments used in the proofs are classical, and come from function theory on Riemann surfaces, and from basic facts on elliptic fibrations with sections. We proceed now to state more precisely the results.

Let C be a smooth plane algebraic curve of degree $d \geq 3$ and $n \in \mathbb{N}$.

THEOREM 1. — *For a very generic choice of distinct points $A_1, \dots, A_n \in C$, there exists no foliation on $\mathbb{P}(2)$ for which the set $C \setminus \{A_1, \dots, A_n\}$ is a leaf and its set of singularities along C is $\{A_1, \dots, A_n\}$.*

COROLLARY 2. — *For a very generic choice of distinct points $A_1, \dots, A_n \in C$, the curve \widehat{C} obtained after performing a sequence of d^2 blow-ups with centers at these points admits no regular foliations.*

The meaning of *very generic* will be explained in Section 1.

In order to state the other results, we restrict to the case $d = 3$ and take $d^2 = 9$ distinct points $A_s, 1 \leq s \leq 9$, along C ; also, we select a straight line $L_\infty \subset \mathbb{P}(2)$, transversal to C , which avoids those points. Let $C \cap L_\infty = \{P_1, P_2, P_3\}$. We then blow up $\mathbb{P}(2)$ once at each of the points A_1, \dots, A_9 .

THEOREM 3. — *The curve \widehat{C} admits a regular foliation with compact leaves if and only if there exists a $a \in \mathbb{N}$ such that $a(\sum_{s=1}^9 A_s - 3 \sum_{j=1}^3 P_j)$ is a principal divisor of C .*

We remark that the condition depends on the straight line L_∞ . The foliations produced by Theorem 3 are pencils of curves of genus one (when seen in $\mathbb{P}(2)$) with base points at A_1, \dots, A_9 . When $a = 1$ the curves in the pencil are generically smooth and cross each other transversely at the base points; these are radial-type singularities for the correspondent foliation. In general, if the curve \widehat{C} admits a regular foliation, the singular points

A_1, \dots, A_9 in $\mathbb{P}(2)$ may not be so simple; there exists only a small sector, for each singularity, where the foliation is of radial type.

As for uniqueness in the situation of Theorem 3 (with $a = 1$), we may state

THEOREM 4. — 1) If \widehat{C} admits a regular foliation such that A_1, \dots, A_9 are singularities of radial type, then $\sum_{s=1}^9 A_s - 3 \sum_{j=1}^3 P_j$ is a principal divisor of C ; conversely, if this divisor is a principal one, there exists only one admitted regular foliation whose singularities (before blow-ups) A_1, \dots, A_9 along C are of radial type. 2) There exists a regular foliation along an elliptic curve which does not come from a pencil of curves.

In part 2) the divisor is still principal but we dropped the radially condition imposed before on the singularities; the dynamics becomes richer.

We acknowledge M. Brunella for helpful conversations, and the referee for its careful reading of the paper (in particular, for suggesting a much more conceptual proof for Theorem 7 and connecting this result to the reference [4]).

1. Divisors and Regular Foliations

Let $C \subset \mathbb{P}(2)$ be a smooth, projective, plane curve of degree $d \in \mathbb{N}$. Given two straight lines L_1, L_2 which cross C transversely at the points P_1^i, \dots, P_d^i , $i=1,2$, the divisors $\sum_{j=1}^d P_j^1$ and $\sum_{j=1}^d P_j^2$ are linearly equivalent. We fix once for all a straight line L_∞ transverse to C to be the line at infinity of $\mathbb{P}(2)$, and let $L_\infty \cap C = \{P_1, \dots, P_d\}$. We consider a subset $\{A_1, \dots, A_n\}$, $n \in \mathbb{N}$ of distinct points taken along $C \setminus L_\infty$, and the set \mathbf{F}_C of foliations on $\mathbb{P}(2)$ satisfying the following conditions:

1. C is \mathcal{F} -invariant for $\mathcal{F} \in \mathbf{F}_C$
2. the singular set of \mathcal{F} along C is exactly $\{A_1, \dots, A_n\}$.

We prove now:

PROPOSITION 5. — \mathbf{F}_C is not empty if and only if there exist $l_1, \dots, l_n \in \mathbb{N}^*$ such that the divisor $\sum_{s=1}^n l_s A_s - k \sum_{j=1}^d P_j$ is a principal divisor of C for $\sum_{s=1}^d l_s = kd$.

Proof. — Let us take $\mathcal{F} \in \mathbf{F}_C$. This foliation is defined by a meromorphic vector field Z in $\mathbb{P}(2)$ with polar divisor $(Z)_\infty = (\deg(\mathcal{F}) - 1)L_\infty$, where $\deg(\mathcal{F})$ is the degree of \mathcal{F} . Since C is \mathcal{F} -invariant, Z is tangent to it, and

we have also $Z(A_s) = 0 \forall 1 \leq s \leq n$. Define l_s as the multiplicity of $Z|_C$ at the point A_s . On the other hand, C is also invariant by the meromorphic vector field $Z_0 = -f_y \partial/\partial x + f_x \partial/\partial y$, where $f = 0$ is a reduced polynomial equation for C . The restriction $Z_0|_C$ has no zeroes in $C \setminus L_\infty$, and poles of order $\deg(C) - 3$ at the points P'_j s; the divisor of the meromorphic function $\xi = (Z|_C)/(Z_0|_C)$ is therefore of the form

$$(\xi) = \sum_{s=1}^n l_s A_s - k \sum_{j=1}^d P_j,$$

where $k = (\sum_{s=1}^d l_s)/d$

For the converse, we need

LEMMA 6. — *Assume that $D = \sum_{s=1}^n l_s A_s - k \sum_{j=1}^d P_j$ is a principal divisor of C , where $l_1, \dots, l_n \in \mathbb{N}^*$ and $\sum_{s=1}^n l_s = k.d$. There exists a polynomial $p(x, y)$ of degree $k \in \mathbb{N}$ such that $p = 0$ intersects C exactly at the points A_1, \dots, A_n with intersection numbers $l_1, \dots, l_n \in \mathbb{N}^*$.*

Proof. — 1) Let g be a meromorphic function of C such that $(g) = D$. Since C is a projective curve, there exist polynomials P, Q satisfying $g = P/Q$ along C . Let us use homogeneous coordinates $[X_0 : X_1 : X_2]$ and keep the notation $g = \frac{P}{Q}$; for simplicity, the zero set of P (allowing multiple components) will be indicated as P .

2) We have that $D = P.C - Q.C$. Since $Q.C > (X_2^k = 0).C$, it follows from Noether's Theorem ([7]) that $Q = AX_2^k + BF$, where $F = 0$ is the homogeneous equation for C and A, B are homogeneous polynomials. We may therefore choose $g = P/AX_2^k$. Now $P.C > A.C$, otherwise we would have poles of g outside L_∞ . Another application of Noether's Theorem lead us to the conclusion that $P = A'A + B'F$, and again we may take $g = A'/X_2^k$. The polynomial $p = A'(x, y, 1)$ satisfies the statement of the Lemma.

We conclude the proof of Proposition 5 by taking the foliation defined as the pencil of curves in $\mathbb{P}(2)$ given by $C_\lambda : f^k + \lambda p = 0, \lambda \in \mathbb{C}$, where $f = 0$ is a reduced polynomial equation for C . \square

Now we can prove Theorem 1. We have to select points in the complement C'_n of the diagonal of the n^{th} -symmetric power of C in a *very generic way*, that is, *outside a countable union of analytic subsets of C'_n* . Let $\vec{l} = (l_1, \dots, l_n) \in (\mathbb{N}^*)^n$ satisfy $\sum_{s=1}^n l_s = kd$, for some $k \in \mathbb{N}$, $J : C \rightarrow J(C)$ be the jacobian map for C and

$$J_{\vec{l}}(A_1, \dots, A_n) = \sum_{s=1}^n l_s J(A_s)$$

Then $\text{cod } J_{\bar{\Gamma}}^{-1}(k \sum_{j=1}^d J(P_j)) \geq 1$. Indeed, the equality

$$l_1 J(A_1) + \sum_{s=2}^n l_s J(A_s) \equiv l_1 J(\tilde{A}_1) + \sum_{s=2}^n l_j J(A_j)$$

in $J(C)$ whenever A_1 and \tilde{A}_1 are close points in C would imply that $l_1(J(A_1) - J(\tilde{A}_1)) \equiv 0$ in $J(C)$, which is impossible since J embeds C into $J(C)$ and $J(C)$ has only a countable number of torsion points. It now follows from Proposition 5 and Abel's Theorem ([5]) that any choice of points in $C'_n \setminus \bigcup_{\bar{\Gamma}} J_{\bar{\Gamma}}^{-1}(k \sum_{j=1}^d J(P_j))$ satisfies the non existence statement of the Theorem.

2. Divisors and Elliptic Fibrations

As we have said in the Introduction, a surface foliation by curves which has no singularities along some invariant compact curve is *regular along the curve*. Equivalently, *the compact curve admits a regular foliation*.

We take the following situation. Suppose C is a smooth plane algebraic curve of degree $d \in \mathbb{N}$, and choose d^2 distinct points $A_1, \dots, A_{d^2} \in C$. We blow up each point $A_s, 1 \leq s \leq d^2$, exactly once. The new surface $\widehat{\mathbb{P}}(2)$ contains the strict transform \widehat{C} of C as a smooth curve of self-intersection number equal to 0, which is a necessary condition for the existence of a foliation on a surface which is regular along some curve ([3]). According to Section 2, this condition is not at all sufficient. Our aim in this Section is to present a condition (in terms of divisors over C) which ensures the existence of a special type of regular foliation along \widehat{C} , namely a foliation whose leaves are compact. Let G_C be the space of these foliations.

Let us consider $\mathcal{G} \in G_C$, and take a small section transversal to \widehat{C} . There exists $a \in \mathbb{N}$ such that any leaf close to \widehat{C} (and different from \widehat{C}) intersects Σ in $a \in \mathbb{N}$ points. We select one of these leaves, say \widehat{L} , and blow it down to the curve L in $\mathbb{P}(2)$. Then L has $a \in \mathbb{N}$ smooth branches going transversely through each point $A_s, 1 \leq s \leq n$, and its reduced polynomial equation, restricted to C , gives us a principal divisor of the form $a(\sum_{s=1}^{d^2} A_s - d \sum_{j=1}^d P_j)$.

When C is a smooth plane cubic, we may state

THEOREM 7. — $G_C \neq \emptyset$ if and only if there exists $a \in \mathbb{N}$ such that

$$a \left(\sum_{s=1}^9 A_s - 3 \sum_{j=1}^3 P_j \right)$$

is a principal divisor of C .

Proof. — 1) First of all, let us remark that the normal bundle $\mathcal{O}_{\widehat{\mathbb{P}}(2)}(\widehat{C})|_{\widehat{C}}$ to \widehat{C} in $\widehat{\mathbb{P}}(2)$ (for simplicity, written as $N_{\widehat{C}}$) coincides with $\mathcal{O}_C(-D)$ where $D := \sum_{s=1}^{s=9} A_s - 3 \sum_{j=1}^{j=3} P_j$. In fact, the canonical divisor K of $\widehat{\mathbb{P}}(2)$ is $\sum_{s=1}^{s=9} E_s - 3L_\infty$ (E_s is the exceptional divisor that arises from blowing up at the point A_s) and $K|_{\widehat{C}} = -N_{\widehat{C}}$ (this follows from the adjunction formula and from the fact that the canonical divisor of \widehat{C} is zero). Therefore aD is a principal divisor iff $N_{\widehat{C}}^a$ is trivial; if we assume that $a \in \mathbb{N}$ is the smallest integer with that property, then $N_{\widehat{C}}^b$ is not trivial for all $1 \leq b < a$, $b \in \mathbb{N}$. In particular, for such values of $b \in \mathbb{N}$ there is no holomorphic section to $N_{\widehat{C}}^b$ since it would have zeroes, contradicting $\deg(bD) = b \deg(D) = 0$.

2) Riemann-Roch's theorem gives $h^0(N_{\widehat{C}}^b) = h^1(N_{\widehat{C}}^b)$, so that $h^1(N_{\widehat{C}}^b) = 0$ for any $1 \leq b < a$. Using the long exact sequence associated to

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}(2)}((b-1)\widehat{C}) \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}(2)}(b\widehat{C}) \longrightarrow N_{\widehat{C}}^b \longrightarrow 0$$

we see that $H^1(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)}((b-1)\widehat{C})) \cong H^1(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)}(b\widehat{C}))$ as long as $1 \leq b < a$. It follows recursively that $H^1(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)}((a-1)\widehat{C})) \cong H^1(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)})$, so that $H^1(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)}((a-1)\widehat{C})) = 0$, and from

$$0 \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}(2)}((a-1)\widehat{C}) \longrightarrow \mathcal{O}_{\widehat{\mathbb{P}}(2)}(a\widehat{C}) \longrightarrow N_{\widehat{C}}^a \longrightarrow 0$$

we get that the map $H^0(\widehat{\mathbb{P}}(2), \mathcal{O}_{\widehat{\mathbb{P}}(2)}(a\widehat{C})) \longrightarrow H^0(\widehat{C}, N_{\widehat{C}}^a)$ is surjective. Therefore $\mathcal{O}_{\widehat{\mathbb{P}}(2)}(a\widehat{C})$ has a holomorphic section u which restricts to \widehat{C} as a nowhere vanishing section of $N_{\widehat{C}}^a$ (since $N_{\widehat{C}}^a$ is trivial). But $\mathcal{O}_{\widehat{\mathbb{P}}(2)}(a\widehat{C})$ has a (canonical) section u_0 which vanishes along \widehat{C} to the order $a \in \mathbb{N}$. Consequently u/u_0 is a holomorphic function from $\widehat{\mathbb{P}}(2)$ to \widehat{C} such that its polar divisor is $a\widehat{C}$ and its zero divisor does not cross \widehat{C} . The fibration defined by u/u_0 belongs to G_C . \square

Remarks. — 1) In the situation of Theorem 7, the curve C is holomorphically diffeomorphic to a complex torus \mathbb{T} via the Jacobian map $J : C \longrightarrow \mathbb{T}$. The set $H = \bigcup_{a \in \mathbb{N}} \{(Z_1, \dots, Z_9) \in \mathbb{T}^9; a \cdot [\sum_{j=1}^9 Z_j - 3J(\sum_{j=1}^3 P_j)] = 0\}$ is an union of hyperplanes of \mathbb{T}^9 ; any point in H with distinct coordinates gives us a divisor aD in C which satisfies the hypothesis of Theorem 3.

2) It should be also noticed that the same arguments of the proof of Theorem 7 apply in the following setting: let C be a smooth elliptic curve

contained in a compact surface X with $h^1(X, \mathcal{O}_X) = 0$; suppose that $C.C = 0$ and also that N_C^a is trivial for some $a \in \mathbb{N}$. Then C is a fiber of multiplicity $a \in \mathbb{N}$ of some elliptic fibration of X .

3. Foliations and Elliptic Fibrations I

This section is dedicated to proving the first part of Theorem 4. We describe again our setting. Let $C \subset \mathbb{P}(2)$ be a smooth algebraic curve of degree 3 (an elliptic curve) which is an element of a pencil of elliptic curves of degree 3 with 9 different points as base points. We write, after the earlier sections, the pencil as the set of curves

$$C_\lambda : f + \lambda p = 0, C = C_0$$

and $\{A_1, \dots, A_9\} = C \cap \{p = 0\}$, $\deg(p) = 3$.

This pencil defines a foliation \mathcal{G} of $\mathbb{P}(2)$ with a rational first integral, and the points A_1, A_2, \dots, A_9 are singularities of radial type. It is easily seen that $\deg(\mathcal{G}) = 4$ (see [8] for the definition of degree of a foliation), therefore \mathcal{G} has 21 singularities (counted with multiplicities); nine of them belong to C . Blowing-up \mathcal{G} at A_1, A_2, \dots, A_9 gives us a foliation $\widehat{\mathcal{G}}$ on the surface $\widehat{\mathbb{P}}(2)$ which is regular along \widehat{C} . The aim is to prove that $\widehat{\mathcal{G}}$ is the unique foliation with this property when the singularities A_1, \dots, A_9 are of radial type (before blow-up's)

The basic idea is to play with the elliptic fibration $\xi : \widehat{\mathbb{P}}(2) \rightarrow \bar{\mathbb{C}}$ whose fibers are the curves \widehat{C}_λ . In our setting, the exceptional divisor E_s which appears after blowing-up the point A_s is a section to the fibration. Although it is not the case here, we observe that when the pencil is given as the set of curves

$$f^a + \lambda q = 0, a \in \mathbb{N}, \deg(q) = 3a$$

(which was studied in the last section), we still have after blowing-up the points A_1, A_2, \dots, A_n a fibration by curves of genus 1; this time the exceptional divisor E_s is no longer a section, but a multisection.

From now on we follow closely the exposition [6] on elliptic fibrations.

Suppose there exists another foliation $\widehat{\mathcal{F}}$ on $\widehat{\mathbb{P}}(2)$ regular along \widehat{C} . There are interesting consequences which will be now explained. Theorem 4, part 1) is proved at the end of the section.

LEMMA 8. — *The functional invariant of the fibration $\widehat{\mathbb{P}}(2) \rightarrow E_s$ is constant, that is, all smooth elliptic fibers are of the same holomorphic type.*

Proof. — 1) Let \widehat{C}_λ be a smooth fiber of \widehat{G} , $\lambda \neq 0$. According to [2], we have

$$c_1(N_{\widehat{F}}).\widehat{C}_\lambda = \chi(\widehat{C}_\lambda) + \text{tang}(\widehat{F}, \widehat{C}_\lambda) = \text{tang}(\widehat{F}, \widehat{C}_\lambda)$$

if \widehat{C}_λ is not \widehat{F} -invariant. Here $N_{\widehat{F}}$ is the normal bundle to \widehat{F} , $c_1(N_{\widehat{F}})$ its Chern class and $\text{tang}(\widehat{F}, \widehat{C}_\lambda)$ the number of tangencies between \widehat{F} and \widehat{C}_λ . Since $\widehat{C}_\lambda = \widehat{C}_0$ homologically, we get

$$c_1(N_{\widehat{F}}).\widehat{C}_\lambda = c_1(N_{\widehat{F}}).\widehat{C}_0.$$

Now \widehat{C}_0 is \widehat{F} -invariant, and again by [2]

$$c_1(N_{\widehat{F}}).\widehat{C}_\lambda = \widehat{C}_0.\widehat{C}_0 + Z(\widehat{F}, \widehat{C}_0),$$

where $Z(\widehat{F}, \widehat{C}_0)$ is the number of zeroes of \widehat{F} along \widehat{C}_0 . We conclude that $\text{tang}(\widehat{F}, \widehat{C}_\lambda) = 0$. Therefore \widehat{F} is transverse to \widehat{G} except along a finite number of fibers (including the singular fibers of \widehat{G}), say $\widehat{C}_0, \widehat{C}_{\lambda_1}, \dots, \widehat{C}_{\lambda_l}$. Any pair of fibers $\widehat{C}_\lambda, \widehat{C}_{\lambda'}$ such that λ and λ' do not belong to $\{0, \lambda_1, \dots, \lambda_l\}$ are holomorphically equivalent: we just lift a path in $\mathbb{C} \setminus \{0, \lambda_1, \dots, \lambda_l\}$ which joins λ to λ' to the leaves of \widehat{F} in order to define a holomorphic diffeomorphism from \widehat{C}_λ to $\widehat{C}_{\lambda'}$. Lemma 8 follows. \square

Consulting the table of behavior of the functional invariant at singular fibers ([11]), and noticing that the elliptic pencil has 9 different base points, we conclude that there are only 3 possibilities for these fibers in our situation: type II (rational cubic with a cusp), type III (a line and a conic tangent to it) and type IV (three lines meeting at the same point). For type III, the functional invariant extends as 1, and as 0 for the other two. Now, the singularities of \widehat{G} appearing in fibers of type II, III and IV have multiplicities (Milnor numbers) 2, 3 and 4 respectively. Since \widehat{G} has 12 singularities (counted with multiplicities), we have then the following possibilities for its singular fibers:

- i) 3 fibers of type IV.
- ii) 4 fibers of type III.
- iii) 2 fibers of type IV and 2 fibers of type II.
- iv) 1 fiber of type IV and 4 fibers of type II.
- v) 6 fibers of type II.

It was said at the end of the proof of Lemma 8 that \widehat{F} is transverse to \widehat{G} except along a finite number of fibers, including the singular fibers of \widehat{G} . In fact, when the singular fiber is not irreducible (types III and IV), this means that at least one of its components is \widehat{F} -invariant. Let us prove that

each irreducible component of a singular fiber \widehat{C}' is $\widehat{\mathcal{F}}$ -invariant; we have to keep in mind that $c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}' = 0$.

1) \widehat{C}' is of type III: let \widehat{C}^1 and \widehat{C}^2 be its components; both have self-intersection numbers equal to -2 . Suppose \widehat{C}^1 is a $\widehat{\mathcal{F}}$ -invariant set, but \widehat{C}^2 is not. One has:

$$\begin{aligned} c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}^1 &= \widehat{C}^1.\widehat{C}^1 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) = -2 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) \\ c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}^2 &= \chi(\widehat{C}^2) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) = 2 + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) \end{aligned}$$

Now $Z(\widehat{\mathcal{F}}, \widehat{C}^1) \geq 0$ because \widehat{C}^1 is smooth, and $\text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) > 0$ because \widehat{C}^1 is tangent to \widehat{C}^2 and it is a $\widehat{\mathcal{F}}$ -invariant set. It follows that

$$c_1(N_{\widehat{\mathcal{F}}}).\widehat{C} = c_1(N_{\widehat{\mathcal{F}}}).(\widehat{C}^1 + \widehat{C}^2) > 0$$

in contradiction with $c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}' = 0$.

2) \widehat{C}' is of type IV: let $\widehat{C}^j, j = 1, 2, 3$ be its components. Again, their self-intersection numbers are equal to -2 . If \widehat{C}^1 is a $\widehat{\mathcal{F}}$ -invariant set but $\widehat{C}^2, \widehat{C}^3$ are not, we have

$$\begin{aligned} c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}^1 &= \widehat{C}^1.\widehat{C}^1 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) = -2 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) \\ c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}^2 &= \chi(\widehat{C}^2) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) = 2 + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) \\ c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}^3 &= \chi(\widehat{C}^3) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^3) = 2 + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^3) \end{aligned}$$

which implies

$$0 = c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}' = 2 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^2) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^3) > 0,$$

contradiction.

If $\widehat{C}^1, \widehat{C}^2$ are $\widehat{\mathcal{F}}$ -invariant sets and \widehat{C}^3 is not, we may write

$$0 = c_1(N_{\widehat{\mathcal{F}}}).\widehat{C}' = -2 + Z(\widehat{\mathcal{F}}, \widehat{C}^1) + Z(\widehat{\mathcal{F}}, \widehat{C}^2) + \text{tang}(\widehat{\mathcal{F}}, \widehat{C}^3)$$

This time $p \in \widehat{C}^1 \cap \widehat{C}^2$ is a singularity of $\widehat{\mathcal{F}}$, so that $Z(\widehat{\mathcal{F}}, \widehat{C}^1) + Z(\widehat{\mathcal{F}}, \widehat{C}^2) \geq 2$ and $\text{tang}(\widehat{\mathcal{F}}, \widehat{C}^3) > 0$, contradiction again.

In conclusion, all the singular fibers of $\widehat{\mathcal{G}}$ are $\widehat{\mathcal{F}}$ -invariant sets; we remind that there is at least one smooth fiber (namely $\widehat{C} = \widehat{C}_0$) which is $\widehat{\mathcal{F}}$ -invariant.

LEMMA 9. — *The (vanishing) orders of \mathcal{F} at the points A_1, A_2, \dots, A_9 are all the same; if $k \in \mathbb{N}$ is this common order, then $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{F}}$ have $k + 2$ fibers of tangency (counted with multiplicities).*

Proof. — 1) We select some singularity A_s ; let us denote by $k \in \mathbb{N}$ the order of \mathcal{F} at A_s (it will turn out that $k \in \mathbb{N}$ does not depend on the singularity). We take local coordinates around A_s as to have equations

$$\dot{x} = x, \dot{y} = y$$

for \mathcal{G} and

$$\dot{x} = (P_k(x, y) + P_{k+1}(x, y) + \dots), \dot{y} = y[Q_{k-1}(x, y) + Q_k(x, y) + \dots]$$

for \mathcal{F} ; $y = 0$ corresponds to C and P_j, Q_j are homogeneous polynomials of degree $j \in \mathbb{N}$.

Let us proceed to find the tangency set between $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ using the equations above. Blowing-up at A_s (with coordinates $x, t = y/x$) gives the equations

$$\dot{x} = 1, \dot{t} = 0$$

for $\widehat{\mathcal{G}}$ and

$$\dot{x} = [P_k(1, t) + xP_{k+1}(1, t) + \dots], \dot{t} = t[Q_k(1, t) - P_{k+1}(1, t) + x(\dots)],$$

We remark that $Q_{k-1}(1, t) - P_k(1, t) \equiv 0$, but $Q_k(1, t) - P_{k+1}(1, t) \neq 0$ (since the exceptional divisor $x = 0$ is not $\widehat{\mathcal{F}}$ -invariant). Also, we may suppose that $P_{k+1}(0, 1) \neq 0$, which amounts simply to assuming that $t = \infty$ is not a tangency point.

Finally, from the equations for $\widehat{\mathcal{F}}$ we get that the tangency set is given (along $x = 0$) by

$$t(Q_k(1, t) - P_{k+1}(1, t)) = 0$$

and so there are $k + 2$ solutions (counted with multiplicities). Observe that the argument independes on the singularity $A_s, 1 \leq s \leq 9$. In particular, the order is the same for all of them. \square

We know also that the foliations $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{F}}$ have at least four common invariant fibers: besides $\widehat{C} = \widehat{C}_0$, there are the singular fibers that appear in the possibilities i), ii), . . . , v). We conclude that

COROLLARY 10. — $k \geq 2$.

The first part of Theorem 4 is then proved.

Remark. — Let C be a plane smooth elliptic curve, and we search for the foliations which leave it invariant and have only radial type singularities $\{A_1, \dots, A_9\} \subset C$. Then necessarily $\sum_{s=1}^9 A_j - 3 \sum_{j=1}^3 P_j$ (where P_1, P_2, P_3 are the points of intersection of the curve with some straight line) is a principal divisor, and we are back to the situation of Theorem 4. See also [10]

4. Foliations and Elliptic Fibrations II

Let us use again the notation introduced in the former section. We want first to find an elliptic fibration $\widehat{\mathcal{G}}$ which comes together with a foliation $\widehat{\mathcal{F}}$ transverse to it except for tangencies along a certain number of singular fibers. Once this is done we will introduce another fiber of tangency, this time a smooth one, by means of holomorphic surgery. Proof of Theorem 5, part 2) will then be achieved. Several steps are needed.

Step 1. — Let us consider the pencil of cubics \mathcal{G}_0 defined by

$$B_\lambda : y^2 - x^3 = \lambda, \lambda \in \mathbb{C}$$

There exists only one base point $[0 : 0 : 1]$ at the line at infinity. If we blow up sufficiently, we get an elliptic fibration $\widehat{\mathcal{G}}_0$ which has two singular fibers: one for $\lambda = 0$, of type *II*, and the other one for $\lambda = \infty$, of type *II**. Eight out of the 9 projective lines which arise from the blow-up's are contained in \widehat{B}_∞ and the remaining one, denoted here by E_9 , is a section to $\widehat{\mathcal{G}}_0$ (by a section we mean a section to the fibration). The pencil \mathcal{G}_0 is transverse to the foliation \mathcal{F}_0 given by

$$d(y^2/x^3) = 0, \text{ or } 2xdy - 3ydx = 0$$

except along B_0 and B_∞ . If \mathcal{F}_0 is submitted to the same sequence of blow-up's applied to \mathcal{G}_0 , it is found that E_9 is a leaf of $\widehat{\mathcal{F}}_0$; we will restrict ourselves to the subset $V = \bigcup_{|\lambda| < 2r} \widehat{B}_\lambda$, $r > 0$ being a small fixed positive real number. Therefore:

- a) $\widehat{\mathcal{G}}_0|_V$ is an elliptic fibration over \mathbb{D} and all fibers are smooth with the exception of the type *II* fiber \widehat{B}_0 .
- b) $\widehat{\mathcal{F}}_0|_V$ is transverse to $\widehat{\mathcal{G}}_0|_V$, except along \widehat{B}_0 . It has two singularities in \widehat{C}_0 : one at the singular point of \widehat{B}_0 , and another one at $E_9 \cap \widehat{B}_0$. This singularity is easily seen to be holomorphically equivalent to the reduced one

$$\xi d\eta + 6\eta d\xi = 0$$

where ξ is a coordinate along \widehat{B}_0 and η is a coordinate along E_9 .

Due to the transversality between the foliations, we may define a "monodromy map" Φ from \widehat{B}_r to itself by lifting the path $|\lambda| = r$ along the leaves of $\widehat{\mathcal{F}}_0$. We observe that $\Phi^6 = \text{Id}$, and that $E_9 \cap \widehat{B}_r$ is a fixed point of Φ of period 6.

We take in $\mathbb{D}_{1+r} \times \widehat{B}_1$ the foliations $\widehat{\mathcal{G}}'_0$ and $\widehat{\mathcal{F}}'_0$ whose leaves are $\{\lambda\} \times \widehat{B}_1$, $\lambda \in \mathbb{D}_{1+r}$ and $\mathbb{D}_{1+r} \times \{p\}$, $p \in \widehat{B}_1$ respectively. As in [9], we may glue together 6 copies of $\widehat{\mathcal{F}}'_0|_V$ to a copy of $\widehat{\mathcal{F}}'_0$ in order to preserve the elliptic fibrations and have a section to the resulting fibration; this section arises by glueing the copies of $E_9 \cap V$ to the same horizontal leaf of $\widehat{\mathcal{F}}'_0$. Such a construction is performed in the following way: we start by glueing $\widehat{\mathcal{F}}'_0|_V$ to $\widehat{\mathcal{F}}'_0$. For this, we choose some holomorphic diffeomorphism h between $\{1\} \times \widehat{B}_1$ and \widehat{B}_r , and extend it to small neighborhoods of $\{1\} \times \widehat{B}_1$ and \widehat{B}_r saturating along the leaves of $\widehat{\mathcal{F}}'_0|_V$ and $\widehat{\mathcal{F}}'_0$ and respecting $\widehat{\mathcal{G}}'_0$ and $\widehat{\mathcal{G}}'_0|_V$. The resulting diffeomorphism is a the desired glueing map. We repeat the same construction using again h to send $\{\lambda_j\} \times \widehat{B}_1$ to \widehat{B}_r , where $\lambda_j = e^{2i\pi j}$, $j = 0, \dots, 5$. We arrive at a surface M' which has an elliptic fibration $\widehat{\mathcal{G}}'$ over a domain of \mathbb{C} diffeomorphic to a disc (we still call it \mathbb{D} for simplicity) and possesses a transversal foliation $\widehat{\mathcal{F}}'$ (except for 6 type II fibers of tangency); besides, the fibration has a section, because 6 copies of $E_9 \cap V$ were glued together to the same horizontal leaf of $\widehat{\mathcal{F}}'_0$.

The monodromy map of $\widehat{\mathcal{F}}'$ associated to $\partial\mathbb{D}$, which is defined for any fiber of $\widehat{\mathcal{G}}'|_{\partial\mathbb{D}}$, is the identity map, due to $\Phi^6 = \text{Id}$ and the fact that we have used the same diffeomorphism h for all the glueing maps. This allows us to extend $\widehat{\mathcal{F}}'$ to a foliation transverse to the fibers of an elliptic fibration (with the exception of 6 fibers) of a compact surface : we just glue it to the trivial horizontal foliation of $(\mathbb{C} \setminus \mathbb{D}) \times \widehat{B}_1$ (always sending the fibers of $\widehat{\mathcal{G}}'$ near $\partial M'$ to the vertical fibers of $(\mathbb{C} \setminus \mathbb{D}) \times \widehat{B}_1$ near $\partial\mathbb{D} \times \widehat{B}_1$).

Let us call M the complex surface obtained from this construction, $\widehat{\mathcal{G}}$ the elliptic fibration and $\widehat{\mathcal{F}}$ the foliation transverse to the fibration. They have in common 6 leaves which are type II singular fibers (and are transverse elsewhere); the section (still called E_9) to $\widehat{\mathcal{G}}$ is invariant for $\widehat{\mathcal{F}}$. Let Q_1, \dots, Q_6 be the points of intersection of E_9 with the singular fibers of $\widehat{\mathcal{G}}$. From b) above one has that the Camacho-Sad index $\text{ind}_{Q_j}(\widehat{\mathcal{F}}, E_9) = -1/6$, $1 \leq j \leq 6$, so that $E_9.E_9 = -1$ (see [3]).

Step 2. — Since M is an elliptic fibration over \mathbb{C} with 6 type II singular fibers, its Euler-Poincaré characteristic is 12. Furthermore, the fibration has a section; it follows (see (III.4.4), (III.4.6) and (IV.1.2) in [11]) that

- 1) M is obtained from $\mathbb{C}P(2)$ blowing-up 9 points.
- 2) The fibration $\widehat{\mathcal{G}}$ comes from a pencil \mathcal{G} of generically smooth cubics in the projective plane with those 9 points as base points (one of them produces the section E_9 , but the others could be infinitely near).

In fact, the remaining 8 points must all be distinct because otherwise either some component of the exceptional divisor, different from E_9 , is part of a singular fiber (which is not the case) or a connected component of the exceptional divisor, still different from E_9 , has at least 2 elements not contained in the fibers. This would imply that the generic member of the pencil is not smooth, which is not the case also. Summarising: M comes from $CP(2)$ after blow-up's at 9 different points A_1, \dots, A_9 producing sections E_1, \dots, E_9 to the fibration $\widehat{\mathcal{G}}$.

Step 3. — The singularities of $\widehat{\mathcal{F}}$, besides the singular points of the singular fibers, are the points Q_1, \dots, Q_6 which belong to E_9 . There exists therefore a component, say E_1 , of the exceptional divisor without any singularities of $\widehat{\mathcal{F}}$. In particular, E_1 is not $\widehat{\mathcal{F}}$ -invariant. Let us choose a smooth fiber \widehat{C} of $\widehat{\mathcal{G}}$ such that $\widehat{C} \cap E_1$ is not a tangency point between $\widehat{\mathcal{F}}$ and E_1 . We want to change $\widehat{\mathcal{F}}$ in a neighborhood of \widehat{C} in order to turn this curve into an invariant one with non trivial holonomy.

Due to the transversality between $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ along that fiber, we may trivialise the fibration in a neighborhood W of \widehat{C} . Let us assume then that this neighborhood is equivalent to $\mathbb{D}_{r_2} \times \widehat{C}$ (\mathbb{D}_{r_2} denotes the disc $\{|\lambda| < r_2\}$, r_2 to be chosen afterwards), $\widehat{\mathcal{G}}$ is the fibration defined by $d\lambda = 0$, $\lambda \in \mathbb{D}_{r_2}$ and finally that $z = 0$ defines E_1 , where $z \in \mathbb{C}$ comes from the uniformization of \widehat{C} (Remark: it can be easily verified that C is equivalent to the quotient of \mathbb{C} by the lattice Γ generated by 1 and $e^{2i\pi/6}$). For simplicity we assume that $\widehat{\mathcal{F}}$ is of the form $dz/d\lambda = 1$ in W . There exist as a consequence a point $(\lambda', 0) \in E_1 \times \{0\}$, close to $(0, 0) \in E_1 \times \{0\}$, two annuli A and B and a holomorphic diffeomorphism ψ such that

- 1) $\lambda' \in A = \{\lambda \in E_1, 0 < r_1 < |\lambda| < r_2 < 1\}$, r_1 to be chosen afterwards.
- 2) $0 \in B \subset \widehat{C}$
- 3) the leaf of $\widehat{\mathcal{F}}|_W$ through a point of $\{\lambda'\} \times B$ cuts $A \times \{0\}$ at its ψ -image; this defines ψ as a diffeomorphism from $\{\lambda'\} \times B$ to $A \times \{0\}$ (in particular, $\psi(\lambda', 0) = (\lambda', 0)$).

Let us now consider the foliation \mathcal{H} in $\mathbb{C} \times \widehat{C}$ defined as

$$dz/d\lambda = \frac{\lambda - \lambda_1}{\lambda}$$

If we choose $\lambda_1 \cdot e^{2i\pi} \in \Gamma$, the lattice associated to \widehat{C} , we get that the monodromy map of \mathcal{H} when we turn along A is the identity map. Once λ_1 is fixed, we choose $0 < r_1 < r_2$ sufficiently big as to have $|\lambda_1| \ll r_1$. The equation that defines \mathcal{H} will be approximately $dz/d\lambda = 1$ in a neighborhood of

$A \times B$. We see that again there exists a diffeomorphism ψ' from $\{\lambda'\} \times B$ to an annulus $A' \times \{0\}$ close to $A \times \{0\}$ such that the leaf of \mathcal{H} through a point of B cuts $A' \times \{0\}$ at its ψ' -image; one has $\psi'(\lambda', 0) = (\lambda', 0)$.

We intend to glue the foliation $\widehat{\mathcal{F}}|_W$ restricted to $\{\lambda \in \mathbb{C}; r_1 < |\lambda| < r_2\} \times \widehat{C}$ to the foliation \mathcal{H} restricted to $A'_e \times \widehat{C}$, where A'_e is the topological disk inside the outer boundary of A' ($\lambda = 0$ belongs to A'_e). We have to define an holomorphic diffeomorphism from $A \times \widehat{C}$ to $A' \times \widehat{C}$ which preserves fibers of $\widehat{\mathcal{G}}$ and sends leaves of $\widehat{\mathcal{F}}|_W$ to leaves of \mathcal{H} . This can be achieved by taking the identity map on $\{\lambda'\} \times \widehat{C}$ and propagating it along leaves and respecting fibers. It is well defined since the monodromies of $\widehat{\mathcal{F}}|_W$ and \mathcal{H} when we turn along A and A' are equal to the identity map. It is important to notice that "respecting fibers" means that a fiber of $\widehat{\mathcal{G}}$ over a point of A is sent to a fiber over a point of A' according to some holomorphic diffeomorphism from A to A' .

Such a glueing construction may produce a complex surface different from M , unless we are able to keep a section to the new elliptic fibration (this follows exactly from the arguments in Step 2 above). In order to guarantee that E_1 is still a section, we proceed as follows. Consider the diffeomorphism $h = \psi' \circ \psi^{-1}$ from $A \times \{0\}$ to $A' \times \{0\}$ ($h(\lambda', 0) = (\lambda', 0)$). We take the identity map on $\{\lambda'\} \times \widehat{C}$ and propagate it along the leaves of $\mathcal{F}|_W$ and \mathcal{H} but now using h : the fiber over a point $(\lambda, 0) \in A \times \{0\}$ has to be sent into the fiber over $(h(\lambda), 0) \in A' \times \{0\}$. The resulting diffeomorphism preserves $\{z = 0\}$, so that the section E_9 persists.

Let $\widehat{\mathcal{F}}'$ denote the new foliation in M . We see that the elliptic curve corresponding to $\lambda = 0$ in the equation of \mathcal{H} is invariant, without singularities and its holonomy group is not trivial; the same can be said of this curve as a leaf of $\widehat{\mathcal{F}}'$. To obtain the plane foliation we look for it is enough to blow-down all the components of the exceptional divisor.

The proof of Theorem 4 is then completed.

Remark. — The elliptic fibration used above corresponds, among the possibilities discussed after Lemma 8, to case v). It would be interesting trying to repeat the construction with the other possibilities.

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