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Existence, uniqueness and stabilization for a nonlinear plate system with nonlinear damping(*)

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RÉSUMÉ. — On étudie l'existence et l'unicité des solutions globales et on établit l'amortissement polynomial de l'énergie pour un système de plaque non linéaire

$$\begin{cases} u_{tt} + \Delta^2 u - \varphi\left(\displaystyle\int_\Omega \left|\nabla u\right|^2 \mathrm{d}x\right)\Delta u + g(u_t) = 0 & \text{in } \Omega \times (0,+\infty) \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \times (0,+\infty) \\ u(0) = u_0 \quad u_t(0) = u_1 & \end{cases}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 3$, à frontière régulière Γ et on considère la fonction φ une fois continûment différentiable avec $\varphi(s) \geq 0$, pour tout $s \geq 0$ et g est continue et non décroissante.

ABSTRACT. — We study the existence and uniqueness of global solutions and the polynomial decay of the energy for the following nonlinear plate system with nonlinear damping

$$\begin{cases} u_{tt} + \Delta^2 u - \varphi\left(\displaystyle\int_\Omega \left|\nabla u\right|^2 \mathrm{d}x\right)\Delta u + g(u_t) = 0 & \text{in } \Omega \times (0,+\infty) \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \times (0,+\infty) \\ u(0) = u_0 \quad u_t(0) = u_1 & \end{cases}$$

where Ω is an open subset of $\mathbb{R}^N$, $N \leq 3$, with smooth boundary Γ and φ is a continuously differentiable real valued function which satisfies $\varphi(s) \geq 0$, for all $s \geq 0$ and g a continuous nondecreasing real function.

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1. Introduction

In this paper we consider the following nonlinear plate system with nonlinear damping:

$$\begin{cases} u_{tt} + \Delta^2 u - \varphi\left(|\nabla u|^2\right) \Delta u + g(u_t) = 0 & \text{in } \Omega \times (0,+\infty) \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \times (0,+\infty) \\ u(0) = u_0 \quad u_t(0) = u_1 \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N \leq 3$) having a smooth boundary $\partial\Omega = \Gamma$ and φ is a continuously differentiable real valued function, which satisfies $\varphi(s) \geq 0$, for all $s \geq 0$. We also consider g a continuous nondecreasing function and we use the notation

$$|\nabla u|^2 = \sum_{j=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_j}(x) \right|^2 \mathrm{d}x \,.$$

The following problem

$$\begin{cases} u_{tt} + \Delta^2 u - \varphi\left(|\nabla u|^2\right) \Delta u = 0 & \text{in } \Omega \times (0,+\infty) \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \times (0,+\infty) \\ u(0) = u_0 \quad u_t(0) = u_1 \end{cases} \tag{1.2}$$

in dimension $N = 1$, is a general mathematical formulation of a problem arising in the dynamic buckling of a hinged extensible beam under an axial force (Eisley [3] and Dickey [2]).

It is interesting to observe that the nonlinearity $\varphi\left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)$ is an approximation by averaging of the classical von Kármán model (Nayfeh–Mook [19]).

The existence of solutions for the Problem (1.2), in the one-dimensional case was studied by Dickey [2]. Pohozaev [20] considered the existence of solutions of the corresponding hyperbolic systems in which, in particular, the fourth order term $\Delta^2 u$ is dropped. In higher dimensions, regular unique solutions were obtained by Medeiros [15].

When the equation of (1.2) does not include the term $\Delta^2 u$ and the dimension $N = 1$, this system describes the vibrations of an elastic stretched string. This case was studied by Lions [14], when $\varphi(s) \geq m_0 > 0$ and under the same assumption, local strong solutions were obtained by Rivera [22]. Menzala [17] solved this problem when $\Omega = \mathbb{R}^N$.

Vasconcellos–Teixeira [25] proved the existence and uniqueness of global strong solutions and the exponential decay of the energy for the following nonlinear damped hyperbolic problem:

$$\begin{cases} u_{tt} - \varphi\left(\left|\nabla u\right|^2\right) \Delta u + g(u) - \Delta u_t = 0 & \\ u = 0 & \text{on } \Gamma \times (0, +\infty) \\ u(0) = u_0 \quad u_t(0) = u_1 \,. & \end{cases} \tag{1.3}$$

The Problem (1.3), without the term $g(u)$, was solved by Medeiros–Miranda [16] and the exponential decay for the energy was proved.

We would like to make a few comments on the related literature. Lagnese [8] considered the modelling of nonlinear plates and studied the stabilization for these systems. We also refer to the book by J. Lagnese [9] for a systematic study of the stabilization of plate systems.

For a study on stability for nonlinear waves and beams with damping on the boundary, we refer, for example, to the works of Lasiecka [12], Lasiecka–Triggiani [13] and Lagnese–Leugering [10], Kormonik [7] and Zuazua [27].

The von Kármán system was studied, for example, by Horn–Lasiecka [6], where the existence and uniqueness of global solutions for the system with nonlinear boundary dissipation was proved. In Horn [5] and Puel–Tucsnak [21] the boundary stabilization for the system was showed. Recently, Bisognin, *et al.* [1] and Menzala–Zuazua [18] proved existence of global solutions and the exponential stability for von Kármán system of plates, in the presence of thermal effects.

The aim of this work is to study the global existence of solutions and the stabilization of (1.1).

The existence and uniqueness of solutions is proved in Section 2, where no restriction is placed on the growth of the damping term g, namely, g is only assumed to be a continuous nondecreasing function with $g(0) = 0$. There, we use Galerkin methods and since g does not have further restrictions, we need special estimates to obtain the convergence of nonlinear terms and we must also consider the dimension $N \leq 3$. This is not relevant from a physical point of view since the system (1.1) is a model built for $N = 2$.

We can find in Lan *et al.* [11] the existence and uniqueness of global solutions for (1.1), when the damping g is a very particular function, namely, $g(s) = |s|^{\alpha-1} s$ with $0 < \alpha < 1$. Therefore, they only consider nonlinearities going at infinity in a sublinear way, so, in this case, it is easy to show that the function g is continuous from $L^2(\Omega)$ into $L^2(\Omega)$.This is not the case in the context we are working on.

In Section 3, we study the stabilization of (1.1) when the damping g satisfies convenient assumptions to control its growth.

Let us consider the energy

$$E(t) = \frac{1}{2}\,\Big[|u_t|^2 + \widehat{\varphi}\Big(|\nabla u|^2\Big) + |\Delta u|^2\Big] \tag{1.4}$$

where u is a regular solution of (1.1) and

$$\widehat{\varphi}(t) = \int_0^t \varphi(s)\,\mathrm{d}s\,.$$

We can prove that

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2} \int_\Omega g\big(u_t(x,s)\big)\, u_t(x,s)\,\mathrm{d}x\,\mathrm{d}s \tag{1.5}$$

and therefore, the energy is a decreasing function of time.

Now, to prove the uniform polynomial decay of the energy (1.4) we perturbe it with a convenient Lyapunov function. We show that the resultant perturbed energy is equivalent to the energy (1.4). The origin of this method can be found in Zuazua [25] (see also [26]).

Finally, in Section 4 some remarks about Problem (1.1) will be made.

2. The existence theorem

Let H be the $L^2(\Omega)$ space with the usual norm $|\cdot|$. W is the Sobolev space $H_0^1(\Omega)$ normed by $\|u\|_W = |\nabla u|$ and V means the space $H_0^2(\Omega)$ with the norm $\|u\|_V = |\Delta u|$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\begin{cases} g(0) = 0 \\ g \text{ is a nondecreasing function.} \end{cases} \tag{2.1}$$

We consider $\varphi \in C^1([0\,,+\infty)\,;\mathbb{R})$ such that

$$\varphi(s) \geq 0 \quad \text{for } s \geq 0\,. \tag{2.2}$$

For each $T > 0$ we put $Q = \Omega \times (0,T)$.

THEOREM 2.1.— *Under the above assumptions, if* $(u_0, u_1) \in V \times H$ *and for every* $T > 0$*, there exists an unique* $u : [0\,,T] \to H$ *such that*

$$u \in C([0\,,T]\,;V) \cap C^1([0\,,T]\,;H) \tag{2.3}$$

$$u_{tt} \in L^1(0,T\,;V') \tag{2.4}$$

$$u_t g(u_t) \in L^1(Q) \quad g(u_t) \in L^1(0,T\,;V') \tag{2.5}$$

$$u_{tt}(t) + \Delta^2 u(t) - \varphi\Big(\|u(t)\|_W^2\Big)\,\Delta u(t) + g\big(u_t(t)\big) = 0 \quad \textit{in } V' \textit{ a.e. in } t \tag{2.6}$$

$$u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) \qquad \textit{in } \Omega\,. \tag{2.7}$$

Proof.— First of all we establish the existence. Let $T > 0$ be fixed and denote by V_m the space generated by $\{w_1, w_2, \ldots, w_m\}$ where the set $\{w_m\,;\, m \in \mathbb{N}\}$ is a "basis" of V. Since $(u_0, u_1) \in V \times H$ we obtain

$$u_0 = \sum_{m=1}^{\infty} a_m w_m \text{ in } V \quad \text{and} \quad u_1 = \sum_{m=1}^{\infty} b_m w_m \text{ in } H\,. \tag{2.8}$$

Then for all $m \in \mathbb{N}$ there exists $T_m \leq T$ and $u_m(t) = \sum_{j=1}^{m} f_j(t) w_j$ defined for $t \in [0\,,T_m)$ such that the following holds:

$$\begin{cases} \big(u_m''(t) \mid v\big) + \big(Au_m(t) \mid Av\big) + \varphi\Big(\|u_m(t)\|_W^2\Big)\big(Au_m(t) \mid v\big) + \\ \qquad\qquad + \Big(g\big(u_m'(t)\big) \mid v\Big) = 0\,, \quad v \in V_m \\ u_m(0) = u_{0m} \quad u_m'(0) = u_{1m} \end{cases} \tag{2.9}$$

where $(\cdot \mid \cdot)$ denotes the usual inner product in H, $A = -\Delta$, $u_{0m} = \sum_{j=1}^{m} a_j w_j$, $u_{1m} = \sum_{j=1}^{m} b_j w_j$ and V' is the dual space of V.

Remark 2.1.— Since $N \leq 3$, we have that V is compactly embedded in $L^\infty(\Omega)$, hence, by (2.1), for each $t > 0$ and $m \in \mathbb{N}$, $g\big(u_m'(t)\big) \in L^\infty(\Omega) \subset H \subset L^1(\Omega)$.

We need *a priori* estimates to extend the solution of the system (2.9) to the whole interval $[0\,,T]$.

Replacing v in (2.9) by $2u'_m(t)$ and integrating from 0 to T_m we have:

$$\left|u'_m(t)\right|^2 + \left\|u_m(t)\right\|_V^2 + \widehat{\varphi}\Big(\left\|u_m(t)\right\|_W^2\Big) + 2\int_0^t \Big(g\big(u'_m(s)\big) \mid u'_m(s)\Big)\,\mathrm{d}s =$$
$$= \left|u_{1m}\right|^2 + \left\|u_{0m}\right\|_V^2 + \widehat{\varphi}\Big(\left\|u_{0m}\right\|_W^2\Big)\,.$$

Since $\{u_{1m}\}$ and $\{u_{0m}\}$ are bounded sequences in H and V respectively, then, by (2.1) and (2.2) there exists a constant $K > 0$ independent of m and t, such that, for all $m \in \mathbb{N}$ and $t \geq 0$

$$\left|u'_m(t)\right|^2 \leq K \tag{2.10}$$

$$\left\|u_m(t)\right\|_V^2 \leq K \tag{2.11}$$

$$\widehat{\varphi}\Big(\left\|u_m(t)\right\|_W^2\Big) \leq K \tag{2.12}$$

$$\int_0^t \Big(g\big(u'_m(s)\big) \mid u'_m(s)\Big)\,\mathrm{d}s \leq K\,. \tag{2.13}$$

Now, we need the following lemmas.

LEMMA 2.1.— *There exists $K_1 > 0$ such that $\left\|g(u'_m)\right\|_{L^1(Q)} \leq K_1$ for all $m \in \mathbb{N}$.*

Proof.— If we define

$$A_m = \Big\{(x,t) \in Q \;\Big|\; \left|u'_m(x,t)\right| \leq 1\Big\}$$

and

$$B_m = \Big\{(x,t) \in Q \;\Big|\; \left|u'_m(x,t)\right| > 1\Big\}\,,$$

then, from (2.1):

$$\int_0^T \int_\Omega \Big|g\big(u'_m(x,t)\big)\Big|\,\mathrm{d}x\,\mathrm{d}t =$$
$$= \int_{A_m}\!\!\int \Big|g\big(u'_m(x,t)\big)\Big|\,\mathrm{d}x\,\mathrm{d}t + \int_{B_m}\!\!\int \Big|g\big(u'_m(x,t)\big)\Big|\,\mathrm{d}x\,\mathrm{d}t \leq$$
$$\leq \int_0^T \int_\Omega \sup_{|s|\leq 1} \left|g(s)\right|\,\mathrm{d}x\,\mathrm{d}t + \int_{B_m}\!\!\int g\big(u'_m(x,t)\big)u'_m(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Hence, by (2.13), we have

$$\int_0^T \int_\Omega \left| g\big(u_m'(x,t)\big)\right| \,\mathrm{d}x\,\mathrm{d}t \le \mu(Q) \sup_{|s|\le 1} \left|g(s)\right| + K \quad \text{for } m \in \mathbb{N}$$

which concludes the lemma. By $\mu(\,\cdot\,)$ we denote the Lebesgue measure in $\mathbb{R}^{N+1}$. □

LEMMA 2.2.— *The sequences $\{u_m'(\,\cdot\,)\}$ and $\{u_m(\cdot)\}$ are Cauchy sequences in $C([0,T];H)$ and $C([0,T];V)$, respectively.*

Proof.— Let $m_2 \ge m_1$ be two natural numbers and

$$\mu_j(t) = \varphi\Big(\left\|u_{m_j}(t)\right\|_W^2\Big), \quad j = 1, 2.$$

Then, by (2.9), $\omega(t) = u_{m_2}(t) - u_{m_1}(t)$ satisfies

$$\begin{aligned} &\big(\omega''(t) \mid v\big) + \big(A\omega(t) \mid Av\big) + \mu_2(t)\big(A\omega(t) \mid v\big) + \\ &\qquad + \Big(g\big(u_{m_2}'(t)\big) - g\big(u_{m_1}'(t)\big) \mid v\Big) = \\ &= -\big(\mu_2(t) - \mu_1(t)\big)\big(Au_{m_1}(t) \mid v\big) \quad \text{for } v \in V_m\,. \end{aligned} \tag{2.14}$$

If we take, in (2.14), $v = 2\omega'$ then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G(t) = \mu_2'(t)\left\|\omega(t)\right\|_W^2 - 2\big(\mu_2(t) - \mu_1(t)\big)\big(Au_{m_1}(t) \mid \omega'\big) + \\ - 2\Big(g\big(u_{m_2}'(t)\big) - g\big(u_{m_1}'(t)\big) \mid \omega'\Big) \end{aligned} \tag{2.15}$$

where

$$G(t) = \left|\omega'(t)\right|^2 + \left\|\omega(t)\right\|_V^2 + \mu_2\left\|\omega(t)\right\|_W^2.$$

Now, by (2.11) and the fact that φ is a function of class C^1, we have constants $C > 0$ and $K > 0$ such that

$$\begin{aligned} &\left|\mu_2(t) - \mu_1(t)\right| \le C\left\|\omega(t)\right\|_W \quad \text{for } t \in [0,T], \\ &\left|\big(Au_{m_1}(t) \mid \omega'(t)\big)\right| \le K\left|\omega'(t)\right|. \end{aligned} \tag{2.16}$$

Hence, from (2.1), (2.15) and (2.16) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G(t) &\le \left|\mu_2'(t)\right| \left\|\omega(t)\right\|_W^2 + 2CK\left\|\omega(t)\right\|_W\left|\omega'(t)\right| \le \\ &\le \Big(c_1\left|\mu_2'(t)\right| + c_2\Big)G(t)\,. \end{aligned} \tag{2.17}$$

Since

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_{m_2}(t)\right\|_W^2\right| \le 2\left\|u_{m_2}(t)\right\|_V \left|u'_{m_2}(t)\right|$$

it follows, by (2,2), (2.10) and (2.11), that

$$\left|\mu_2'(t)\right| \le c \quad \text{for } t > 0\,. \tag{2.18}$$

Replacing (2.18) in (2.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,G(t) \le K_2\,G(t) \quad \text{for } t > 0\,.$$

So,

$$G(t) \le e^{K_2 T}\,G(0)\,, \quad t \in [\,0\,,T\,]\,. \tag{2.19}$$

On the other hand, by (2.2) and (2.8)

$$G(0) \to 0 \quad \text{as } m_1,\, m_2 \to +\infty\,.$$

Then, by (2.2) and (2.19) we prove the Lemma 2.2. □

By Lemma 2.2 there exists $u : [\,0\,,T\,] \to V$ such that

$$u_m \to u \quad \text{in } C\big([\,0\,,T\,]\,;\,V\big) \tag{2.20}$$

$$u'_m \to u_t \quad \text{in } C\big([\,0\,,T\,]\,;\,H\big)\,. \tag{2.21}$$

So (2.20) and the continuity of φ:

$$\varphi\Big(\left\|u_m(t)\right\|_W^2\Big) \longrightarrow \varphi\Big(\left\|u(t)\right\|_W^2\Big) \quad \text{in } C\big([\,0\,,T\,]\big)\,. \tag{2.22}$$

LEMMA 2.3.— *For each $T > 0$, $g(u_t) \in L^1(Q)$ and $\left\|g(u_t)\right\|_{L^1(Q)} \le K_1$, where K_1 is obtained in Lemma 2.1.*

Proof.— By (2.1) and (2.21) there exists a subsequence $\{u'_\nu\}$ of $\{u'_m\}$ such that

$$g\big(u'_\nu(x,t)\big) \longrightarrow g\big(u_t(x,t)\big) \quad \text{a.e. in } Q \tag{2.23}$$

$$0 \le g\big(u'_\nu(x,t)\big)\,u'_\nu(x,t) \longrightarrow g\big(u_t(x,t)\big)\,u_t(x,t) \quad \text{a.e. in } Q\,.$$

Hence, by (2.13) and Fatou's lemma we have

$$\int_0^T \int_\Omega u_t(x,t) g\big(u_t(x,t)\big)\, \mathrm{d}x\, \mathrm{d}t \le K \quad \text{for } T > 0\,. \tag{2.24}$$

Now, using (2.24), the proof follows similarly as Lemma 2.1. □

LEMMA 2.4.— *$g(u'_\nu) \to g(u_t)$ in $L^1(0,T\,;\,V')$.*

Proof.— To prove this Lemma we need the following Theorem.

THEOREM (Strauss [24]).— *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite Lebesgue measure, and $\big(w_\nu\big)_{\nu\in\mathbb{N}}$ a sequence of measurable functions, $w_\nu : \Omega \to \mathbb{R}$, $\nu \in \mathbb{N}$.*

Let $f_\nu : \mathbb{R} \to \mathbb{R}$, $\nu \in \mathbb{N}$ be such that

(a) f_ν, $\nu \in \mathbb{N}$ is bounded on bounded subsets of $\mathbb{R}$;

(b) $f_\nu \circ w_\nu$ is measurable and

$$\int_\Omega \Big| f_\nu\big(w_\nu(x)\big)\Big|\,\big|w_\nu(x)\big|\, \mathrm{d}x \le C\,, \quad \forall\, \nu \in \mathbb{N}\,;$$

(c) $f_\nu \circ w_\nu \to v$ a.e. in Ω.

Then $v \in L^1(\Omega)$ and

$$\lim_{\nu\to+\infty} \int_\Omega \Big| f_\nu\big(w_\nu(x)\big) - v(x)\Big|\, \mathrm{d}x = 0\,.$$

From the assumptions on g, (2.13) and (2.23), it follows, by the above theorem, that

$$g(u'_\nu) \longrightarrow g(u_t) \quad \text{in } L^1(Q)\,. \tag{2.25}$$

So, since $L^1(\Omega)$ is continuously embedded on V', we obtain the Lemma 2.4. □

To prove (2.6), we first observe that, by (2.20)

$$\Delta^2 u_m \to \Delta^2 u \quad \text{in } C\big([\,0\,,T\,]\,;\,V'\big)\,. \tag{2.27}$$

Fixing $j \leq \nu$ and integrating (2.9) from 0 to $t < T$ we have

$$\big(u'_\nu(t) \mid w_j\big) = \big(u_{1\nu} \mid w_j\big) - \left\{ \int_0^t \langle \Delta^2 u_\nu(s) \mid w_j \rangle_{V'V} \, ds + \right.$$
$$+ \int_0^t \varphi\Big(\|u_\nu(s)\|_W^2 \Big) \big(A u_\nu(s) \mid w_j\big) \, ds +$$
$$\left. + \int_0^t \langle g(u_\nu(s)) \mid w_j \rangle_{V'V} \, ds \right\} .$$

Using (2.20), (2.21), (2.22), (2.27) and the density of the set $\{w_m \, ; \, m \in \mathbb{N}\}$ in V, we can pass to the limit as $\nu \to +\infty$ to obtain

$$\big(u_t(t) \mid v\big) = \big(u_1 \mid v\big) +$$
$$- \left\{ \int_0^t \Big(\langle \Delta^2 u(s) \mid v \rangle_{V'V} + \varphi\Big(\|u(s)\|_W^2 \Big) \big(A u(s) \mid v\big) \Big) \, ds + \right.$$
$$\left. + \int_0^t \Big\langle g\big(u_t(s)\big) \mid v \Big\rangle_{V'V} \right\} ds \quad \text{for } v \in V$$

and so we can obtain (2.6).

Finally, by (2.8), (2.20) and (2.21) we have (2.7). □

We end by proving the uniqueness. Consider $u, v : [0, T] \to H$ satisfying (2.3)–(2.7) and define $w(t) = u(t) - v(t)$, $t \in [0, T]$.

Then, by (2.6) and (2.7) we have

$$w_{tt} + A^2 w + \varphi\Big(\|u\|_W^2 \Big) A w + \big(g(u_t) - g(v_t)\big) + $$
$$+ \Big(\varphi\Big(\|u\|_W^2 \Big) - \varphi\Big(\|v\|_W^2 \Big) \Big) \, A v = 0 \tag{2.28}$$

$$w(0) = w'(0) = 0 \, . \tag{2.29}$$

We take $s \in [0, T]$ fixed and we define

$$z(t) = \begin{cases} - \displaystyle\int_t^s w(\alpha) \, d\alpha & \text{if } 0 \leq t < s \\ 0 & \text{if } t \geq s. \end{cases}$$

Then $z(t) \in H_0^2(\Omega)$ for each $t \in [0, T]$ and moreover

$$z(t) = w_1(t) - w_1(s) \, , \quad \text{where } w_1(t) = \int_0^T w(\alpha) \, d\alpha \tag{2.30}$$

$$z(s) = 0 \, , \quad z'(t) = w(t) \, . \tag{2.31}$$

From (2.28) we obtain

$$\int_0^s \Big\{ \langle w''(t) \mid z(t) \rangle_{V'V} + \langle A^2 w(t) \mid z(t) \rangle_{V'V} + \\ + \Big(\varphi\Big(\|u(t)\|_W^2 \Big) Aw \mid z(t) \Big) + \Big\langle g\big(u_t(t)\big) - g\big(v_t(t)\big) \mid z(t) \Big\rangle_{V'V} + \\ + \Big(\varphi\Big(\|u\|_W^2 \Big) - \varphi\Big(\|v\|_W^2 \Big) \Big) \big(Av(t) \mid z(t) \big) \Big\} \, dt = 0 \,.$$

For $t < s$ by (2.29) and (2.31), we have:

$$\int_0^s \langle w''(t) \mid z(t) \rangle_{V'V} \, dt = -\frac{1}{2} \left| w(s) \right|^2 . \tag{2.32}$$

From (2.30) and (2.31) we obtain:

$$\int_0^s \langle A^2 w(t) \mid z(t) \rangle_{V'V} \, dt = -\frac{1}{2} \left| A w_1(s) \right|^2 . \tag{2.33}$$

We claim that

$$\int_0^s \langle g(u_t(t)) - g(v_t(t)) \mid z(t) \rangle_{V'V} \, dt \le 0.$$

In fact, let X be the set

$$X = \big\{ (x,t) \in \Omega \times [0\,,\, s) \;\big|\; u_t(x,t) \ge v_t(x,t) \big\} \,.$$

Then, by (2.29) we have $w(x,t) \ge 0$ in X.

So, from (2.31) and mean value theorem we obtain $z(x,t) \le 0$ in X.

Therefore, since g is a nondecreasing function we have

$$\big(g\big(u_t(x,t)\big) - g\big(v_t(x,t)\big) \big) \; z(x,t) \le 0 \quad \text{in } X \,.$$

In a similar way we can prove the above inequality if (x,t) belongs to

$$\Big\{ (x,t) \in \Omega \times [0\,,\, s) \;\Big|\; u_t(x,t) < v_t(x,t) \Big\} \,.$$

Now, using (2.32) and (2.33) we have

$$\frac{1}{2} \left| w(s) \right|^2 + \frac{1}{2} \left| A w_1(s) \right|^2 \le \\ \le \int_0^s \varphi\Big(\|u(t)\|_W^2 \Big) \Big| \big(Aw(t) \mid z \big) \Big| \, dt + \\ + \int_0^s \Big| \varphi\Big(\|u(t)\|_W^2 \Big) - \varphi\Big(\|v(t)\|_W^2 \Big) \Big| \; \Big| \Big(Av(t) \mid z(t) \Big) \Big| \, dt \,. \tag{2.34}$$

By (2.3), the assumptions on φ and (2.30), we have that there exists $C > 0$ such that

$$\frac{1}{2}\left|w(s)\right|^2 + \frac{1}{2}\left|Aw_1(s)\right|^2 \leq 2C\int_0^s \left|w(t)\right|\left(\left|Aw_1(t)\right| + \left|Aw_1(s)\right|\right) dt\,.$$

Taking $\gamma = \max\{6c^2T\,,\,1/3T\}$, we obtain

$$\frac{1}{6}\left(\left|w(s)\right|^2 + \left|Aw_1(s)\right|^2\right) \leq \gamma\int_0^s \left(\left|w(t)\right|^2 + \left|Aw_1(t)\right|^2\right) dt\,.$$

So, using Gronwall Inequality, we conclude the uniqueness. □

COROLLARY 2.1.— *Under the hypothesis of Theorem 2.1 there exists an unique* $u \in L^\infty(0,+\infty\,;V)$ *such that*

$$u \in C(0,+\infty\,;V) \cap L^\infty(0,+\infty\,;V)$$

$$u_t \in C(0,+\infty\,;H) \cap L^\infty(0,+\infty\,;H)\,, \quad u_{tt} \in L^1_{\mathrm{loc}}(0,+\infty\,;V')$$

$$g(u_t) \in L^1_{\mathrm{loc}}(0,+\infty\,;V')\,, \quad u_t g(u_t) \in L^1(\Omega \times (0,+\infty))$$

$$u_{tt} + \Delta^2 u - \varphi\left(\left\|u(t)\right\|_W^2\right)\Delta u + g(u_t) = 0 \quad \textit{in } V' \textit{ a.e. with respect to } t$$

$$u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) \quad \textit{in } \Omega\,.$$

3. Stabilization

This section deals with the decay of the energy (1.4) associated with the system (1.1). For this we need additional assumptions about the functions φ and g.

Let $\varphi \in C^1([0\,,+\infty)\,;\mathbb{R})$ be such that

$$\varphi(s) > 0 \quad \text{for } s \geq 0\,. \tag{3.1}$$

The function g satisfies (2.1) and there exist $p > 0$, $\lambda > 0$ and $c_0 > 0$ such that

$$g(s)s \geq c_0\left|s\right|^{p+1} \quad \text{if } |s| \leq 1 \tag{3.2}$$

$$g(s)s \geq c_0\left|s\right|^2 \quad \text{if } |s| \geq 1 \tag{3.3}$$

$$\left|g(s)\right| \leq c_0\left|s\right|^\lambda \quad \text{if } |s| \leq 1\,. \tag{3.4}$$

Remark 3.1.— From (3.1) we obtain, for each $K > 0$, a constant $m_K > 0$ such that

$$\widehat{\varphi}(t) \leq m_K\, t\varphi(t)\,, \quad t \in [\,0\,,\,K\,]\,.$$

Remark 3.2.— From (3.2) and (3.4) we have $p \geq \lambda$.

THEOREM 3.1.— *Under the above hypothesis, if* $(u_0, u_1) \in V \times H$, *then:*

- *if* $p = \lambda = 1$ *there exists* $\gamma > 0$ *such that*

$$E(t) \leq 4E(0)\exp(-\gamma t) \quad \textit{for } t > 0\,; \tag{3.5}$$

- *if* $\lambda \geq 1$ *and* $p > 1$ *there exists* $\delta > 0$ *such that*

$$E(t) \leq 4\Big\{\delta t + \big(E(0)\big)^{-(p-1)/2}\Big\}^{-2/(p-1)} \quad \textit{for } t > 0\,; \tag{3.6}$$

- *if* $\lambda < 1$ *there exists* $\mu > 0$ *such that*

$$E(t) \leq 4\Big\{\mu t + \big(E(0)\big)^{-(p+1-2\lambda)/2\lambda}\Big\}^{-2\lambda/(p+1-2\lambda)} \quad \textit{for } t > 0\,. \tag{3.7}$$

Proof.— It is sufficient to obtain (3.5), (3.6) and (3.7) for the approximated solutions $u_m(t)$, because the convergences showed in the proof of the Theorem 2.1 imply the above result for the solution u. From now on, by simplicity, we will denote u_m by u.

We observe, from the definition of $E(t)$, that

$$E'(t) = -\int_\Omega g\big(u_t(x,t)\big)u_t(x,t)\,\mathrm{d}x, \quad t > 0 \tag{3.8}$$

and so, by (2.1)

$$E(t) \leq E(0) \quad \text{for } t > 0\,. \tag{3.9}$$

Let $\rho(t) = \int_\Omega u(x,t)\,u_t(x,t)\,\mathrm{d}x$, since V is continuously embedded in H, there exists $c_1 > 0$ such that

$$\big|\rho(t)\big| \leq c_1\, E(t) \quad \text{for } t \geq 0\,. \tag{3.10}$$

Now, we consider two separated cases.

3.1 Case $\lambda \geq 1$

LEMMA 3.1.— *There exist positive constants c_2, c_3 and c_4 such that*

$$\rho'(t) \leq c_4 |u_t(t)|^2 - \frac{1}{2} \|u(t)\|_V^2 - \frac{1}{2} \frac{m_K}{c_2} \widehat{\varphi}\Big(\|u(t)\|_W^2\Big) - c_3 E'(t) . \quad (3.11)$$

Proof.— We have $\rho'(t) = |u'(t)|^2 + (u(t) \mid u_{tt}(t))$, so replacing in (2.9) v by $u(t)$, we obtain

$$\rho'(t) = |u_t(t)|^2 - \|u(t)\|_V^2 - \varphi\Big(\|u(t)\|_W^2\Big) \|u(t)\|_W^2 - \Big(g\big(u_t(t)\big) \mid u(t)\Big) . \quad (3.12)$$

By (2.11) and Remark 3.1, there exists $c_2 = \max\Big\{|\varphi(s)| \, ; \, s \in [0, K]\Big\}$ such that

$$-\varphi\Big(\|u(t)\|_W^2\Big) \|u(t)\|_W^2 \leq -\frac{m_K}{2c_2} \widehat{\varphi}\Big(\|u(t)\|_W^2\Big) . \quad (3.13)$$

On the other hand, it follows by Remark 2.1 that there exists $C > 0$ such that

$$\Big|\Big(g\big(u_t(t)\big) \mid u(t)\Big)\Big| \leq C \Big|g\big(u_t(t)\big)\Big|_{L^1(\Omega)} \|u(t)\|_V$$

and so, for $c = |\Omega|^{1/2}$

$$\Big|g\big(u_t(t)\big)\Big|_{L^1(\Omega)} \leq c \left\{ \int_{|u_t| \leq 1} \Big|g\big(u_t(x,t)\big)\Big|^2 \, \mathrm{d}x \right\}^{1/2} + \\ + \int_{|u_t| \geq 1} g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x .$$

Hence by (3.4), $\lambda \geq 1$,

$$\Big|g\big(u_t(t)\big)\Big|_{L^1(\Omega)} \leq c |u_t(t)| + \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x .$$

Then, by (2.11), we have

$$\Big|\big(g(u_t) \mid u(t)\big)\Big| \leq C^2 \frac{c^2}{2} |u_t|^2 + \frac{1}{2} \|u(t)\|_V^2 + \\ + \sqrt{K} \, C \int_\Omega g\big(u_t(x,t)\big) \, u_t(x,t) \, \mathrm{d}x . \quad (3.14)$$

Now, replacing (3.13) and (3.14) in (3.12) and using (3.8) we obtain (3.11) where $c_3 = \sqrt{K}\,C$ and $c_4 = 1 + C^2 c^2/2$.

Let we define $\sigma(t) = \big(E(t)\big)^{(p-1)/2}\rho(t)$, then we have

$$\sigma'(t) = \frac{p-1}{2}\,\big(E(t)\big)^{(p-3)/2}E'(t)\rho(t) + \big(E(t)\big)^{(p-1)/2}\rho'(t)\,. \tag{3.15}$$

Since $E'(t) < 0$, it follows by (3.9) and (3.10):

$$\frac{p-1}{2}\,\big(E(t)\big)^{(p-3)/2}E'(t)\rho(t) \le -c_1\frac{p-1}{2}\,\big(E(0)\big)^{(p-1)/2}E'(t)\,. \tag{3.16}$$

So, replacing (3.11) and (3.16) in (3.15) we obtain

$$\begin{aligned}\sigma'(t) \le -c_5\,E'(t) + \big(E(t)\big)^{(p-1)/2}\times \\ \times\left(c_4\big|u_t(t)\big|^2 - \frac{1}{2}\,\big\|u(t)\big\|_V^2 - \frac{1}{2}\,\frac{m_K}{c_2}\,\widehat{\varphi}\Big(\big\|u(t)\big\|_W^2\Big)\right)\end{aligned} \tag{3.17}$$

where $c_5 = (c_1\,(p-1)/2 + \sqrt{K}\,C)\big(E(0)\big)^{(p-1)/2} > 0$.

For each $\varepsilon > 0$, we consider

$$E_\varepsilon(t) = (1+\varepsilon c_5)E(t) + \varepsilon\sigma(t)\,. \tag{3.18}$$

Remark 3.3. — We can conclude that there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2}\,\big(E_\varepsilon(t)\big)^{(p+1)/2} \le \big(E(t)\big)^{(p+1)/2} \le 2\big(E_\varepsilon(t)\big)^{(p+1)/2}\,, \quad t>0\,,\ 0<\varepsilon\le\varepsilon_0\,.$$

(i) If we consider $p = 1$ (then $\lambda = 1$) we have, by (3.2), (3.3) and (3.17)

$$E_\varepsilon'(t) \le (\varepsilon c_4 - c_0)\big|u_t\big|^2 - \frac{\varepsilon}{2}\,\big\|u(t)\big\|_V^2 - \frac{\varepsilon}{2}\,\frac{m_K}{c_2}\,\widehat{\varphi}\Big(\big\|u(t)\big\|_W^2\Big)$$

if we take

$$\varepsilon < \min\left\{\varepsilon_0, \frac{c_0}{1+c_4}\right\} \quad \text{and} \quad c_6 = \min\left\{1, \frac{m_K}{c_2}\right\}$$

then

$$E_\varepsilon'(t) \le -\varepsilon c_6 E(t)\,.$$

So, by Remark 3.3 we have (3.5).

(ii) If we consider $p > 1$ we have, by (3.17) and (3.18)

$$\begin{aligned} E'_\varepsilon(t) \leq E'(t) + \varepsilon\big(E(t)\big)^{(p-1)/2} \times \\ \times \left(c_4\big|u_t(t)\big|^2 - \frac{1}{2}\left\|u(t)\right\|_V^2 - \frac{m_K}{2c_2}\,\widehat{\varphi}\Big(\left\|u(t)\right\|_W^2\Big)\right) . \end{aligned} \tag{3.19}$$

By (3.2) and (3.3), there exists $c_7 > 0$, which depends of Ω and $E(0)$, such that

$$\int_\Omega g\big(u_t(x,t)\big)\, u_t(x,t)\,\mathrm{d}x \geq c_7\big|u_t(t)\big|^{p+1} \tag{3.20}$$

then, by (3.8) and (3.19)

$$E'_\varepsilon(t) \leq -c_7\big|u_t(t)\big|^{p+1} + \varepsilon\left(c_4 + \frac{1}{2}\right)\big(E(t)\big)^{(p-1)/2}\big|u_t(t)\big|^2 - \varepsilon c_6\big(E(t)\big)^{(p+1)/2}.$$

Now, using Young's inequality, with $\mu > 0$ such that $\mu^{(p+1)/(p-1)}(c_4 + 1/2) < c_6/2$, we obtain

$$E'_\varepsilon(t) \leq -\left(c_7 - \varepsilon\frac{2c_4+1}{2\mu^{(p+1)/2}}\right)\,\big|u_t(t)\big|^{p+1} - \frac{\varepsilon}{2}\,c_6\big(E(t)\big)^{(p+1)/2} .$$

Taking

$$\varepsilon < \min\left\{\varepsilon_0, \frac{2\mu^{(p+1)/2}c_7}{2c_4+1}\right\},$$

it follows, by Remark 3.3 that

$$E'_\varepsilon \leq -\frac{\varepsilon}{4}\,c_6\big(E_\varepsilon(t)\big)^{(p+1)/2} .$$

So,

$$E_\varepsilon(t) \leq \left(\frac{p-1}{8}\,\varepsilon c_6 t + \big(E_\varepsilon(0)\big)^{-(p-1)/2}\right)^{-2/(p-1)} . \tag{3.21}$$

Now, (3.6) follows from Remark 3.3 and (3.21).

3.2 Case $\lambda < 1$

In this case, the remark 3.2 implies that $p + 1 - 2\lambda > 0$.

Let us define $\psi(t) = \big(E(t)\big)^{(p+1-2\lambda)/2\lambda}\rho(t)$.

Then, by (3.8), (3.9) and (3.10):

$$\psi'(t) \leq -c_8 E'(t) + \big(E(t)\big)^{(p+1-2\lambda)/2\lambda} \rho'(t) \tag{3.22}$$

where

$$c_8 = c_1 \, \frac{p+1-2\lambda}{2\lambda} \, \big(E(0)\big)^{(p+1-2\lambda)/2\lambda} .$$

It follows by (3.12) and (3.13)

$$\rho'(t) \leq |u_t(t)|^2 - \|u(t)\|_V^2 - \frac{m_K}{2c_2} \, \widehat{\varphi}\Big(\|u(t)\|_W^2\Big) + \Big|\Big(g\big(u_t(t)\big) \mid u(t)\Big)\Big| \, . \tag{3.23}$$

On the other hand, by (3.2) and (3.4) there exists a positive constant, α, which depends of Ω and c_0 such that

$$\int_\Omega \Big|g\big(u_t(x,t)\big)\Big| \, \mathrm{d}x \leq$$

$$\leq \alpha \left\{ \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x \right\}^{\lambda/(p+1)} + \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x \, .$$

Replacing the above inequality in (3.23) and using (2.11) and (3.8), we have:

$$\rho'(t) \leq |u_t(t)|^2 - \|u(t)\|_V^2 - \frac{m_K}{2c_2} \, \widehat{\varphi}\Big(\|u(t)\|_W^2\Big) +$$

$$+ \overline{C} \left\{ \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x \right\}^{\lambda/(p+1)} \|u(t)\|_V - c_3 E'(t) \, .$$

Taking $c_9 = c_3\big(E(0)\big)^{(p+1-2\lambda)/2\lambda}$ and replacing the above inequality in (3.22):

$$\psi'(t) \leq -(c_8 + c_9) E'(t) +$$

$$+ \frac{3}{2} \, \big(E(t)\big)^{(p+1-2\lambda)/2\lambda} |u_t(t)|^2 - c_6 \big(E(t)\big)^{(p+1)/2\lambda} +$$

$$+ \overline{C}\sqrt{2} \, \big(E(t)\big)^{(p+1-\lambda)/2\lambda} \left\{ \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x \right\}^{\lambda/(p+1)} . \tag{3.24}$$

On the other hand, by Young's Inequality and (3.8)

$$\big(E(t)\big)^{(p+1-\lambda)/2\lambda} \left\{ \int_\Omega g\big(u_t(x,t)\big) u_t(x,t) \, \mathrm{d}x \right\}^{\lambda/(p+1)} \leq$$

$$\leq \beta^{(p+1)/(p+1-\lambda)} \big(E(t)\big)^{(p+1)/2\lambda} \, \frac{p+1-\lambda}{p+1} - \left(\frac{1}{\beta}\right)^{(p+1)/\lambda} \frac{\lambda}{p+1} \, E'(t) \, . \tag{3.25}$$

We choose $\beta > 0$ such that

$$\overline{C}\sqrt{2}\,\beta^{(p+1)/(p+1-\lambda)}\,\frac{p+1-\lambda}{p+1} < \frac{c_6}{2}$$

and

$$E(t)^{(p+1-2\lambda)/2\lambda}\big|u_t(t)\big|^2 \le \frac{p+1-2\lambda}{p+1}\,\gamma^{(p+1)/(p+1-2\lambda)}\big(E(t)\big)^{(p+1)/2\lambda} + \\ + \frac{2\lambda}{p+1}\left(\frac{1}{\gamma}\right)^{(p+1)/2\lambda}\big|u_t(t)\big|^{(p+1)/\lambda} \tag{3.26}$$

for $\gamma > 0$, such that

$$\frac{3}{2}\,\frac{p+1-2\lambda}{p+1}\,\gamma^{(p+1)/(p+1-2\lambda)} < \frac{c_6}{4}\,.$$

Now, since $p + 1/\lambda > p+1$ using (2.10) and (3.20) there exists $R > 0$ such that

$$\big|u_t(t)\big|^{(p+1)/\lambda} < -R\,E'(t)\,. \tag{3.27}$$

Taking

$$c_{10} = \overline{C}\sqrt{2}\left(\frac{1}{\beta}\right)^{(p+1)/\lambda}\frac{\lambda}{p+1}\,, \quad c_{11} = \frac{3\lambda}{p+1}\left(\frac{1}{\gamma}\right)^{(p+1)/2\lambda} R$$

and replacing (3.25), (3.26) and (3.27) in (3.24) it follows that

$$\psi'(t) \le -c_{12}E'(t) - \frac{c_6}{4}\,E(t)^{(p+1)/2\lambda}\,. \tag{3.28}$$

where $c_{12} = c_8 + c_9 + c_{10} + c_{11}$.

For each $\varepsilon > 0$, we define

$$E_\varepsilon(t) = (1+\varepsilon c_{12})E(t) + \varepsilon\psi(t)\,, \quad t \ge 0\,. \tag{3.29}$$

Then there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2}\big(E_\varepsilon(t)\big)^{(p+1)/2\lambda} \le \big(E(t)\big)^{(p+1)/2\lambda} \\ \le 2\big(E_\varepsilon(t)\big)^{(p+1)/2\lambda}, \quad t \ge 0\,,\ 0 < \varepsilon \le \varepsilon_0\,. \tag{3.30}$$

Taking the derivative of (3.29), using (3.28) and (3.30) we deduce that

$$E'_\varepsilon(t) \le -\frac{\varepsilon c_6}{4}\big(E(t)\big)^{(p+1)/2\lambda} \le -\frac{\varepsilon c_6}{8}\big(E_\varepsilon(t)\big)^{(p+1)/2\lambda}, \quad 0 < \varepsilon \le \varepsilon_0\,.$$

So,

$$E_\varepsilon(t) \le \left(\frac{p+1-2\lambda}{2\lambda}\,\frac{\varepsilon c_6}{8}\,t + \big(E_\varepsilon(0)\big)^{-(p+1-2\lambda)/2\lambda}\right)^{-2\lambda/(p+1-2\lambda)},$$

$t > 0$, $0 < \varepsilon \le \varepsilon_0$. Finally (3.7) follows from (3.30) and the above inequality. □

4. Final remarks

If we consider the inhomogeneous system

$$\begin{cases} u_{tt}(t) + \Delta^2 u(t) - \varphi\Big(\|u(t)\|_W^2\Big)\,\Delta u(t) + g\big(u_t(t)\big) = f(t) & \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \times (0,T) \\ u(0) = u_0\,, \quad u_t(0) = u_1\,. & \end{cases} \tag{4.1}$$

where f belongs to $L^2(0,T\,;\,H)$, we can prove, using the same approach as in Theorem 2.1, the existence of solutions satisfying (2.3)-(2.7).

It should be noted that the presence of the nonlinear damping term g in (1.1), without restriction on its growth, contributes to some technical difficulties at the level of the existence theory. Therefore, we needed to consider the dimension $N \leq 3$ (Remark 2.1).

To prove the Theorem 2.1, in dimension $N \geq 4$ we consider, for instance, $g(s) = |s|^{q-1}s$, where

$$\begin{cases} 1 < q \leq \dfrac{N+4}{N-4} & \text{if } N > 4 \\ q > 1 & \text{if } N = 4. \end{cases}$$

We can also consider the sublinear case, as it was mentioned in Section 1, that is $g(s) = |s|^{q-1}s$, $0 < q < 1$. In both cases, the proof of the Theorem 2.1 follows in the same way.

The Remark 3.1 still holds if we replace the assumption on (3.1), by the following:

$$\varphi(0) = 0\,, \quad \varphi(s) > 0 \text{ if } s > 0 \quad \text{and} \quad \varphi'(0) \neq 0\,.$$

It is important to observe that the assumptions (3.2) and (3.3) on the damping g ensure, respectively, its coercivity at the origin and at the infinity. On the other hand, the assumption (3.4) controls the growth of g. The decay order of the energy (1.4) depends only on the constants p and λ, *i.e.*, it depends on the behaviour of g at the origin.

The estimates on the decay rates of the energy, obtained from Theorem 3.1, are probably optimal, but this question has not been proved yet even in the simplest case of the semilinear damped wave equation.

Finally, we claim that the Theorem 3.1 can be proved, if $N \geq 4$, using the same method, provided solutions do exist. In fact, it is sufficient to make the following additional hypothesis on g:

$$|g(s)| \leq C_0 |s|^q \quad \text{if } |s| \geq 1$$

where $1 < q < +\infty$ if $N = 4$ and $1 < q \leq (N+4)/(N-4)$ if $N > 4$.

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