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# Residue formulas for meromorphic functions on surfaces(*)

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RÉSUMÉ. — On calcule, pour le feuilletage singulier défini par une fonction méromorphe sur une surface complexe, les résidus de Baum–Bott et on énonce les théorèmes des résidus. En les appliquant au cas des feuilletages provenant de polynômes de deux variables, on obtient quelques formules, en particulier une formule de D. T. Lê et une "formule de nombre de Milnor" pour une application holomorphe possédant des fibres non réduites.

ABSTRACT. — For a singular foliation defined by a meromorphic function on a complex surface, we compute the Baum–Bott residues and describe the residue theorems. Applying these to the case of foliations arising from polynomials in two variables, we obtain various formulas, in particular a formula of D. T. Lê and a "Milnor number formula" for a holomorphic map having non-reduced fibers.

Let $X$ be a complex manifold of dimension $n$. A dimension one (singular) holomorphic foliation $\mathcal{E}$ on $X$ is defined locally by a holomorphic vector field and its singular set $S(\mathcal{E})$ is defined by patching together the zero sets of the vector fields defining $\mathcal{E}$. For a compact connected component $Z$ of $S(\mathcal{E})$ (of codimension in $X$ greater than one) and a symmetric homogeneous polynomial $\psi$ of degree $n$, there is the "Baum–Bott residue" $\mathrm{Res}_\psi(\mathcal{E}, Z)$, which is a complex number determined by the behavior of the foliation near $Z$. If $X$ is compact, the sum of these residues over the components of $S(\mathcal{E})$ is

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equal to the characteristic class $\psi(TX - E)$ of the virtual bundle $TX - E$, where $TX$ denotes the holomorphic tangent bundle of $X$ and $E$ the line bundle associated with $\mathcal{E}$, the tangent bundle of the foliation ([BB1] and [BB2]).

When $X$ is a complex surface ($n = 2$), the foliation $\mathcal{E}$ is also defined locally by a holomorphic 1-form. Thus if we have a meromorphic function $\varphi$ on $X$, considering its differential $\mathrm{d}\varphi$, we have a naturally defined foliation (see Section 2 below) whose leaves are the level sets of $\varphi$ and whose singular points include the critical points and the indeterminacy points of $\varphi$. In this article, we compute the above residues and examine the residue formula for foliations defined this way. In Section 1, we recall basic facts about the Baum–Bott residues and write down the global invariant in the residue formula in terms of the conormal bundle $F$ of the foliation (the "annihilator" of $E$). Since the dimension of $X$ is two, we have essentially two kinds of residues, one for $\psi = \sigma_1^2$ and the other for $\psi = \sigma_2$, where $\sigma_i$ denotes the elementary symmetric function of degree $i$. The residue for $\sigma_2$ at a singular point gives the index of the vector field defining $\mathcal{E}$ near the point and if $\mathcal{E}$ is defined by a global vector field, the residue formula reduces to the Poincaré–Hopf theorem. We describe, in Section 2, how a foliation is defined from a given meromorphic function $\varphi$ on $X$. If the critical points of $\varphi$ (away from its pole divisor) are isolated, the line bundle $F$ turns out to be the dual of the bundle determined by the pole divisor $D$ of $\mathrm{d}\varphi$ (Lemma 2.1). It is not difficult to find the residues at a singular point away from the pole divisor of $\varphi$. It is also possible to compute the residue for $\sigma_1^2$ at a singular point on the pole divisor explicitly. The residue formula in this case is simply a formula to express the self-intersection number of $D$ as a sum of local contributions at the singular points (on $D$) of the foliation. We may compute the residue for $\sigma_2$ at a point on the pole divisor in some cases which will be useful in the following section. In Section 3, we apply these results to foliations on the projective plane $\mathbf{P}^2$, or on its modifications, arising from a polynomial $f$ in two variables. Let $\varphi_0$ denote the rational function on $\mathbf{P}^2$ obtained by extending $f$. First, considering the foliation on $\mathbf{P}^2$ defined by $\varphi_0$, we obtain a formula of D. T. Lê (Theorem 3.2). Then, if we remove the indeterminacy of $\varphi_0$ by a sequence of blowing-ups of $\mathbf{P}^2$, $\varphi_0$ is modified to a meromorphic function $\varphi$ which gives a fibration of the blown-up surface $X$ over $\mathbf{P}^1$. Considering the foliation on $X$ defined by $\varphi$, we obtain a formula (Theorem 3.8), which may be interpreted as a "Milnor number formula" for a map with non-reduced fibers.

## 1. Baum–Bott residues of singular foliations on surfaces

Let $X$ be a complex analytic manifold of dimension two (a complex surface). A dimension one complex analytic singular foliation $\mathcal{E}$ on $X$ is determined by a system $\{(U_\alpha, v_\alpha)\}$, where $\{U_\alpha\}$ is an open covering of $X$ and, for each $\alpha$, $v_\alpha$ is a holomorphic vector field on $U_\alpha$ such that $v_\beta = e_{\alpha\beta} v_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomophic function $e_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. We denote the set of zeros of $v_\alpha$ on $U_\alpha$ by $S(v_\alpha)$; $S(v_\alpha) = \{p \in U_\alpha \mid v_\alpha(p) = 0\}$, and call it the singular set of $v_\alpha$. Since $S(v_\alpha)$ and $S(v_\beta)$ coincide in $U_\alpha \cap U_\beta$, the union $\bigcup_\alpha S(v_\alpha)$ is an analytic set in $X$, which we call the singular set of the foliation $\mathcal{E}$ and denote by $S(\mathcal{E})$. We say that $\mathcal{E}$ is reduced if $S(\mathcal{E})$ consists of isolated points. Since the system $\{e_{\alpha\beta}\}$ satisfies the cocycle condition, it determines a line bundle, which we denote by $E$ and call the tangent bundle of the foliation.

Singular foliations can also be defined in terms of holomorphic 1-forms. Thus a codimension one complex analytic singular foliation $\mathcal{F}$ on $X$ is determined by a system $\{(U_\alpha, \omega_\alpha)\}$, where $\omega_\alpha$ is a holomorphic 1-form on $U_\alpha$, such that $\omega_\beta = f_{\alpha\beta}\omega_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomophic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. As in the case of vector fields, we can define the singular set $S(\mathcal{F})$ by patching the singular (zero) sets $S(\omega_\alpha)$ of $\omega_\alpha$ together and we may talk about the reduciblity of $\mathcal{F}$. We denote by $F$ the line bundle determined by the cocycle $\{f_{\alpha\beta}\}$ and call it the conormal bundle of the foliation.

The two definitions above are equivalent, as long as we consider reduced foliations, in the sense that there is a natural one-to-one correspondence between the reduced dimension one foliations and the reduced codimension one foliations [Sw]. In fact, the correspondence is given by taking the annihilator of each other, namely, if $\mathcal{E} = \{(U_\alpha, v_\alpha)\}$ is a (reduced) dimension one foliation, it corresponds to the (reduced) codimension one foliation $\mathcal{F} = \{(U_\alpha, \omega_\alpha)\}$ with $\langle v_\alpha, \omega_\alpha \rangle = 0$ on $U_\alpha$ and vice versa. Note that in the above correspondence, we have $S(\mathcal{E}) = S(\mathcal{F})$ and the integral curves of the vector field $v_\alpha$ are the solutions of the differential equation $\omega_\alpha = 0$. In what follows we consider only reduced foliations.

Let $\mathcal{E}$ be a dimension one singular foliation on $X$. For each point $p$ in $S(\mathcal{E})$ and a homogeneous and symmetric polynomial $\psi$ of degree two, we have the Baum–Bott residue $\mathrm{Res}_\psi(\mathcal{E}, p)$ of $\mathcal{E}$ at $p$ for $\psi$, which is a complex number given as follows ([BB1], [BB2]).

Suppose that $U$ is a coordinate neighborhood and that $p$ is the origin of a coordinate system $(x, y)$ on $U$ and is an isolated zero of the vector field $v$ defining $\mathcal{E}$ on $U$. We write

$$v = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

with $a$ and $b$ holomorphic functions on $U$, and let $A$ be the Jacobian matrix $\partial(a,b)/\partial(x,y)$. For elementary symmetric polynomials $\sigma_i$, $i = 1, 2$, in two variables, we set

$$\sigma_1(A) = \operatorname{tr} A \quad \text{and} \quad \sigma_2(A) = \det A\,.$$

If $\psi$ is a homogeneous and symmetric polynomial of degree two, it is written as $\psi = \widetilde{\psi}(\sigma_1, \sigma_2)$, for some polynomial $\widetilde{\psi}$. We set $\psi(A) = \widetilde{\psi}(\sigma_1(A), \sigma_2(A))$. Then the Baum–Bott residue $\operatorname{Res}_\psi(\mathcal{E}, p)$ is given by the Grothendieck residue symbol

$$\operatorname{Res}_\psi(\mathcal{E}, p) = \operatorname{Res}_p \begin{bmatrix} \psi(A)\, dx \wedge dy \\ a,\ b \end{bmatrix},$$

which is represented by the integral

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{\psi(A)\, dx \wedge dy}{ab}\,.$$

In the above, $\Gamma$ is the 2-cycle in $U$ defined by

$$\Gamma = \Big\{(x,y) \,\Big|\, |a(x,y)| = |b(x,y)| = \varepsilon\Big\},$$

for a sufficiently small positive number $\varepsilon$, and is oriented so that the form $d(\arg a) \wedge d(\arg b)$ is positive.

In particular, if $\psi = \sigma_2$, since $\sigma_2(A)\, dx \wedge dy = da \wedge db$, the residue $\operatorname{Res}_{\sigma_2}(\mathcal{E}, p)$ is the intersection number $(a, b)_p$ at $p$ of the divisors defined by $a$ and $b$ [GH, Chap. 5], which is equal to the index of the vector field $v$ at $p$.

Now we denote by $TX$ the holomorphic tangent bundle of $X$. A section of the bundle $E$ is represented by a collection $\{s_\alpha\}$ of functions satisfying $s_\alpha = e_{\alpha\beta} s_\beta$. Thus the collection $\{s_\alpha v_\alpha\}$ defines a section of $TX$. Hence we have a bundle map $E \to TX$, which is injective exactly on $X \setminus S(\mathcal{E})$.

We have the following residue theorem.

THEOREM 1.1 (Baum-Bott).— *If the complex surface $X$ is compact, then*

$$\sum_{p\in S(\mathcal{E})} \mathrm{Res}_{\psi}(\mathcal{E},p) = \psi(TX-E) \frown [X]\,,$$

*where, denoting by $c_1 = c_1(TX-E)$ and $c_2 = c_2(TX-E)$ the first and second Chern classes of the virtual bundle $TX-E$, we set $\psi(TX-E) = \widetilde{\psi}(c_1, c_2)$.*

Recall that, if we denote by $c(X)$ and $c(E)$ the total Chern classes of $TX$ and $E$, the total Chern class of $TX-E$ is given by $c(TX-E) = c(X)/c(E)$. Hence we have

$$\begin{aligned} c_1(TX-E) &= c_1(X) - c_1(E)\,,\\ c_2(TX-E) &= c_2(X) - c_1(X)c_1(E) + c_1^2(E)\,. \end{aligned}$$

If we denote by $\mathcal{F}$ the codimension one foliation corresponding to $\mathcal{E}$ and by $F$ the associated line bundle, we have the following lemma.

LEMMA 1.2.— *We have $F = E \otimes K$, where $K$ denotes the canonical bundle of $X$.*

*Proof.*— We assume that each $U_\alpha$ is a coordinate neighborhood and let $(x_\alpha, y_\alpha)$ be a coordinate system on $U_\alpha$. If we write

$$v_\alpha = a_\alpha \frac{\partial}{\partial x_\alpha} + b_\alpha \frac{\partial}{\partial y_\alpha}\,,$$

then $\mathcal{F}$ is defined by $\omega_\alpha = b_\alpha\,\mathrm{d}x_\alpha - a_\alpha\,\mathrm{d}y_\alpha$ on $U_\alpha$. Let the systems $\{e_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$ be defined by $v_\beta = e_{\alpha\beta} v_\alpha$ and $\omega_\beta = f_{\alpha\beta}\omega_\alpha$, as before, so that they define the bundles $E$ and $F$, respectively. From $v_\beta = e_{\alpha\beta} v_\alpha$, we have

$$a_\beta = e_{\alpha\beta}\left(a_\alpha \frac{\partial x_\beta}{\partial x_\alpha} + b_\alpha \frac{\partial x_\beta}{\partial y_\alpha}\right), \quad b_\beta = e_{\alpha\beta}\left(a_\alpha \frac{\partial y_\beta}{\partial x_\alpha} + b_\alpha \frac{\partial y_\beta}{\partial y_\alpha}\right).$$

Substituting these in $\omega_\beta = b_\beta\,\mathrm{d}x_\beta - a_\beta\,\mathrm{d}y_\beta$, we get

$$\omega_\beta = e_{\alpha\beta} \det\left(\frac{\partial(x_\beta, y_\beta)}{\partial(x_\alpha, y_\alpha)}\right)\omega_\alpha\,,$$

which proves the lemma. □

For line bundles $L_1$ and $L_2$ on a compact complex manifold $X$, we denote $(c_1(L_1) \smile c_1(L_2)) \frown [X]$ by $L_1 \cdot L_2$. Then, noting that $c_1(X) = -c_1(K)$, we have, from Theorem 1.1 and Lemma 1.2, the following formulas.

PROPOSITION 1.3.— *If $X$ is compact,*

$$\sum_{p \in S(\mathcal{E})} \mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = F^2 ,$$

$$\sum_{p \in S(\mathcal{E})} \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \chi(X) + F^2 - K \cdot F ,$$

*where $\chi(X)$ denotes the Euler number of $X$.*

## 2. Singular foliations defined by meromorphic functions

Let $\varphi$ be a meromorphic function on a complex surface $X$. We take a coordinate covering $\{U_\alpha\}$ of $X$ so that, on each $U_\alpha$, the differential $\mathrm{d}\varphi$ of $\varphi$ is written as

$$\mathrm{d}\varphi = \varphi_\alpha \omega_\alpha ,$$

where $\varphi_\alpha$ is a meromorphic function and $\omega_\alpha$ is a holomorphic 1-form with isolated zeros on $U_\alpha$. Then the system $\{(U_\alpha, \omega_\alpha)\}$ defines a (reduced) codimension one singular foliation $\mathcal{F}$. The associated line bundle $F$ is defined by the cocycle $\{f_{\alpha\beta}\}$, $f_{\alpha\beta} = \varphi_\alpha / \varphi_\beta$. The leaves of this foliation are the level sets of $\varphi$.

We denote by $D^{(0)}$ and $D^{(\infty)}$, respectively, the zero and pole divisors of $\varphi$; $(\varphi) = D^{(0)} - D^{(\infty)}$. Let $D^{(0)} = \sum_{j=1}^s n_j D_j^{(0)}$ and $D^{(\infty)} = \sum_{i=1}^r m_i D_i^{(\infty)}$ be the irreducible decompositions with $n_i$ and $m_j$ positive integers. For a divisor $D$ on $X$, we denote by $|D|$ the support of $D$ and by $[D]$ the line bundle determined by $D$.

LEMMA 2.1.— *If the critical points of $\varphi$ in $X \setminus |D^{(\infty)}|$ are all isolated, then we have*

$$F = \left[ - \sum_{i=1}^r (m_i + 1) D_i^{(\infty)} \right] .$$

*Proof.*— First note that the assumption implies that $D^{(0)}$ is reduced ($n_j = 1$ for all $j$). At each point $p$ in $X$, we express the germ $\varphi$ as $\varphi = f/g$

with $f$ and $g$ relatively prime holomorphic function germs defining $D^{(0)}$ and $D^{(\infty)}$, respectively, at $p$. Let $g = g_1^{m_1} \cdots g_r^{m_r}$ be a decomposition so that $D_i^{(\infty)}$ is defined by $g_i$. Note that $f$ or $g_i$ may be a unit or may be reducible but they are reduced (i.e., not divisible by the square of a non-unit). We compute

$$\mathrm{d}\varphi = \frac{1}{g_1^{m_1+1} \cdots g_r^{m_r+1}}\, \omega\,,$$

where $\omega$ is the holomorphic 1-form germ given by

$$\omega = g_1 \cdots g_r \,\mathrm{d}f - f\left(\sum_{i=1}^{r} m_i g_1 \cdots \widehat{g}_i \cdots g_r \,\mathrm{d}g_i\right).$$

We claim that the zero of $\omega$ is (at most) isolated and thus $\mathcal{F}$ is defined by $\omega$ near the point $p$. In fact, if $\omega = h\omega'$ for some non-unit $h$ and a holomorphic 1-form germ $\omega'$, then $h$ must be divisible by a factor $g'$ of some $g_i$ which is a non-unit. This implies that $\mathrm{d}g_i = g'\theta$ for some holomorphic 1-form germ $\theta$, which is a contradiction, since $g_i$ is reduced. □

*Remark* 2.2.— Under the assumption of Lemma 2.1, the singular points of $\mathcal{F}$ in $X \setminus |D^{(\infty)}|$ are the critical points of $\varphi$ and the singular points of $\mathcal{F}$ in $|D^{(\infty)}|$ include the intersection points of $D^{(0)}$ and $D_i^{(\infty)}$ (indeterminacy points), the intersection points of $D_i^{(\infty)}$ and $D_j^{(\infty)}$, $i \neq j$, and the singular points of $D_i^{(\infty)}$.

Hereafter throughout this section, we assume that the critical points of $\varphi$ on $X \setminus |D^{(\infty)}|$ are all isolated. We denote by $\mathcal{E}$ the dimension one foliation corresponding to $\mathcal{F}$.

LEMMA 2.3.— *For a singular point $p$ of $\mathcal{E}$ in $X \setminus |D^{(\infty)}|$, we have*

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0 \quad \textit{and} \quad \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\varphi)\,,$$

*where $\mu_p(\varphi)$ denotes the Milnor number of $\varphi$ at $p$.*

*Proof.*— By the assumption, near a point $p$ in $X \setminus |D^{(\infty)}|$, $\mathcal{F}$ is defined by $\mathrm{d}\varphi$. Hence, if we denote by $(x, y)$ a coordinate system near $p$, $\mathcal{E}$ is defined by the holomorphic vector field

$$v = \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}\,.$$

The Jacobian matrix A of $v$ is

$$A = \begin{pmatrix} \dfrac{\partial^2 \varphi}{\partial x \partial y} & \dfrac{\partial^2 \varphi}{\partial y^2} \\ -\dfrac{\partial^2 \varphi}{\partial x^2} & -\dfrac{\partial^2 \varphi}{\partial y \partial x} \end{pmatrix}.$$

Thus, since $\sigma_1(A) = \operatorname{tr}(A) = 0$, the residue $\operatorname{Res}_{\sigma_1^2}(\mathcal{E}, p)$ is equal to 0. To compute $\operatorname{Res}_{\sigma_2}(\mathcal{E}, p)$, put $a = \partial\varphi/\partial y$ and $b = -\partial\varphi/\partial x$. Then, since $\sigma_2(A) = \det(A)$,

$$\operatorname{Res}_{\sigma_2}(\mathcal{E}, p) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_\Gamma \frac{da \wedge db}{ab} = (a, b)_p = \mu_p(\varphi)\,. \;\square$$

In what follows, we denote by $(D_1 \cdot D_2)_p$ the intersection number of divisors $D_1$ and $D_2$ at a point $p$ and by $D_1 \cdot D_2$ the (total) intersection number, which is equal to $[D_1] \cdot [D_2]$.

LEMMA 2.4.— *For a singular point $p$ of $\mathcal{E}$ in $|D^{(\infty)}|$, we have*

$$\operatorname{Res}_{\sigma_1^2}(\mathcal{E}, p) = \sum_{i=1}^{r} \frac{(m_i+1)^2}{m_i} \left(D^{(0)} \cdot D_i^{(\infty)}\right)_p + \\ - \sum_{1 \le i < j \le r} \frac{(m_i - m_j)^2}{m_i m_j} \left(D_i^{(\infty)} \cdot D_j^{(\infty)}\right)_p.$$

*Thus if $p$ is not an intersection point of $D^{(0)}$ and $D_i^{(\infty)}$ or of $D_i^{(\infty)}$ and $D_j^{(\infty)}$ with $m_i \neq m_j$, we have* $\operatorname{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0$.

*Proof.*— If we denote by $(x, y)$ a coordinate system near $p$, $\mathcal{E}$ is defined by the vector field

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$$

with

$$a = g_1 \cdots g_r \frac{\partial f}{\partial y} - f\left(\sum_{i=1}^{r} m_i g_1 \cdots \widehat{g_i} \cdots g_r \frac{\partial g_i}{\partial y}\right),$$

$$b = -g_1 \cdots g_r \frac{\partial f}{\partial x} + f\left(\sum_{i=1}^{r} m_i g_1 \cdots \widehat{g_i} \cdots g_r \frac{\partial g_i}{\partial x}\right)$$

(see the proof of Lemma 2.1). For the matrix $A = \partial(a,b)/\partial(x,y)$, we compute

$$\operatorname{tr} A = -\sum_{i=1}^{r}(m_i+1)G_i \det\frac{\partial(f,g_i)}{\partial(x,y)} + \sum_{1\le i<j\le r}(m_i-m_j)fG_{ij}\det\frac{\partial(g_i,g_j)}{\partial(x,y)},$$

where $G_i = g_1\cdots\widehat{g}_i\cdots g_r$ and $G_{ij} = g_1\cdots\widehat{g}_i\cdots\widehat{g}_j\cdots g_r$. We have

$$\operatorname{Res}_{\sigma_1^2}(\mathcal{E},p) = \operatorname{Res}_p\begin{bmatrix}(\operatorname{tr} A)^2\,\mathrm{d}x\wedge\mathrm{d}y\\ a,\ b\end{bmatrix} = \operatorname{Res}_p\begin{bmatrix}\operatorname{tr} A\cdot\tau\\ a,\ b\end{bmatrix}, \tag{2.1}$$

where

$$\tau = -\sum_{i=1}^{r}(m_i+1)\tau_i + \sum_{1\le i<j\le r}(m_i-m_j)\tau_{ij}$$

with $\tau_i = G_i\,\mathrm{d}f\wedge\mathrm{d}g_i$ and $\tau_{ij} = fG_{ij}\,\mathrm{d}g_i\wedge\mathrm{d}g_j$. Now, for a fixed $i$, we may write

$$\begin{pmatrix}a\\ b\end{pmatrix} = C\begin{pmatrix}f\\ g_i\end{pmatrix},\quad C = \begin{pmatrix}-\displaystyle\sum_{k=1}^{r} m_kG_k\frac{\partial g_k}{\partial y} & G_i\dfrac{\partial f}{\partial y}\\ \displaystyle\sum_{k=1}^{r} m_kG_k\frac{\partial g_k}{\partial x} & -G_i\dfrac{\partial f}{\partial x}\end{pmatrix}.$$

Also, for $j\neq i$, we may write

$$\begin{pmatrix}a\\ b\end{pmatrix} = C_j\begin{pmatrix}f\\ g_i\end{pmatrix},\quad C_j = \begin{pmatrix}-\displaystyle\sum_{k\neq j} m_kG_k\frac{\partial g_k}{\partial y} & G_i\dfrac{\partial f}{\partial y} - fm_jG_{ij}\dfrac{\partial g_j}{\partial y}\\ \displaystyle\sum_{k\neq j} m_kG_k\frac{\partial g_k}{\partial x} & -G_i\dfrac{\partial f}{\partial y} + fm_jG_{ij}\dfrac{\partial g_j}{\partial x}\end{pmatrix}.$$

Then we compute

$$\operatorname{tr} A\cdot G_i = -\left(\frac{m_i+1}{m_i} + \sum_{j\neq i}\frac{m_i-m_j}{m_im_j}\right)\det C + \sum_{j\neq i}\frac{m_i-m_j}{m_im_j}\det C_j\,.$$

Thus, using one of the properties of the Grothendieck residue symbol, we have

$$\operatorname{Res}_p\begin{bmatrix}\operatorname{tr} A\cdot\tau_i\\ a,\ b\end{bmatrix} = -\frac{m_i+1}{m_i}\operatorname{Res}_p\begin{bmatrix}\mathrm{d}f\wedge\mathrm{d}g_i\\ f,\ g_i\end{bmatrix}. \tag{2.2}$$

Next, for $i$ and $j$ with $i < j$, we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C' \begin{pmatrix} g_i \\ g_j \end{pmatrix},$$

$$C' = \begin{pmatrix} G_i \dfrac{\partial f}{\partial y} - f \displaystyle\sum_{k \neq i} m_k G_{ik} \dfrac{\partial g_k}{\partial y} & -f m_i G_{ij} \dfrac{\partial g_i}{\partial y} \\ -G_i \dfrac{\partial f}{\partial x} + f \displaystyle\sum_{k \neq i} m_k G_{ik} \dfrac{\partial g_k}{\partial x} & f m_i G_{ij} \dfrac{\partial g_i}{\partial x} \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} = C'' \begin{pmatrix} g_i \\ g_j \end{pmatrix},$$

$$C'' = \begin{pmatrix} -f m_j G_{ij} \dfrac{\partial g_j}{\partial y} & G_j \dfrac{\partial f}{\partial y} - f \displaystyle\sum_{k \neq j} m_k G_{jk} \dfrac{\partial g_k}{\partial y} \\ f m_j G_{ij} \dfrac{\partial g_j}{\partial x} & -G_j \dfrac{\partial f}{\partial x} + f \displaystyle\sum_{k \neq j} m_k G_{jk} \dfrac{\partial g_k}{\partial x} \end{pmatrix}.$$

Also, for $\ell \neq i, j$, we may write

$$\begin{pmatrix} a \\ b \end{pmatrix} = C'_\ell \begin{pmatrix} g_i \\ g_j \end{pmatrix}$$

with

$$C'_\ell = \begin{pmatrix} -f \left( m_j G_{ij} \dfrac{\partial g_j}{\partial y} + m_\ell G_{i\ell} \dfrac{\partial g_\ell}{\partial y} \right) & G_j \dfrac{\partial f}{\partial y} - f \displaystyle\sum_{k \neq j, \ell} m_k G_{jk} \dfrac{\partial g_k}{\partial y} \\ f \left( m_j G_{ij} \dfrac{\partial g_j}{\partial x} + m_\ell G_{i\ell} \dfrac{\partial g_\ell}{\partial x} \right) & -G_j \dfrac{\partial f}{\partial x} + f \displaystyle\sum_{k \neq j, \ell} m_k G_{jk} \dfrac{\partial g_k}{\partial x} \end{pmatrix}.$$

Then we compute

$$\operatorname{tr} A \cdot f G_{ij} = \frac{m_i + 1}{m_i} \det C' + \left( \sum_{\ell \neq i, j} \frac{m_\ell + 1}{m_\ell} - \frac{m_j + 1}{m_j} \right) \det C'' + \\ - \sum_{\ell \neq i, j} \frac{m_\ell + 1}{m_\ell} \det C'_\ell .$$

Thus we have

$$\operatorname{Res}_p \begin{bmatrix} \operatorname{tr} A \cdot \tau_{ij} \\ a,\ b \end{bmatrix} = - \frac{m_i - m_j}{m_i m_j} \operatorname{Res}_p \begin{bmatrix} dg_i \wedge dg_j \\ g_i,\ g_j \end{bmatrix}. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have the lemma. □

In what follows, we set $D = \sum_{i=1}^{r}(m_i + 1)D_i^{(\infty)}$, which may be called the pole divisor of $\mathrm{d}\varphi$. If $X$ is compact, by Lemmas 2.1, 2.3 and 2.4, the first formula in Proposition 1.3 becomes

$$D^2 = \sum_p \left( \sum_{i=1}^{r} \frac{(m_i+1)^2}{m_i} I_i(p) - \sum_{1\le i<j\le r} \frac{(m_i - m_j)^2}{m_i m_j} I_{ij}(p) \right), \qquad (2.4)$$

where $I_i(p) = \left(D^{(0)} \cdot D_i^{(\infty)}\right)_p$ and $I_{ij}(p) = \left(D_i^{(\infty)} \cdot D_j^{(\infty)}\right)_p$, and the sum for $p$ is taken over the intersection points of $D^{(0)}$ and $D_i^{(\infty)}$ and of $D_i^{(\infty)}$ and $D_j^{(\infty)}$. Note that (2.4) also follows from the fact that $D^{(0)} - D^{(\infty)}$ is linearly equivalent to 0. Also, from the second formula in Proposition 1.3, we have the following formula.

PROPOSITION 2.5.— *Let $\varphi$ be a meromorphic function on a compact complex surface $X$. If the critical points of $\varphi$ (away from the pole divisor) are all isolated, we have*

$$\sum_{p\in S(\mathcal{E})\cap(X\setminus|D|)} \mu_p(\varphi) + \sum_{p\in S(\mathcal{E})\cap|D|} \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \chi(X) + D^2 + K\cdot D\,.$$

*Remark* 2.6.— Following [K], we call the quantity $(1/2)(D^2 + K\cdot D) + 1$ the virtual genus of a divisor $D$. Then we may define the "virtual Euler number" $\chi'(D)$ of $D$ by

$$\chi'(D) = -(D^2 + K\cdot D)\,.$$

With this, the right hand side of the formula in Proposition 2.5 is written as $\chi(X) - \chi'(D)$.

Now we compute $\mathrm{Res}_{\sigma_2}(\mathcal{E}, p)$ for a singular point $p$ of $\mathcal{E}$ in $|D| = |D^{(\infty)}|$ in some special cases.

*Case (I)*

Let $p$ be an intersection point of $D^{(0)}$ and $D_i^{(\infty)}$ and assume that $D_i^{(\infty)}$ is non-singular at $p$ with no other components of $D^{(\infty)}$ passing through $p$. We may take a coordinate system $(x, y)$ near $p$ so that $g_i(x, y) = x$. We may write $\varphi = f/x^{m_i}$ with $f$ defining $D^{(0)}$ near $p$ and see that the holomorphic 1-form

$$\omega = \left(x\frac{\partial f}{\partial x} - m_i f\right) \mathrm{d}x + x\frac{\partial f}{\partial y}\,\mathrm{d}y$$

defines the foliation near $p$ (see the proof of Lemma 2.1). We have

$$\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \left(x\frac{\partial f}{\partial y}\,,\, m_i f - x\frac{\partial f}{\partial x}\right)_p = I_1 + I_2\,,$$

where

$$I_1 = \left(x\,,\, m_i f - x\frac{\partial f}{\partial x}\right)_p \quad \text{and} \quad I_2 = \left(\frac{\partial f}{\partial y}\,,\, m_i f - x\frac{\partial f}{\partial x}\right)_p\,.$$

First we have

$$I_1 = (x, f)_p = \left(D^{(0)} \cdot D_i^{(\infty)}\right)_p\,. \tag{2.5}$$

In order to calculate $I_2$, let $h$ be a (local) irreducible component of $\partial f/\partial y$ at $p$ and $\pi(t) = \big(x(t), y(t)\big)$ a uniformalization of the curve $h = 0$. Then, since

$$\frac{\partial f}{\partial y}\big(\pi(t)\big) = 0\,,$$

we have

$$\frac{\mathrm{d}f}{\mathrm{d}t}\big(\pi(t)\big) = \frac{\partial f}{\partial x}\big(\pi(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}\,. \tag{2.6}$$

LEMMA 2.7.— *The germs $f$ and $\partial f/\partial y$ are relatively prime at $p$.*

*Proof.*— Since $f$ is reduced and is regular in $y$, we have the lemma by the Weierstrass preparation theorem. □

LEMMA 2.8.— *The germs $\partial f/\partial x$ and $\partial f/\partial y$ and the germs $x$ and $\partial f/\partial y$ are relatively prime at $p$.*

*Proof.*— By Lemma 2.7, $f\big(\pi(t)\big) \not\equiv 0$. Hence

$$\frac{\partial f}{\partial x}\big(\pi(t)\big) \not\equiv 0 \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \not\equiv 0$$

by (2.6). □

Now if we write

$$f(\pi(t)) = \sum_{n\geq q} a_n t^n\,, \quad \frac{\partial f}{\partial x}(\pi(t)) = \sum_{n\geq r} b_n t^n \quad \text{and} \quad x(t) = \sum_{n\geq s} c_n t^n\,,$$

with $a_q \neq 0$, $b_r \neq 0$ and $c_s \neq 0$, from (2.6), we get

$$q = r + s \quad \text{and} \quad n a_n = \sum_{k=s}^{n-r} k b_{n-k} c_k\,, \qquad n \geq q\,.$$

Thus we may write

$$\left(m_i f - x\frac{\partial f}{\partial x}\right)(\pi(t)) = \sum_{n\geq q}\left(m_i a_n - \sum_{k=s}^{n-r} b_{n-k}c_k\right) t^n\,.$$

If we denote the order of this power series by $q+\delta$ with $\delta$ a non-negative integer, we have

$$\left(h, m_i f - x\frac{\partial f}{\partial x}\right)_p = q + \delta\,.$$

Now let

$$\frac{\partial f}{\partial y} = h_1^{\nu_1}\cdots h_\ell^{\nu_\ell}$$

be the irreducible decomposition and apply the previous argument for each $h_k$, $k = 1, \ldots, \ell$. Then writing $q$ and $\delta$ for $h_k$ by $q_k$ and $\delta_k$ and recalling that

$$q_k = \left(h_k\,, x\frac{\partial f}{\partial x}\right)_p\,,$$

we get

$$\begin{aligned} I_2 &= \left(\frac{\partial f}{\partial y}\,, x\frac{\partial f}{\partial x}\right)_p + \delta_p = \left(\frac{\partial f}{\partial y}\,, \frac{\partial f}{\partial x}\right)_p + \left(\frac{\partial f}{\partial y}\,, x\right)_p + \delta_p \\ &= \mu_p(f) + \left(D^{(0)}\cdot D_i^{(\infty)}\right)_p - 1 + \delta_p\,, \end{aligned}$$

where $\delta_p = \sum_{k=1}^{\ell} \nu_k \delta_k$. Combined with (2.5), we get the following proposition.

PROPOSITION 2.9.— *Let $\varphi$ be a meromorphic function on a complex surface $X$ whose critical points in $X \setminus |D^{(\infty)}|$ are all isolated. For an intersection point $p$ of $D^{(0)}$ and $D_i^{(\infty)}$ such that $D_i^{(\infty)}$ is non-singular at $p$ with no other components of $D^{(\infty)}$ passing through $p$, we have*

$$\mathrm{Res}_{\sigma_2}(\mathcal{E},p) = \mu_p(f) + \delta_p + 2\Big(D^{(0)} \cdot D_i^{(\infty)}\Big)_p - 1\,,$$

*where $f$ is a defining equation of $D^{(0)}$ near $p$.*

Note that, in general, we have $\delta_p = 0$.

*Case (II)*

Let $p$ be an intersection point of $D_i^{(\infty)}$ and $D_j^{(\infty)}$ and assume that $D_i^{(\infty)}$ and $D_j^{(\infty)}$ intersect transversally at $p$ with $D^{(0)}$ or any other component of $D^{(\infty)}$ not passing through $p$. We may take a coordinate system $(x, y)$ near $p$ so that $g_i(x,y) = x$ and $g_j(x,y) = y$. We may write $\varphi = 1/(xy)$ and see that the holomorphic 1-form

$$\omega = y\,\mathrm{d}x + x\,\mathrm{d}y$$

defines the foliation near $p$. Then we have

$$\mathrm{Res}_{\sigma_2}(\mathcal{E},p) = 1\,. \tag{2.7}$$

## 3. Foliations arising from polynomials

In this section we apply the formulas in the previous section to the case of foliations on the two dimensional projective space $\mathbf{P}^2$ or on its modifications which are defined by compactifying polynomials in two variables.

Let $f(x,y)$ be a polynomial of degree $d$ with complex coefficients. We regard $f(x,y)$ as a function on $\mathbf{C}^2$ and extend it to a meromorphic (rational) function $\varphi_0$ on $\mathbf{P}^2$. If we denote by $(\zeta_0 : \zeta_1 : \zeta_2)$ homogeneous coordinates on $\mathbf{P}^2$, the rational function $\varphi_0$ is given by

$$\varphi_0(\zeta_0 : \zeta_1 : \zeta_2) = f(\zeta_1/\zeta_0, \zeta_2/\zeta_0) = \frac{\widetilde{f}(\zeta_0,\zeta_1,\zeta_2)}{\zeta_0^d}\,,$$

where, denoting by $f_k$ the homogeneous piece of $f$ of degree $k$,

$$\widetilde{f}(\zeta_0,\zeta_1,\zeta_2) = \zeta_0^d f_0 + \zeta_0^{d-1} f_1(\zeta_1,\zeta_2) + \cdots + \zeta_0 f_{d-1}(\zeta_1,\zeta_2) + f_d(\zeta_1,\zeta_2)\,.$$

We assume that the critical points of $f$ are all isolated. Thus the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are relatively prime and the polynomial $f$ is reduced.

Let $\mathcal{F}$ denote the singular foliation on $\mathbf{P}^2$ determined by $\varphi_0$. If we denote by $L_\infty$ the "infinite line" $\zeta_0 = 0$, then the pole divisor of $\varphi_0$ is $dL_\infty$. Thus the line bundle $F$ associated with $\mathcal{F}$ is given by $F = [-(d+1)L_\infty]$ (Lemma 2.1).

Let $U_i$ denote the coordinate neighborhood $\{\zeta_i \neq 0\}$ in $\mathbf{P}^2$, for $i = 0, 1, 2$. On the "finite part" $U_0$, $\mathcal{F}$ is defined by $\mathrm{d}f$, since, by assumption, the critical points of $f$ are all isolated. We have $S(\mathcal{F}) \cap U_0 = C(f)$, the set of critical points of $f$ in $U_0 = \mathbf{C}^2$. Now we find 1-forms defining $\mathcal{F}$ on the infinite parts of $\mathbf{P}^2$. We work on the coordinate neighborhood $U_2$, however, it is similar on $U_1$. In fact, changing the coordinate system on $\mathbf{C}^2$, if necessary, we may asuume that $f_d(x,y)$ is not divisible by $y$. Then the singular points of $\mathcal{F}$ on $L_\infty$ are all in $U_2$. We take

$$(u, v) = \left(\frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2}\right)$$

as a coordinate system on $U_2$. Then the function $\varphi_0$ is written as, on $U_2$,

$$\varphi_0 = \frac{\widehat{f}(u,v)}{u^d},$$

where $\widehat{f}(u,v) = \widetilde{f}(u,v,1)$. Hence, on $U_2$, $\mathcal{F}$ is defined by

$$\omega = \left(u\frac{\partial \widehat{f}}{\partial u} - d \cdot \widehat{f}\right) \mathrm{d}u + u\frac{\partial \widehat{f}}{\partial v}\,\mathrm{d}v$$

(see the proof of Lemma 2.1). Note that the points in $S(\omega) \cap L_\infty$ are given by

$$u = 0, \quad \text{and} \quad f_d(v,1) = 0.$$

Thus, if $f_d(x,y) = \prod_{i=1}^k (b_i x - a_i y)^{d_i}$, $\sum_{i=1}^k d_i = d$, is the factorization of $f_d$, there are $k$ singular points $p_i = (0 : a_i : b_i)$, $i = 1, \ldots, k$, of $\mathcal{F}$ on $L_\infty$. We call $d_i$ the multiplicity of $f$ at a point at infinity $p_i$ and denote it by $\mathrm{m}_{p_i}(f)$. It is equal to the intersection number of the divisor of $\widehat{f}$ and $L_\infty$ at $p_i$.

Let $\mathcal{E}$ be the dimension one foliation corresponding to $\mathcal{F}$. On the finite part $U_0$ of $\mathbf{P}^2$, the vector field

$$v_0 = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$$

defines $\mathcal{E}$ and on the infinite part $U_2$, the vector field

$$v_2 = u\frac{\partial \widehat{f}}{\partial v}\frac{\partial}{\partial u} - \left(u\frac{\partial \widehat{f}}{\partial u} - d\cdot \widehat{f}\right)\frac{\partial}{\partial v}$$

defines $\mathcal{E}$.

For a singular point $p$ in $L_\infty \cap U_2$, we have, from Proposition 2.9,

$$\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\widehat{f}) + \delta_p + 2\mathrm{m}_p(f) - 1\,, \tag{3.1}$$

where $\delta_p$ is a non-negative integer defined as in Proposition 2.9 with $f$ replaced by $\widehat{f}$. If the curve $\widehat{f} = 0$ is generic in the family $\widehat{f} - \lambda u^d = 0$, $\lambda \in \mathbf{C}$, the number $\delta_p$ is equal to the "value of a jump in Milnor number at infinity" of D. T. Lê.

Since $\chi(\mathbf{P}^2) = 3$, $D = (d+1)L_\infty$, $K = -3L_\infty$, $L_\infty^2 = 1$ and $\sum \mathrm{m}_p(f) = d$, from Proposition 2.5 and (3.1), we have the following formula.

THEOREM 3.1.— *For a polynomial $f$ of degree $d$, we have*

$$\sum_{p\in C(f)} \mu_p(f) + \sum_{i=1}^{k} \mathrm{Res}_{\sigma_2}(\mathcal{E}, p_i) = d^2 - d + 1\,,$$

*or equivalently*

$$\sum_{p\in C(f)} \mu_p(f) + \sum_{i=1}^{k} \left(\mu_{p_i}(\widehat{f}) + \delta_{p_i} - 1\right) = d^2 - 3d + 1\,,$$

*where, letting $f_d(x,y) = \prod_{i=1}^{k} \left(b_i x - a_i y\right)^{d_i}$ be the factorization of the highest degree homogeneous piece $f_d$ of $f$, $p_i$ denotes the point on $L_\infty$ given by $p_i = (0 : a_i : b_i)$.*

This formula, together with a nice interpretation of the numbers $\delta_p$ as mentioned above, is also obtained by D. T. Lê (private communication).

For the other residue $\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p)$ at $p$ in $S(\mathcal{E}) \cap L_\infty$, we have, by Lemma 2.4,

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = \frac{(d+1)^2 \mathrm{m}_p(f)}{d}$$

and (2.4) becomes

$$(d+1)^2 = \sum_{p\in S(\mathcal{E})\cap L_\infty} \frac{(d+1)^2 \mathrm{m}_p(f)}{d}\,. \tag{3.2}$$

This is a tautology, since $\sum \mathrm{m}_p(f) = d$. However, it is interpreted as a formula to allocate the self-intersection number of the pole divisor $(d+1)L_\infty$ of $\mathrm{d}\varphi$ to the singular points of the foliation $\mathcal{E}$.

*Example 3.2.* — For the polynomial $f(x,y) = x - x^2y$, the singular points of $\mathcal{E}$ are $p_1 = (0:1:0)$ and $p_2 = (0:0:1)$. We have

$$\mathrm{Res}_{\sigma_2}(\mathcal{E}, p_1) = 1\,, \qquad \mathrm{Res}_{\sigma_2}(\mathcal{E}, p_2) = 6\,,$$
$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p_1) = \frac{16}{3}\,, \qquad \mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p_2) = \frac{32}{3}\,.$$

*Example 3.3.* — For $f(x,y) = y^n - x^m$ $(2 \le n \le m)$, the singular points of $\mathcal{E}$ are $p = (1:0:0)$ and $p_1 = (0:0:1)$. We have

$$\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(f) = (n-1)(m-1)\,, \qquad \mathrm{Res}_{\sigma_2}(\mathcal{E}, p_1) = m^2 - mn + n\,,$$
$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0\,, \qquad \mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p_1) = (m+1)^2\,.$$

Next we consider the compactification $\pi : X \to \mathbf{P}^2$ of $f$ as constructed by D. T. Lê and C. Weber in [LW]. Following [LW], the set $A(f)$ of atypical values of $f$ is expressed as $A(f) = D(f) \cup I(f)$, where $D(f)$ is the set of critical values of $f$ and $I(f)$ is determined by the behavior of $f$ at infinity (see [Fr] for more details). Then the compactification $\pi : X \to \mathbf{P}^2$ is obtained from $\mathbf{P}^2$ by a finite sequence of blowing-ups of "points at infinity" and has the following properties [LW]:

(1) $X$ is a compact complex surface and $\pi$ is a proper holomorphic map inducing a biholomorphic map of $X \setminus \pi^{-1}(L_\infty)$ onto $\mathbf{P}^2 \setminus L_\infty = \mathbf{C}^2$.

(2) $\pi^{-1}(L_\infty)$ is a union of projective lines with normal crossings.

(3) The meromorphic function $\varphi = \varphi_0 \circ \pi$ does not have indeterminacy points, where $\varphi_0 = \widetilde{f}/\zeta_0^d$. Thus we may think of $\varphi : X \to \mathbf{P}^1$ as a holomorphic map.

(4) For $\lambda \in \mathbf{C} - I(f)$, $\pi$ gives an imbedded resolution of the singularities of the curve $C_\lambda : \widetilde{f} - \lambda\zeta_0^d = 0$ on $L_\infty$.

Moreover, if we denote by $\mathcal{A}$ and $\mathcal{A}_\infty$, respectively, the intersection graphes of the divisor $\pi^{-1}(L_\infty)$ and the pole divisor of $\varphi$,

(5) $\mathcal{A}$ is a connected tree and $\mathcal{A}_\infty$ is a connected sub-tree of $\mathcal{A}$.

(6) Each connected component of $\mathcal{A} \setminus \mathcal{A}_\infty$ is a bamboo which contains a unique dicritical component (a component of $\pi^{-1}(L_\infty)$ on which $\varphi$ is not constant).

Let $\mathcal{E}$ be the foliation on $X$ determined by $\varphi$ and let $D^{(\infty)} = \sum_{i=1}^r m_i D_i^{(\infty)}$ be the pole divisor of $\varphi$. For simplicity, we assume that the critical points of $\varphi$ (away from $|D^{(\infty)}|$, even on $|\pi^{-1}(L_\infty)| \setminus |D^{(\infty)}|$) are all isolated. Thus for any finite value $\lambda$ (even for an atypical value of $f$), the divisor $\varphi = \lambda$ is reduced. Note that this assumption is satisfied if each component of $\mathcal{A} \setminus \mathcal{A}_\infty$ contains only one vertex (the dicritical component). Then there are two types of singularities of $\mathcal{E}$:

(a) critical points of $\varphi$ on $X \setminus |D^{(\infty)}|$,

(b) intersection points in $D^{(\infty)}$.

If $p$ is a singular point of type (a), the residues are given by Lemma 2.3. Note that if $p$ is in $X \setminus \pi^{-1}(L_\infty) \simeq \mathbf{P}^2 \setminus L_\infty = \mathbf{C}^2$, we have $\mu_p(\varphi) = \mu_{\pi(p)}(f)$.

Let $p$ be a singular point of type (b). If $p$ is the intersection point of $D_i^{(\infty)}$ and $D_j^{(\infty)}$, we have, by Lemma 2.4 and (2.7),

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = -\frac{(m_i - m_j)^2}{m_i m_j}, \quad \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = 1. \tag{3.3}$$

If we again set $D = \sum_{i=1}^r (m_i + 1) D_i^{(\infty)}$ (the pole divisor of $d\varphi$), the residue formula (2.4) becomes

$$D^2 = -\sum_{1 \le i < j \le r} \frac{(m_i - m_j)^2}{m_i m_j} \delta_{ij}, \tag{3.4}$$

where $\delta_{ij} = 1$, if $D_i^{(\infty)}$ meets $D_j^{(\infty)}$, and $\delta_{ij} = 0$ otherwise.

If we recall that the critical values of $\varphi$ are atypical values of $f$ [LW], we see that the sum of the residues for $\sigma_2$ over the singular points $p$ of type (a) may be expressed as $\sum_{\lambda \in A(f)} \mu(X_\lambda)$, where $X_\lambda$ denotes the (reduced) curve $\varphi = \lambda$ and $\mu(X_\lambda)$ its total Milnor number. Denoting by $\ell$ the number of intersection points in $D^{(\infty)}$, which is the number of 1-simplices in $\mathcal{A}_\infty$, we have, from (3.3) and Proposition 2.5, the following formula.

THEOREM 3.4.— *In the above situation, we have*

$$\sum_{\lambda \in A(f)} \mu(X_\lambda) + \ell = \chi(X) - \chi'(D).$$

In the above, the sum $\sum_{\lambda \in A(f)} \mu(X_\lambda)$ can also be written as

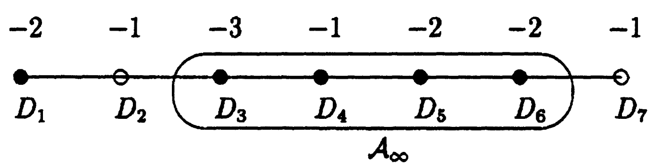
$$\sum_{p \in C(f)} \mu_p(f) + \sum_{p \in C(\varphi) \cap |\pi^{-1}(L_\infty)|} \mu_p(\varphi),$$

where $C(\varphi)$ denotes the critical set of $\varphi$ restricted to $X \setminus |D^{(\infty)}|$. Note that the value of $\varphi$ at a point $p$ in the second sum is in $I(f)$. If we denote by $n$ the number of blowing-ups to obtain $X$ (= (number of vertices in $\mathcal{A}$) $-$ 1), we have $\chi(X) = n + 3$. Recall also that $\chi'(D) = -(D^2 + K \cdot D)$ (Remark 2.6). We may represent $K$ by a divisor with support in $\pi^{-1}(L_\infty)$.

*Remark* 3.5.— The formula in Theorem 3.4 may be thought of as a "Milnor number formula" in the presence of multiple (non-reduced) fibers. In fact, if $D^{(\infty)}$ is reduced, the formula coincides with the Milnor number formula [Fl, Example 14.1.5] for the map $\varphi : X \to \mathbf{P}^1$, since, noting that $D = 2D^{(\infty)}$ in this case and recalling $\left(D^{(\infty)}\right)^2 = 0$, we have $\chi'(D) = 2\chi'(D^{(\infty)}) = \chi(\mathbf{P}^1)\chi(X_t)$, where $X_t$ is the (non-singular) curve $\varphi = t$ for $t \in \mathbf{C} \setminus A(f)$.

The following example of compactification is due to D. T. Lê and C. Weber. The residue formulas in this case are examined in [Sg].

*Example 3.6.*— Let $f(x, y) = x - x^2 y$. The polynomial $f$ has no critical points and the rational function $\varphi_0(\zeta_0 : \zeta_1 : \zeta_2) = \zeta_1(\zeta_1\zeta_2 - \zeta_0^2)/\zeta_0^3$ has indeterminancy points at $(0 : 0 : 1)$ and $(0 : 1 : 0)$. We see that $A(f) = I(f) = \{0\}$. The intersection graph $\mathcal{A}$ of $\pi^{-1}(L_\infty)$ is as follows:




The integers in the first row denote the self-intersection numbers. $D_2$ and $D_7$ are the dicritical components, $D_4$ is the proper transform of $L_\infty$ and the value of $\varphi$ on $D_1$ is 0, which is atypical. We have

$$D^{(\infty)} = D_3 + 3D_4 + 2D_5 + D_6$$

and

$$K = -D_1 - 2D_2 - 2D_3 - 3D_4 - 2D_5 - D_6 .$$

Hence

$$D = 2D_3 + 4D_4 + 3D_5 + 2D_6 .$$

Thus we have $D^2 = -2$ and $K \cdot D = -2$. Therefore, $\chi(X) - \chi'(D) = 3 + 6 - 2 - 2 = 5$. On the other hand, the foliation defined by $\varphi$ has 5 singular points $p_i$, $i = 1, \ldots, 5$, where $p_1$ and $p_2$ are the critical points of $\varphi$ and are on $D_1$ and $p_i$ is the intersection point of $D_i$ and $D_{i+1}$ for $i = 3, 4, 5$. We compute the residues and obtain the following table:

| | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $\mathrm{Res}_{\sigma_1^2}$ | 0 | 0 | $-\frac{4}{3}$ | $-\frac{1}{6}$ | $-\frac{1}{2}$ |
| $\mathrm{Res}_{\sigma_2}$ | 1 | 1 | 1 | 1 | 1 |

Thus we see that (3.4) and the formula in Theorem 3.4 are satisfied.

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