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# On Bergman completeness of pseudoconvex Reinhardt domains (*)

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RÉSUMÉ. — Dans cet article nous donnons une description précise des domaines de Reinhardt pseudoconvexes qui sont complets au sens de Bergman.

ABSTRACT. — In the paper we give a precise description of Bergman complete bounded pseudoconvex Reinhardt domains.

## 0. Introduction

The study of the boundary behaviour of the Bergman kernel has a very comprehensive literature. Very closely related to this problem is the problem of Bergman completeness of a domain. It is well-known that any Bergman complete domain is pseudoconvex (see [Bre]). The converse is not true (take the punctured unit disc in $\mathbb{C}$). The problem which pseudoconvex domains are Bergman complete has a long history (see e.g. [Ohs], [Jar-Pfl]). The most general result, stating that any bounded hyperconvex domain is Bergman complete, has been recently obtained independently by Z. Błocki

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and P. Pflug (see [Bło-Pfl]) and by G. Herbort (see [Her]). The result generalizes earlier results on Bergman completeness. An example of a bounded non-hyperconvex domain, which is Bergman complete can be found already in dimension one (see [Chen]). Another example of such a domain is given in [Her]; the example is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Motivated by the last example we solve in the paper entirely the problem, which bounded Reinhardt domains are Bergman complete. Since the concept of hyperconvexity in the class of bounded Reinhardt domains is completely understood (see [Car-Ceg-Wik] and [Zwo 2]) the paper gives us complete answer about the mutual relationship between hyperconvexity and the Bergman completeness in the considered class of domains. The condition which satisfy the Bergman complete bounded pseudoconvex Reinhardt domains is expressed in simple geometrical properties of some convex cones associated to Reinhardt pseudoconvex domains.

The paper is the next one in understanding the completeness with respect to invariant distances in the class of Reinhardt pseudoconvex domains (see [Pfl 2], [Fu], [Zwo 1] and [Zwo 2]).

## 1. Definitions and statement of results

Let us denote by $E$ the unit disc in $\mathbb{C}$. By $A_+$ (respectively, $A_-$) we denote the non-negative (respectively, non-positive) numbers from $A$. Put also $A_*^n := (A_*)^n$, where $A_* := A \setminus \{0\}$.

Let $D$ be a domain in $\mathbb{C}^n$. Let us denote by $L_h^2(D)$ square integrable holomorphic functions on $D$. $L_h^2(D)$ is a Hilbert space. Let $\{\varphi_j\}_{j\in J}$ ($\#J \leqslant \aleph_0$) be an orthonormal basis of $L_h^2(D)$. Then we define

$$K_D(z) := \sum_{j\in J} |\varphi_j(z)|^2,\ z \in D.$$

Let us call $K_D$ *the Bergman kernel of* $D$. For domains $D$ such that for any $z \in D$ there is $f \in L_h^2(D)$ with $f(z) \neq 0$ (for example for $D$ bounded) we have

$$K_D(z) = \sup\{\frac{|f(z)|^2}{||f||^2_{L^2(D)}} : f \in L_h^2(D), f \not\equiv 0\}$$

If $D$ is such that $K_D(z) > 0$, $z \in D$ then $\log K_D$ is a smooth plurisubharmonic function. In this case we define

$$\beta_D(z;X) := \left(\sum_{j,k=1}^{n} \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k\right)^{1/2},\ z \in D,\ X \in \mathbb{C}^n.$$

$\beta_D$ is a pseudometric called *the Bergman pseudometric.*

For $w, z \in D$ we put

$$b_D(w,z) := \inf\{L_{\beta_D}(\alpha)\}$$

where the infimum is taken over piecewise $C^1$-curves $\alpha : [0,1] \mapsto D$ joining $w$ and $z$ and $L_{\beta_D}(\alpha) := \int_0^1 \beta_D(\alpha(t); \alpha'(t))dt$.

We call $b_D$ *the Bergman pseudodistance of* $D$.

The Bergman distance (as well as the Bergman metric) is invariant with respect to biholomorphic mappings. In other words, for any biholomorphic mapping $F : D \mapsto G$ $(D, G \subset\subset \mathbb{C}^n)$ we have

$$b_G(F(w), F(z)) = b_D(w,z),$$
$$\beta_G(F(w); F'(w)X) = \beta_D(w; X),\ w, z \in D, X \in \mathbb{C}^n.$$

A bounded domain $D$ is called *Bergman complete* if any $b_D$-Cauchy sequence is convergent to some point in $D$ with respect to the standard topology of $D$.

As already mentioned any bounded Bergman complete domain is pseudoconvex (see [Bre]). Let us recall that a bounded domain $D$ is called *hyperconvex* if it admits a continuous negative plurisubharmonic exhaustion function. Now we may formulate the following very general result:

THEOREM 1 (see [Bło-Pfl], [Her]). — *Let $D$ be a bounded hyperconvex domain in $\mathbb{C}^n$. Then $D$ is Bergman complete.*

It is known that the converse implication in Theorem 1 does not hold (see [Chen] and [Her]). Our aim is to find a precise description of Bergman complete bounded Reinhardt pseudoconvex domains. Let us underline that we make no use of Theorem 1. Before we can formulate the result let us recall some standard definitions and results on pseudoconvex Reinhardt domains and let us introduce the notions necessary for formulating our theorem.

A domain $D \subset \mathbb{C}^n$ is called *Reinhardt* if $(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \in D$ for any $z \in D$, $\theta_j \in \mathbb{R}$, $j = 1, \ldots, n$.

For a point $z \in \mathbb{C}_*^n$ we put $\log|z| := (\log|z_1|, \ldots, \log|z_n|)$. We denote $\log D := \{\log|z| : z \in D \cap \mathbb{C}_*^n\}$.

Let us denote:

$$V_j := \{z \in \mathbb{C}^n : z_j = 0\},\ j = 1, \ldots, n;$$
$$V_I := V_{j_1} \cap \ldots \cap V_{j_k},\ \text{where } I = \{j_1, \ldots, j_k\},\ 1 \leqslant j_1 < \ldots < j_k \leqslant n.$$

We have the following description of pseudoconvex Reinhardt domains.

PROPOSITION 2 (see [Vla], [Jak-Jar]). — *Let $D$ be a Reinhardt domain. Then $D$ is a pseudoconvex Reinhardt domain if and only if*

$$\log D \textit{ is convex and for any } j \in \{1,\dots,n\}$$
$$\textit{if } D \cap V_j \neq \emptyset \textit{ and } (z', z_j, z'') \in D \textit{ then } (z', \lambda z_j, z'') \in D \textit{ for any } \lambda \in \bar{E}.$$

From the above description we have following properties. Assume that $D$ is a Reinhardt pseudoconvex domain and assume that for some $j \in \{1,\dots,n\}$ we have $D \cap V_j \neq \emptyset$. Then for the mapping

$$\pi_j : D \ni z \mapsto (z_1,\dots,z_{j-1},0,z_{j+1},\dots,z_n) \in V_j$$

we have the property

$$\pi_j(D) = D \cap V_j.$$

In particular, $\pi_j(D)$ is a Reinhardt pseudoconvex domain in $\mathbb{C}^{n-1}$ (after trivial identification). We may go further and we may formulate the following result.

Assume that $D \cap V_I \neq \emptyset$, $I = \{j_1,\dots,j_k\}$, $1 \leqslant j_1 < \dots < j_k \leqslant n$, $k < n$. Define the mapping $(\pi_I(z))_j := 0$ if $j \in I$ and $z_j$ otherwise. Then $\pi_I(D) = D \cap V_I$ and $\pi_I(D)$ is a Reinhardt pseudoconvex domain in $\mathbb{C}^{n-k}$.

In view of the above considerations convex domains in $\mathbb{R}^n$ play an important role in the study of pseudoconvex Reinhardt domains. It turns out that while considering different classes of holomorphic functions the special role is played by cones associated to the logarithmic image of the domain.

We say that $C \subset \mathbb{R}^n$ is a *cone with vertex at $a$* if for any $v \in C$ we have $a + t(v-a) \in C$ whenever $t > 0$. If, in the sequel, we do not specify the vertex of the cone, then we shall mean a cone with vertex at 0.

Following [Zwo 3] for a convex domain $\Omega \subset \mathbb{R}^n$ and a point $a \in \Omega$ let us define

$$\mathfrak{C}(\Omega, a) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \Omega\}.$$

It is easy to see that $\mathfrak{C}(\Omega, a)$ is a closed convex cone (with vertex at 0). Note that

$$\mathfrak{C}(\Omega, a) = \bigcup_{C + a \subset \Omega, C - \text{cone}} C = \text{the largest cone contained in } (\Omega - a).$$

Moreover, $\mathfrak{C}(\Omega, a) = \mathfrak{C}(\Omega, b)$ for any $a, b \in \Omega$. Therefore, we may well define $\mathfrak{C}(\Omega) := \mathfrak{C}(\Omega, a)$ for some (any) $a \in \Omega$.

Note that assuming $0 \in \Omega$ we have that $\mathfrak{C}(\Omega) = h^{-1}(0)$, where $h$ is the Minkowski functional of $\Omega$. It is also easy to see that

$$\mathfrak{C}(\Omega) = \{0\} \text{ if and only if } \Omega \subset\subset \mathbb{R}^n.$$

For a pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ we define $\mathfrak{C}(D) := \mathfrak{C}(\log D)$. Let us also define

$$\begin{aligned} \tilde{\mathfrak{C}}(D) &:= \{v \in \mathfrak{C}(D) : \overline{\exp(a + \mathbb{R}_+ v)} \subset D\}, \\ \mathfrak{C}'(D) &:= \mathfrak{C}(D) \setminus \tilde{\mathfrak{C}}(D), \end{aligned}$$

where $a$ is some point from $\log D$.

Let us remark that the definition of $\tilde{\mathfrak{C}}(D)$ (and, consequently, that of $\mathfrak{C}'(D)$) does not depend on the choice of $a \in \log D$ (exactly as in the case of $\mathfrak{C}(D)$). It follows easily from the above mentioned properties of pseudoconvex Reinhardt domains.

Our aim is the following result:

THEOREM 3. — *For a bounded pseudoconvex Reinhardt domain $D$ in $\mathbb{C}^n$ the following two conditions are equivalent:*

$$D \text{ is Bergman complete,} \tag{i}$$

$$\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset. \tag{ii}$$

The existence of vectors from $\mathfrak{C}'(D)$, which are rational (equivalently are from $\mathbb{Z}^n$) means that we may embed a punctured disc in $D$ in such a way that the mapping extends to a mapping defined on the whole disc and the extension maps 0 to a point from the boundary (actually, this mapping is a monomial mapping whose powers come from the element of $\mathfrak{C}'(D) \cap \mathbb{Z}^n$).

In our study the key role will be played by the following criterion:

THEOREM 4 (see [Kob]). — *Let $D$ be a bounded domain. Assume that there is a subspace $\mathcal{E} \subset L^2_h(D)$ with $\bar{\mathcal{E}} = L^2_h(D)$ such that for any $f \in \mathcal{E}$, $z^0 \in \partial D$ and for any sequence $\{z^\nu\}_{\nu=1}^\infty \subset D$ $z^\nu \to z^0$ there is a subsequence $\{z^{\nu_j}\}$ such that*

$$\frac{|f(z^{\nu_j})|}{\sqrt{K_D(z^{\nu_j})}} \to 0 \text{ as } j \to \infty. \tag{KC}$$

*Then $D$ is Bergman complete.*

As already mentioned the first non-hyperconvex pseudoconvex Reinhardt domain, which is Bergman complete was given by G. Herbort (see

[Her]). His example was the one with $\mathfrak{C}(D) = \tilde{\mathfrak{C}}(D) = \{0\} \times \mathbb{R}_-$ and his proof was based on the Kobayashi Criterion applied to some subspace defined with the help of weight functions.

The case $n = 2$ in Theorem 3 was proven in [Zwo 3]. In the present paper we use the same methods (however, simplified and generalized) as those given there; especially we verify (KC) for a linear subspace of finite linear combinations of monomials square-integrable on $D$. It is the simplest possible subspace, which may be tested in the condition (KC).

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ put $z^\alpha := z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$ for $z \in \mathbb{C}^n$ such that $z_j \neq 0$ if $\alpha_j < 0$. Then we define

$$\mathcal{E} := \operatorname{Span}\{z^\alpha : z^\alpha \in L_h^2(D)\}.$$

We know that $\bar{\mathcal{E}} = L_h^2(D)$.

Note that in order to verify the property (KC) at some $z^0$ for $\mathcal{E}$ it is sufficient to show that this property holds for all functions $z^\alpha \in L_h^2(D)$.

## 2. Auxiliary results

Below we give some results concerning the description of monomial mappings in pseudconvex Reinhardt domains and some results on diophantine approximation in convex cones, which will be crucial in our considerations. Lemmas 5–7 come from [Zwo 3]; nevertheless, for the sake of completeness we give their proofs here.

LEMMA 5 (cf. [Zwo 3]). — *Let $D$ be a pseudoconvex Reinhardt domain. Let $\alpha \in \mathbb{Z}^n$. Then*

$$z^\alpha \in L_h^2(D) \textit{ if and only if } \langle \alpha + \mathbf{1}, v \rangle < 0 \textit{ for any } v \in \mathfrak{C}(D),\ v \neq 0.$$

*Proof.* — Assume that $a = (1, \dots, 1) \in D$. First we prove the following

CLAIM. — Assume that $\mathfrak{C}(D) \neq \{0\}$. Then for any $\varepsilon > 0$ there is a cone $T$ such that $(\log D) \setminus T$ is bounded and

if $v \in T$, $||v|| = 1$ then there exists $w \in \mathfrak{C}(D)$ such that $||w|| = 1$ and $||v - w|| < \varepsilon$.

*Proof of the Claim.* — Let $h$ be a Minkowski functional of $\log D$. $\log D$ is convex, so $h$ is continuous. Recall that $h^{-1}(0) = \mathfrak{C}(D)$. From the continuity of $h$ we get that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\{w \in \mathbb{R}^n : h(w) \leqslant$

$\delta, ||w|| = 1\} \subset \{w \in \mathbb{R}^n : ||w|| = 1$ and there is $v \in \mathfrak{C}(D)$, $||v|| = 1$, $||w - v|| < \varepsilon\}$.

Now take the cone $T$ to be the smallest cone containing the set $\{w \in \mathbb{R}^n : h(w) \leqslant \delta, ||w|| = 1\}$. Note that $(\log D)\setminus T$ is bounded. If it were not the case, then there would be $x_\nu \to \infty$ such that $x_\nu \in (\log D)\setminus T$, so $h(x_\nu) < 1$, consequently $h(\frac{x_\nu}{||x_\nu||}) < \frac{1}{||x_\nu||}$, so $x_\nu \in T$ for $\nu$ large enough – contradiction. □

If $\mathfrak{C}(D) = \{0\}$ then the result is trivial. Assume that $\mathfrak{C}(D) \neq \{0\}$. Fix $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L^2_h(D)$. Let $v \in \mathfrak{C}(D)$, $v \neq 0$. We may assume that $|v_n| = 1$. There is an open bounded set $U \subset \mathbb{R}^{n-1}$ such that $0 \in U \times \{0\}$ and $U \times \{0\} + \mathbb{R}_+ v \subset \log D$. We have

$$\infty > \int_D |z^\alpha|^2 = \int_{D\cap\mathbb{C}^n_*} |z^\alpha|^2 = (2\pi)^n \int_{\log D} e^{2\langle\alpha+\mathbf{1},x\rangle} dx_1 \dots dx_n \geqslant$$

$$(2\pi)^n \int_0^\infty \Big(\int_{U\times\{0\}+x_n v} e^{2\langle\alpha+\mathbf{1},x\rangle} dx_1 \dots dx_{n-1}\Big) dx_n = M \int_0^\infty e^{2x_n\langle\alpha+\mathbf{1},v\rangle} dx_n,$$

from which we get the desired inequality $\langle\alpha + \mathbf{1}, v\rangle < 0$.

Assume now that $\langle\alpha + \mathbf{1}, v\rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$. Then there is some $\delta > 0$ such that $\langle\alpha+\mathbf{1}, v\rangle \leqslant -\delta$ for any $v \in \mathfrak{C}(D)$, $||v|| = 1$. Now using Claim we get the existence of a cone $T$ fulfilling among others the following inequality:

$$\langle\alpha + \mathbf{1}, v\rangle \leqslant -\delta/2,\ v \in T,\ ||v|| = 1$$

or

$$\langle\alpha + \mathbf{1}, v\rangle \leqslant (-\delta/2)||v||,\ v \in T.$$

It follows from the description of $T$ ($(\log D) \setminus T$ is bounded) that

$$\int_{\log D} e^{2\langle\alpha+\mathbf{1},x\rangle} dx < \infty \text{ if and only if } \int_T e^{2\langle\alpha+\mathbf{1},x\rangle} dx < \infty.$$

And now let us estimate the last expression

$$\int_T e^{2\langle\alpha+\mathbf{1},x\rangle} dx \leqslant \int_T e^{-\delta||x||} dx \leqslant \int_{\mathbb{R}^n} e^{-\delta||x||} dx < \infty,$$

which finishes the proof of the lemma. □

LEMMA 6 (cf. [Zwo 3]). — *Let $H$ be a $k$-dimensional vector subspace of $\mathbb{R}^n$ such that $H \cap \mathbb{Q}^n = \{0\}$. Let $\{v^1, \dots, v^k\}$ be a vector base of $H$. Then the set*

$$\{(\langle \alpha, v^1 \rangle, \dots, \langle \alpha, v^k \rangle) : \alpha \in \mathbb{Z}^n\}$$

*is dense in $\mathbb{R}^k$.*

*Proof.* — Certainly, $k < n$. It is easy to see that there is a vector subspace $\tilde{H} \supset H$ of dimension $n-1$ such that $\tilde{H} \cap \mathbb{Q}^n = \{0\}$. Therefore, we lose no generality assuming that $k = n-1$.

Moreover, we lose no generality assuming that for a matrix

$$\tilde{V} := \begin{bmatrix} v_1^1 & \dots & v_1^{n-1} \\ \cdot & \dots & \cdot \\ v_{n-1}^1 & \dots & v_{n-1}^{n-1} \end{bmatrix}$$

we have $\det \tilde{V} \neq 0$.

For $j = 1, \dots, n-1$ we find $t^j \in \mathbb{R}^{n-1}$ such that $\tilde{V} t^j = e^j \in \mathbb{R}^{n-1}$. Put $w^j := \sum_{k=1}^{n-1} t_k^j v^k$, $j = 1, \dots, n-1$. We have $w_l^j = \delta_{jl}$, $j, l = 1, \dots, n-1$. Certainly, $w^j \in H$, $j = 1, \dots, n-1$. It follows from the assumption of the lemma that the set $\{w_n^1, \dots, w_n^{n-1}\}$ is $\mathbb{Z}$-linearly independent (that is if $\sum_{j=1}^{n-1} s_j w_n^j \in \mathbb{Z}$ for some $s_j \in \mathbb{Z}$ then $s_j = 0$). Then in in view of multidimensional Kronecker Approximation Theorem (see e.g. [Hla-Sch-Tas]) the set

$$\{(\alpha_n w_n^1 - [\alpha_n w_n^1], \dots, \alpha_n w_n^{n-1} - [\alpha_n w_n^{n-1}]) : \alpha_n \in \mathbb{Z}\}$$

is dense in $[0,1)^{n-1}$. But $\langle \alpha, w^j \rangle = \alpha_j + \alpha_n w_n^j$; therefore,

$$\{(\langle \alpha, w^1 \rangle, \dots, \langle \alpha, w^{n-1} \rangle) : \alpha \in \mathbb{Z}^n\} \quad \text{is dense in } \mathbb{R}^{n-1}. \tag{1}$$

Put $T := [t^1, \dots, t^{n-1}] \in \mathbb{R}^{(n-1)\times(n-1)}$. We have that $\det T \neq 0$. We have that

$$[w^1, \dots, w^{n-1}] = [v^1, \dots, v^{n-1}] T.$$

Consequently,

$$(\langle \alpha, v^1 \rangle, \dots, \langle \alpha, v^{n-1} \rangle) = (\langle \alpha, w^1 \rangle, \dots, \langle \alpha, w^{n-1} \rangle) T^{-1},$$

which, in view of (1), finishes the proof of the lemma. $\square$

LEMMA 7 (cf. [Zwo 3]). — *Let $D$ be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Fix $z^0 \in \partial D$ satisfying the following condition:*

$$\textit{for any } j \in \{1,\dots,n\} \textit{ if } z_j^0 = 0 \textit{ then } D \cap V_j \neq \emptyset$$

*(this condition is satisfied if, for instance, $z^0 \in \mathbb{C}_*^n$).*

*Then the condition (KC) is satisfied at $z^0$ (for the subspace $\mathcal{E}$).*

*Proof.* — First note that for any $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L_h^2(D)$ we have that $\alpha_j \geqslant 0$ if $z_j^0 = 0$. Therefore, it is sufficient to show that $K_D(z) \to \infty$ as $z \to z^0$. Let $I := \{j : z_j^0 = 0\}$. Without loss of generality we may assume that $I = \{1,\dots,s\}$. We easily see that $s < n$. Then $D \subset \mathbb{C}^s \times \pi_I(D)$ (we identify $\pi_I(D)$ with a subset of $\mathbb{C}^{n-s}$, if $s = 0$ then $\pi_I := \mathrm{id}$). Note that the assumptions of the criterion from [Pfl 1] ('outer cone condition') are satisfied for the domain $\tilde{D}$ (and consequently also for $D$), where $\tilde{D}$ is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^{n-s}$, $\pi_I(D) \subset \tilde{D}$, $\pi_I(z^0) \in \partial\tilde{D}$ and $\partial\tilde{D}$ is $C^2$ near $\pi_I(z^0)$, which finishes the proof. The existence of such $\tilde{D}$ follows from the convexity of $\log \pi_I(D)$ and the fact that $\pi_I(z^0) \in \partial\pi_I(D) \cap \mathbb{C}_*^{n-s}$.
□

LEMMA 8. — *Let $\beta, v \in \mathbb{R}^n$, $||v|| = 1$, $\{x^\nu\}_{\nu=1}^\infty \subset \mathbb{R}^n$ be such that $||x^\nu|| \to \infty$, $\frac{x^\nu}{||x^\nu||} \to v$ as $\nu \to \infty$, and $\langle \beta, v\rangle < 0$. Then*

$$\langle x^\nu, \beta\rangle \to -\infty,\ \nu \to \infty.$$

*Proof.* — Suppose that the Lemma does not hold. Then we may assume without loss of generality that $\langle x^\nu, \beta\rangle \geqslant M$ for some $M > -\infty$, $\nu = 1, 2, \dots$. Therefore,

$$\langle \frac{x^\nu}{||x^\nu||}, \beta\rangle \geqslant \frac{M}{||x^\nu||}.$$

Passing with $\nu$ to infinity we get that $\langle v, \beta\rangle \geqslant 0$ – contradiction. □

Since $\mathfrak{C}(D) \subset \mathbb{R}_-^n$, $\mathfrak{C}(D)$ contains no straight lines. The latter property is invariant with respect to linear isomorphisms and is closely related to the hyperbolicity of a domain $D$ (see [Zwo 1]); therefore, although it may be formulated a little more generally, we assume in Lemmas 9 and 10 that the cones contain no straight lines.

LEMMA 9. — *Let $C$ be a convex closed cone such that $C \cap \mathbb{Q}^n = \{0\}$ and let $C$ contain no straight lines. Let $v \in \mathrm{int}_{\mathrm{Span}\, C} C$. Then for any $\delta > 0$ there is $\beta \in \mathbb{Z}^n$ such that:*

$$\langle \beta, v\rangle > 0,$$
$$|\langle \beta, w\rangle| < \delta \textit{ for any } w \in C,\ ||w|| = 1.$$

*Proof.* — Denote by $\mathcal{U}$ the largest vector subspace of $\operatorname{Span} C$ among those spanned by vectors from $\mathbb{Z}^n$. Because of the assumptions of the lemma we have that $v \notin \mathcal{U}$.

Let $\{v^1, \dots, v^r\}$ be a vector basis of $\operatorname{Span} C$ such that $\{v^1, \dots, v^s\}$ is a vector basis of $\mathcal{U}$ and $v^r = v$. Certainly, $s < r$. Since there is $M$ large enough such that for any $w = \sum_{j=1}^r t_j v^j \in C$, $||w|| = 1$ we have that $|t_j| \leqslant M < \infty$, $j = 1, \dots, r$, it is sufficient to find $\beta \in \mathbb{Z}^n$ such that $0 \leqslant \langle \beta, v^j \rangle < \tilde{\delta}$, $j = 1, \dots, r$ and $\langle \beta, v^r \rangle > 0$, where $\tilde{\delta} := \frac{\delta}{rM} > 0$.

Let $A \in \mathbb{Z}^{n \times n}$ be a linear isomorphism of $\mathbb{R}^n$ such that $\mathcal{U} = A(\mathbb{R}^s \times \{0\}^{n-s})$. Since $\langle \gamma, Aw \rangle = \langle A^* \gamma, w \rangle$ for any $\gamma \in \mathbb{Z}^n$, $w \in \mathbb{R}^n$ we see that we may transportate the problem to that with $\mathcal{U} = \mathbb{R}^s \times \{0\}^{n-s}$ (possibly with other value of $\tilde{\delta}$).

Therefore, we assume that $\mathcal{U} = \mathbb{R}^s \times \{0\}^{n-s}$.

Note that $\operatorname{Span} C \cap (\mathbb{R}^s \times \mathbb{Q}^{n-s}) = \mathbb{R}^s \times \{0\}^{n-s}$ and the system of vectors $\{(v^j_{s+1}, \dots, v^j_n), j = s+1, \dots, r\}$ is linearly independent. Consequently, we get the existence of $\beta \in \mathbb{Z}^n$ such that $\beta_j = 0$, $j = 1, \dots, s$ and $0 < \langle \beta, v^j \rangle < \tilde{\delta}$, $j = s+1, \dots, r$ (use Lemma 6 applied to $\pi_{1,\dots,s}(\operatorname{Span} C) \subset \{0\}^s \times \mathbb{R}^{n-s}$). □

LEMMA 10. — *Let $C \subset \mathbb{R}^n$ be a convex closed cone such that $C \cap \mathbb{Q}^n = \{0\}$ and let $C$ contain no straight lines. Then for any $\delta > 0$, $v \in C$, $v \neq 0$ there is $\beta \in \mathbb{Z}^n$ such that:*

$$\langle \beta, v \rangle > 0,$$
$$\langle \beta, w \rangle < \delta \text{ for any } w \in C,\ ||w|| = 1.$$

*Proof.* — Denote by $H = H(C, v)$ a maximal vector subspace of $\operatorname{Span} C$ among those for which $v \in \operatorname{int}_H(C \cap H)$ (one may verify that $H$ is well-defined and unique). It follows from convexity of $C$ that

$$\text{for any } w \in \operatorname{int}_H(C \cap H) \text{ we have that } H(C, v) = H(C, w). \tag{2}$$

We shall need the following:

CLAIM 1. — *There are a sequence of vector subspaces $H =: H_k \subset H_{k-1} \subset \dots \subset H_1 \subset H_0 = \operatorname{Span} C$ such that $\dim H_j = \dim H_{j+1} + 1$, $j = 0, \dots, k-1$ and vectors $v^j \in H_j \setminus H_{j+1}$ orthogonal to $H_{j+1}$ such that*

$$C \cap H_j \subset \{u + tv^j : u \in H_{j+1}, t \leqslant 0\}. \tag{3}$$

*Proof of Claim 1.*— Note that to prove Claim 1 it is sufficient to show that having given a vector subspace $H_j$ of $\operatorname{Span} C$ such that $H \subsetneq H_j$ (or $j < k$) we can find a vector subspace $H_{j+1}$ with $H \subset H_{j+1} \subset H_j$ and $\dim H_j = \dim H_{j+1} + 1$, and a vector $v^j$ such that (3) is satisfied.

There are two possibilities:

If $\operatorname{Span}(C \cap H_j) \neq H_j$ then we define $H_{j+1}$ as any vector hyperplane of $H_j$ containing $\operatorname{Span}(C \cap H_j)$.

If $\operatorname{Span}(C \cap H_j) = H_j$ then one may easily verify that $H \cap \operatorname{int}_{H_j}(C \cap H_j) = \emptyset$ (it easily follows from definition of $H$ and (2)). Then by the Hahn-Banach theorem there exists a supporting hyperplane $H_{j+1}$ of $C \cap H_j$ in $H_j$ containing $H$, from which we easily get the desired $v^j$ and (3). $\square$

It follows from Lemma 9 that there is $\tilde{\beta} \in \mathbb{Z}^n$ such that $\langle \tilde{\beta}, v \rangle > 0$ and $\langle \tilde{\beta}, w \rangle < \delta$ for any $||w|| = 1$, $w \in C \cap H$. Therefore, applying induction and Claim 1 to finish the proof of Lemma 10 it is sufficient to prove the following:

CLAIM 2.— *Assume that there is $\tilde{\beta}$ as desired in Lemma 10 for $C \cap H_{j+1}$ and $v$ ($j < k$). Then there is $\beta \in \mathbb{Z}^n$ as desired in Lemma 10 for $C \cap H_j$ and $v$ (the subspaces $H_j$ and $H_{j+1}$ are those appearing in Claim 1).*

*Proof of Claim 2.*— Put $M_1 := \sup\{\langle \tilde{\beta}, w \rangle : w \in H_{j+1}, ||w|| = 1\} < \infty$.

In view of the Dirichlet pigeon-hole theorem (see e.g. [Har-Wri]) we have that for any positive integer $N$ there are $\beta_l^N, q = q(N) \in \mathbb{Z}$, $q > 0$ such that $\beta_l^N - qv_l^j = \varepsilon(l, N) \in (-1/N, 1/N)$, $l = 1, \dots, n$; moreover, $q$ may be chosen so that it tends to infinity as $N$ tends to infinity. Denote $\varepsilon^N := (\varepsilon(1, N), \dots, \varepsilon(n, N))$. Then we have $\beta^N = qv^j + \varepsilon^N$.

We claim that for large $N$ $\beta := \tilde{\beta} + \beta^N$ satisfies the desired property.

First note that because $v^j$ is orthogonal to $v$ ($v \in H \subset H_{j+1}$) we have

$$\langle \tilde{\beta} + \beta^N, v \rangle = \langle \tilde{\beta}, v \rangle + \langle \varepsilon^N, v \rangle.$$

Since the second summand in the formula above tends to 0 as $N$ goes to infinity the last expression is positive for $N$ large enough.

Suppose that the second property of the lemma does not hold for infinetly many $N$, i.e. without loss of generality we may write that for any $N$ there are

$$t^N \leqslant 0,\ u^N \in H_{j+1},\ w^N = u^N + t^N v^j \in C,\ ||w^N|| = 1$$

such that $\langle \tilde{\beta} + \beta^N, w^N \rangle \geqslant \delta$.

Take $\tilde{\delta} < \delta$ such that $\langle\tilde{\beta}, u\rangle \leqslant \tilde{\delta}$, $u \in H_{j+1} \cap C$, $||u|| = 1$.

There is $M_2$ such that for any $N$ we have $||u^N|| \leqslant M_2$, $-M_2 \leqslant t^N \leqslant 0$.

Then we have ($v^j$ is orthogonal to $u^N$)

$$\delta \leqslant \langle\tilde{\beta}+\beta^N, w^N\rangle = \langle\tilde{\beta}, u^N\rangle + t^N\langle\tilde{\beta}, v^j\rangle + \langle\varepsilon^N, u^N\rangle + t^N\langle\varepsilon^N, v^j\rangle + t^N q\langle v^j, v^j\rangle. \tag{4}$$

Without loss of generality we may assume that $t^N \to t$. Moreover, we may assume that $u^N \to u \in H_{j+1}$ and, therefore, $w^N \to w \in C$, $||w|| = 1$.

We claim that $t = 0$. Suppose the contrary, so $t < 0$. Then the first four summands in the last expression of (4) are bounded from above and the last expression tends to $-\infty$ – contradiction.

Consequently, $w^N \to u$, $||u|| = 1$, $u \in C \cap H_{j+1}$. Note that in view of (4) (making use of the fact that $t^N \leqslant 0$, $q > 0$), we have

$$\delta \leqslant \langle\tilde{\beta}, u^N\rangle + t^N\langle\tilde{\beta}, v^j\rangle + \langle\varepsilon^N, u^N\rangle + t^N\langle\varepsilon^N, v^j\rangle.$$

Passing with $N$ to infinity we get $\delta \leqslant \langle\tilde{\beta}, u\rangle \leqslant \tilde{\delta}$ – contradiction. □ □

## 3. Proof of Theorem 3

The proof of implication ((i)$\Longrightarrow$(ii)) is simple and is to find in [Zwo 3]. For the sake of completeness we give it here, too.

*Proof of implication ((i)$\Longrightarrow$(ii)).* — Suppose that there is $v \in \mathfrak{C}'(D) \cap \mathbb{Q}^n$. Certainly, $v \neq 0$. We assume that $a \in \log D$ from the definition of $\mathfrak{C}(D)$ is equal to $(0, \ldots, 0)$. Without loss of generality we may assume that $v \in \mathbb{Z}^n_-$ and $v_1, \ldots, v_n$ are relatively prime.

It is sufficient to show that the Bergman length $L_{\beta_D}$ of the curve $(t^{-v_1}, \ldots, t^{-v_n})$, $0 < t < 1$ is finite.

Denote $\varphi(\lambda) := (\lambda^{-v_1}, \ldots, \lambda^{-v_n})$, $\lambda \in E_*$. Certainly, $\varphi \in \mathcal{O}(E_*, D)$. Put $u(\lambda) := K_D(\varphi(\lambda))$. Then we have (use Lemma 5)

$$u(\lambda) = \sum_{\alpha\in\mathbb{Z}^n:\langle\alpha+1,v\rangle<0} a_\alpha|\lambda|^{-2\langle\alpha,v\rangle} = \sum_{j=j_0}^{\infty} b_j|\lambda|^{2j},$$

where $b_{j_0} \neq 0$ (note that $j_0 > \langle 1, v\rangle$ and it is possible that many of $b_j$'s in the formula above vanish).

Note that

$$\beta_D^2(\varphi(\lambda);\varphi'(\lambda)) = \frac{\partial^2 \log u(\lambda)}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2}{\partial\lambda\partial\bar{\lambda}}\left(\log\sum_{j=j_0}^{\infty} b_j|\lambda|^{2j-2j_0}\right).$$

The last expression tends to some constant $C \in \mathbb{R}$, which finishes the proof. □

*Proof of implication ((ii)$\Longrightarrow$(i)).* — We prove that the condition (KC) is satisfied in all $z^0 \in \partial D$ for $\mathcal{E}$. Recall that we already know that (KC) is satisfied for all points $z^0 \in \partial D$ such that

$$\text{for any } j \text{ (if } z_j^0 = 0 \text{ then } V_j \cap D \neq \emptyset) \tag{5}$$

(see Lemma 7); in particular, for all points from $\partial D \cap \mathbb{C}_*^n$.

Additionally, note that to prove the property (KC) (at $z^0$) it is sufficient to consider sequences $\{z^\nu\}_{\nu=1}^\infty \subset D \cap \mathbb{C}_*^n$.

Without loss of generality $(1,\dots,1) \in D$. Take some $v \in \mathfrak{C}(D) \cap \mathbb{Q}^n$, $v \neq 0$. Since $v \in \tilde{\mathfrak{C}}(D)$ (assumption of the theorem), we get from definition of $\tilde{\mathfrak{C}}(D)$ that

$$\lim_{t\to\infty}(\exp(tv_1),\dots,\exp(tv_n)) =: w \in D.$$

Note that $w_j = 0$ if $v_j < 0$ and $w_j = 1$ if $v_j = 0$. In particular,

$$\text{for any } v \in \mathfrak{C}(D) \cap \mathbb{Q}^n \text{ (if } v_j < 0 \text{ then } D \cap V_j \neq \emptyset). \tag{6}$$

Without loss of generality we may assume that $D \cap V_j \neq \emptyset$, $j = 1,\dots,k$, $D \cap V_j = \emptyset$, $j = k+1,\dots,n$. Because we are interested in these $z^0$ for which (5) is not satisfied we may assume that $k < n$.

In view of our assumptions (and properties of pseudoconvex Reinhardt domains) we know that

$$\mathbb{R}_-^k \times \{0\}^{n-k} \subset \mathfrak{C}(D). \tag{7}$$

Note that

$$\text{for any } v \in \mathfrak{C}(D) \setminus (\mathbb{R}^k \times \{0\}^{n-k}) \text{ we have that } v \notin \mathbb{R}^k \times \mathbb{Q}^{n-k}. \tag{8}$$

Actually, suppose that there exists $v \in \mathfrak{C}(D) \cap (\mathbb{R}_-^k \times \mathbb{Q}_-^{n-k})$, $v_j < 0$ for some $j > k$. Then adding some vector from $\mathbb{R}_-^k \times \{0\}^{n-k}$ we get a vector (we denote it with the same letter) from $\mathfrak{C}(D) \cap \mathbb{Q}^n$ (with $v_j < 0$ for some $j > k$). Then in view of (6) $D \cap V_j \neq \emptyset$ – contradiction.

Denote $\tilde{x} := \pi(x)$, where $\pi(x) := (0, \ldots, 0, x_{k+1}, \ldots, x_n)$, $x \in \mathbb{R}^n$. Note that $\pi(\mathfrak{C}(D))$ is a closed convex cone in $\{0\}^k \times \mathbb{R}_-^{n-k}$ and in view of (8) $\pi(\mathfrak{C}(D)) \cap (\{0\}^k \times \mathbb{Q}^{n-k}) = \{0\}$.

Consider a point $z^0 \in \partial D$, not satisfying (5), in particular,

$$\text{there is } j > k \text{ such that } z_j^0 = 0, \tag{9}$$

and a sequence $z^\nu \to z^0$, $z^\nu \in D \cap \mathbb{C}_*^n$. Put $x^\nu := \log |z^\nu|$. Without loss of generality we may assume that $\frac{x^\nu}{||x^\nu||} \to v \in \mathfrak{C}(D)$. Certainly, $||x^\nu|| \to \infty$.

Fix $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L_h^2(D)$. Define $\delta := \inf\{-\langle \alpha + 1, w\rangle : w \in \mathfrak{C}(D), ||w|| = 1\} > 0$ (use Lemma 5).

Below we consider two cases:

Case (I). $v_j < 0$ for some $j > k$.

We claim that it is sufficient to find $\beta \in \mathbb{Z}^n$ such that

$$\langle \beta, w\rangle < \delta \text{ for any } w \in \mathfrak{C}(D),\ ||w|| = 1 \text{ and } \langle \beta, v\rangle > 0. \tag{10}$$

In fact, then $z^{\alpha+\beta} \in L_h^2(D)$ (use Lemma 5) and

$$\frac{|(z^\nu)^\alpha|}{\sqrt{K_D(z^\nu)}} \leqslant ||z^{\alpha+\beta}||_{L^2(D)} \frac{|(z^\nu)^\alpha|}{|(z^\nu)^{\alpha+\beta}|} = ||z^{\alpha+\beta}||_{L^2(D)} |(z^\nu)^{-\beta}|.$$

The last expression tends to zero (use Lemma 8).

Therefore, we prove the existence of $\beta \in \mathbb{Z}^n$ such that (10) is satisfied.

Use Lemma 10 (applied to $\pi(\mathfrak{C}(D))$ and $\pi(v)$) to get the existence of $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$ such that $\langle \beta, v\rangle = \langle \beta, \pi(v)\rangle > 0$ and $\langle \beta, w\rangle = ||\pi(w)|| \langle \beta, \frac{\pi(w)}{||\pi(w)||}\rangle < \delta$ for any $w \in \mathfrak{C}(D)$, $||w|| = 1$ with $\pi(w) \neq 0$. Since $\langle \beta, w\rangle = 0$ if $\pi(w) = 0$, this finishes the proof.

Case (II). $v_{k+1} = \ldots = v_n = 0$.

Put $\tilde{x} := \pi(x)$. Without loss of generality $\frac{\tilde{x}^\nu}{||\tilde{x}^\nu||} \to \tilde{w} \in \{0\}^k \times \mathbb{R}^{n-k}$. In view of (9) we have $||\tilde{x}^\nu|| \to \infty$.

Note that it is sufficient to find $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$ such that $\langle \beta, \tilde{w}\rangle > 0$ and $\langle \beta, w\rangle = ||\pi(w)|| \langle \beta, \frac{\pi(w)}{||\pi(w)||}\rangle < \delta$, where $\delta$ is as earlier, $w \in \mathfrak{C}(D)$, $w \neq 0$. Then similarly as earlier we have that $z^{\alpha+\beta} \in L_h^2(D)$ and

$$\frac{|(z^\nu)^\alpha|}{\sqrt{K_D(z^\nu)}} \leqslant ||z^{\alpha+\beta}||_{L^2(D)} |(z^\nu)^{-\beta}| = ||z^{\alpha+\beta}||_{L^2(D)} |(z_{k+1}^\nu, \ldots, z_n^\nu)^{-(\beta_{k+1}, \ldots, \beta_n)}|.$$

The last expression tends to zero (remember about convergence $||\tilde{x}^\nu|| \to \infty$ and then use Lemma 8).

If $\tilde{w} \in \pi(\mathfrak{C}(D))$ then the existence of such $\beta$ follows from Lemma 10 applied to $\pi(\mathfrak{C}(D))$ and $\tilde{w}$.

If $\tilde{w} \notin \pi(\mathfrak{C}(D))$ then define $\tilde{C}$ to be the smallest convex cone containing $\mathfrak{C}(D)$ and $-\tilde{w}$. It is easy to see that $\tilde{C} \neq \{0\}^k \times \mathbb{R}^{n-k}$ (e.g. $\tilde{w} \notin \tilde{C}$). Consequently, the set

$$\{\beta \in \{0\}^k \times \mathbb{R}^{n-k} : \langle \beta, u\rangle < 0, u \in \tilde{C} \setminus \{0\}\}$$

is a non-empty convex open cone (in $\{0\}^k \times \mathbb{R}^{n-k}$). Therefore, it contains $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$. In particular, $\langle \beta, -\tilde{w}\rangle < 0$, $\langle \beta, \frac{\pi(w)}{||\pi(w)||}\rangle < 0 < \delta$ for any $w \in \mathfrak{C}(D)$, $\pi(w) \neq 0$, which finishes the proof. □

## 4. Concluding remarks

In view of the results from [Pfl 2], [Fu] and [Zwo 2] we may precisely give the relations between hyperconvexity, Carathéodory completeness, Kobayashi completeness and Bergman completeness in the class of bounded pseudoconvex Reinhardt domains.

Summarizing, the following properies are satisfied:

– all bounded Reinhardt pseudoconvex domains are Kobayashi complete;

– hyperconvexity is equivalent to Carathéodory completeness and they are equivalent to the following property:

$$\text{for any } j = 1, \dots, n \text{ if } \bar{D} \cap V_j \neq \emptyset \text{ then } D \cap V_j \neq \emptyset;$$

– Bergman completeness is equivalent to the equality $\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset$.

In view of the above remarks we may easily produce a great variety of bounded Reinhardt domains, which are Bergman complete but not hyperconvex.

Note that the proper choice of the subspace $\mathcal{E}$ and the Kobayashi criterion reduce the problem of the proof of Bergman completeness of the considered domains to a problem from diophantine approximation on cones $\mathfrak{C}(D)$.

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