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Solutions of the equation $f_y u_x - f_x u_y = g$[*]

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RÉSUMÉ. — On étudie le problème d'existence de solutions de l'équation aux dérivées partielles $f_y u_x - f_y u_y = g$ localement, au voisinage d'un point singulier isolé, dans le cadre analytique réel. On suppose que la fonction f a un minimum local à l'origine.

ABSTRACT. — We study locally, on a neighborhood of an isolated singular point, the existence of solutions of the partial differential equation $f_y u_x - f_x u_y = g$, in the real analytic case. We suppose that the function f has a minimum at the origin.

0. Introduction

We consider the equation

$$f_y u_x - f_x u_y = g \,. \tag{1}$$

where f, g are real analytic functions in a neighborhood of $(0,0) \in \mathbb{R}^2$.

We will suppose that:

i) $f(0,0) = 0$ and $f > 0$ outside the origin, and

ii) the ideal J_f generated by f_x, f_y is of finite codimension as an $\mathbb{R}$-vector space, in $\mathbb{R}\{x,y\}$.

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We say that a solution u of (1) is:

a) *regular* if u is analytic in a neighborhood of the origin.

b) *singular* if u is analytic in a neighborhood of the origin, but not necessarily at the origin.

We consider $\mathbb{R}\{x,y\}$ an $\mathbb{R}\{t\}$-module with the definition

$$\varphi(t) \cdot h(x,y) = \varphi\big(f(x,y)\big)h(x,y)\,.$$

Let Σ be the $\mathbb{R}$-vector space of germs at $(0,0)$ of those analytic functions g such that (1) has a singular solution.

Let Γ be the $\mathbb{R}$-vector space of germs at $(0,0)$ of those analytic functions g such that (1) has a regular solution.

Since

$$f_y\big(fu\big)_x - f_x\big(fu\big)_y = f(f_y u_x - f_x u_y)\,,$$

$\Gamma \subset \Sigma$ are submodules of $\mathbb{R}\{x,y\}$.

The purpose of this article is to study the quotient $T = \Sigma/\Gamma$.

We will show that its structure is related to the action of the monodromy of f at 0 (extending f to an analytic function in a neighborhood of $(0,0) \in \mathbb{C}^2$) over the vanishing cycle γ generated by the cycle in $\mathbb{R}^2$ defined by $f = \epsilon$ ($\epsilon > 0$ sufficiently small). More precisely, let E be the $\mathbb{C}$-vector space of these vanishing cycles and let

$$\mu = \dim_{\mathbb{C}} E = \dim_{\mathbb{R}} \mathbb{R}\{x,y\}/J_f\,,$$

the Milnor number of f at 0.

Let k be the dimension of the subspace of E generated by γ, $L\gamma$, $L^2\gamma$, ... where $L : E \to E$ is the monodromy. We have the following results.

THEOREM 1.— *T is a free $\mathbb{R}\{t\}$-module of rank $\nu = \mu - k$.*

THEOREM 2.— $\nu = \dim_{\mathbb{R}} \Sigma/(\Sigma \cap J_f)$.

Observation.— Since

$$\int_{f=\epsilon} x\,\mathrm{d}y = \int_{f\leq\epsilon} \mathrm{d}x \wedge \mathrm{d}y > 0\,,$$

we have that $\gamma \neq 0$. Then, $\nu \leq \mu - 1$.

Examples

1) $f = x^2 + y^2$. In this case $\mu = 1$ and $\nu = 0$. The existence of regular solutions is equivalent to that of singular solutions.
2) $f = x^2 + y^4$. In this case $\mu = 3$. The monodromy is induced by the map

$$(x, y) \longrightarrow (e^{\pi i}x, e^{\pi i/2}y) = (-x, iy)\,.$$

It follows that $L^2\gamma = -\gamma$. Since the eigenvalues of L are 1, i, $-i$ and γ is an integer cycle, if γ were an eigenvector of L, we would have $L\gamma = \gamma$, which is impossible. Then, γ and $L\gamma$ are independent and $k = 2$. Then $\nu = 1$. (Using the theorem in Section 1, we see that the class of y is a generator of T.)

1. Solutions to the equation

$$f_y u_x - f_x u_y = g \tag{1}$$

THEOREM 1.1.— *The equation (1) has a singular solution if and only if*

$$\int_{f\leq\epsilon} g(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0\,, \quad \forall\,\epsilon > 0 \text{ small enough}\,. \tag{2}$$

First, we are going to define a change of coordinates to simplify the resolution of the equation (1).

Since the problem is local, we are going to suppose, throughout this work, that the neighborhoods of the origin are all sufficiently small.

Let U be a neighborhood of the origin where f is analytic and where there is no critical point of f different from $(0,0)$. Let $V \subset U$ be a simply connected neighborhood of the origin such that $\overline{V} \subset U$.

LEMMA 1.1.— *If $\epsilon > 0$ is small enough, $f\big|_V = \epsilon$ is a simple closed curve $\mathbf{C}_\epsilon$ around the origin.*

Consider the composed function

$$\mathbb{R} \xrightarrow{\exp} S^1 \xrightarrow{\sigma} \mathbf{C}_\epsilon \,,$$

where $\exp(x) = e^{2\pi i x}$ and σ is an analytic diffeomorphism. The analytic curve $\gamma = \sigma \circ \exp$ is periodic with period 1 and is a parametrization of the curve $\mathbf{C}_\epsilon$.

Integrating the vector field $-\operatorname{grad} f$, we obtain:

$$\Phi : (-\lambda, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \lambda > 0$$

such that

$$\frac{\partial \Phi}{\partial t}(t,\theta) = -\operatorname{grad} f\big(\Phi(t,\theta)\big) \quad \text{and} \quad \Phi(0,\theta) = \gamma(\theta)\,.$$

LEMMA 1.2.— $\lim_{t\to\infty} \Phi(t,x) = (0,0)$.

It is a direct application of the Liapunov criterium.

Consider

$$\varphi : (0,\epsilon) \times \mathbb{R} \longrightarrow \mathbb{R}^2,$$

defined by $\varphi(\rho,\theta) = \Phi(t,\theta)$, where $t > 0$ is such that $f\big(\Phi(t,\theta)\big) = \rho$, i.e., $f\big(\varphi(\rho,\theta)\big) = \rho$.

Since for each $(\rho,\theta) \in (0,\epsilon) \times \mathbb{R}$, there exists a unique $t \in (0,\infty)$ such that $\Phi(t,\theta) = \rho$ (Lemma 1.2), φ is well defined.

Moreover, φ is locally one to one, since $\varphi(\rho,\theta) = \varphi(\rho',\theta')$ if and only if $\rho = \rho'$ and $\theta - \theta' \in \mathbb{Z}$.

Using the implicit function theorem we have the two following lemmas.

LEMMA 1.3.— *φ is a real analytic function, periodic with period 1, in the variable θ.*

LEMMA 1.4.— *φ is a local diffeomorphism.*

LEMMA 1.5.— *The diffeomorphism φ transforms the equation $\mathrm{d}u \wedge \mathrm{d}f = g\,\mathrm{d}x \wedge \mathrm{d}y$ into*

$$\widetilde{u}_\theta \,\mathrm{d}\theta \wedge \mathrm{d}\rho = -\widetilde{g} J \,\mathrm{d}\theta \wedge \mathrm{d}\rho\,,$$

where J is the Jacobian of the change of coordinates, $\widetilde{u} = u \circ \varphi$ and $\widetilde{g} = g \circ \varphi$.

If we choose $\epsilon > 0$ small enough, we may suppose that $h = -\widetilde{g}J$ is analytic in $(0, \epsilon) \times \mathbb{R}$. Besides, h is periodic, with period 1, in the second variable.

LEMMA 1.6.— *The function $\widetilde{u}(\rho, \theta)$ given by:*

$$\widetilde{u}(\rho, \theta) = \int_0^1 \theta h(\rho, t\theta)\,\mathrm{d}t\,,$$

is analytic in $(0, \epsilon) \times \mathbb{R}$ and verifies $\widetilde{u}_\theta = h$.

Proof.— Integration by parts. □

LEMMA 1.7.— *If*

$$\int_0^1 h(\rho, \theta)\,\mathrm{d}\theta = 0 \quad \textit{for all } 0 < \rho < \epsilon \textit{ and } \epsilon \textit{ small enough}\,,$$

then the function $\widetilde{u}$ in Lemma 1.6 is periodic in θ, with period 1.

Proof.— Actually,

$$\begin{aligned}
\widetilde{u}(\rho, \theta + 1) &= \int_0^1 (\theta + 1) h\big(\rho, t(\theta + 1)\big)\,\mathrm{d}t \\
&= \int_0^{\theta+1} h(\rho, \nu)\,\mathrm{d}\nu \\
&= \int_0^{\theta} h(\rho, \nu)\,\mathrm{d}\nu + \int_\theta^{\theta+1} h(\rho, \nu)\,\mathrm{d}\nu \\
&= \int_0^{\theta} h(\rho, \nu)\,\mathrm{d}\nu + \int_0^{1} h(\rho, \nu)\,\mathrm{d}\nu \\
&= \int_0^1 \theta h(\rho, t\theta)\,\mathrm{d}t = \widetilde{u}(\rho, \theta)\,. \ \square
\end{aligned}$$

Proof of Theorem 1.1

Suppose (1) has a singular solution u.

Choose $\epsilon > 0$ such that the curve $f = \epsilon$ is entirely contained inside the region where u is regular, and satisfies Lemma 1.1.

For $0 < \delta < \epsilon$,

$$\begin{aligned}\int_{\delta \le f \le \epsilon} g(x, y)\, \mathrm{d}x\, \mathrm{d}y &= \int_{\delta \le f \le \epsilon} (f_y u_x - f_x u_y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{f=\epsilon} u\, \mathrm{d}f - \int_{f=\delta} u\, \mathrm{d}f = 0\,.\end{aligned}$$

Hence,

$$\int_{f \le \epsilon} g\, \mathrm{d}x\, \mathrm{d}y = \lim_{\delta \to 0} \int_{\delta \le f \le \epsilon} g\, \mathrm{d}x\, \mathrm{d}y = 0\,.$$

On the other hand, if (2) holds,

$$\begin{aligned}0 = \int_{f \le \epsilon} g\, \mathrm{d}x\, \mathrm{d}y &= \int_0^\epsilon \int_0^1 \widetilde{g} J\, \mathrm{d}\theta\, \mathrm{d}\rho \\ &= \int_0^\epsilon \int_0^1 -h(\rho, \theta)\, \mathrm{d}\theta\, \mathrm{d}\rho\,.\end{aligned}$$

And then,

$$\int_0^1 h(\rho, \theta)\, \mathrm{d}\theta = 0 \quad \text{if } 0 < \rho < \epsilon\,.$$

By lemmas 1.6 and 1.7, there exists $\widetilde{u}$ analytic in $(0, \epsilon) \times \mathbb{R}$ periodic with period 1 in θ, that verifies the equation $\widetilde{u}_\theta = -\widetilde{g} J$. The $\widetilde{u}$ passes to quotient and defines u by $\widetilde{u} = u \circ \varphi$. The function u is a singular solution to (1), by lemmas 1.4 and 1.5.□

Observe that, if (2) holds, then the equation (1) always has a singular solution that can not be extended to a regular solution. In fact, even though the equation has a regular solution u, the solution $v = u + 1/f$ can not be extended to a regular solution.

2. Considerations on the Gauss–Manin connection

Let f be an analytic function in a neighborhood of $(0, 0) \in \mathbb{C}^n$, $n \ge 2$, with an isolated singularity at the origin.

Let μ be the Milnor number (*cf.* [5]) of the germ of the function f at 0.

Denoting Ω_0^p the sheaf of germs of the p-forms that are holomorphic in a neighborhood of the origin in $\mathbb{C}^n$, we define

$$G = \frac{\Omega_0^n}{\mathrm{d}f \wedge \mathrm{d}\Omega_0^{n-2}}.$$

The operation $\circ : \mathbb{C}\{t\} \times G \to G$ defined by

$$\varphi(t) \circ [\omega] = [\varphi(f)\omega],$$

where $[\cdot]$ indicates the class in G, gives G the structure of a $\mathbb{C}\{t\}$-module.

THEOREM 2.1.— (Brieskorn, Sebastiani) *G is a free $\mathbb{C}\{t\}$-module of* rank μ.

Proof (cf. [3, theorem 5.1])

Consider F, the $\mathbb{C}\{t\}$-submodule of G given by

$$F = \frac{\mathrm{d}f \wedge \Omega_0^{n-1}}{\mathrm{d}f \wedge \mathrm{d}\Omega_0^{n-2}}.$$

G/F is a torsion module.

Define the connection

$$\mathrm{D} : F \longrightarrow G, \quad \mathrm{D}[\mathrm{d}f \wedge \theta] = [\mathrm{d}\theta].$$

It is clear that D is well defined, for if $\mathrm{d}f \wedge \alpha \in [\mathrm{d}f \wedge \theta]$, then there exists $\beta \in \Omega_0^{n-2}$ such that

$$\mathrm{d}f \wedge \alpha = \mathrm{d}f \wedge \theta + \mathrm{d}f \wedge \mathrm{d}\beta.$$

Thus, by the de Rham's lemma,

$$\alpha = \theta + \mathrm{d}\beta + \mathrm{d}f \wedge \gamma \quad \text{for a certain } \gamma \in \Omega_0^{n-2}.$$

It goes without saying that D is $\mathbb{C}$-linear and that it is a connection.

We are going to show that D is a bijective connection.

First, note that if

$$\mathrm{D}[\mathrm{d}f \wedge \theta] = [\mathrm{d}\theta] \in \mathrm{d}f \wedge \mathrm{d}\Omega_0^{n-2},$$

then

$$\mathrm{d}\theta = \mathrm{d}f \wedge \mathrm{d}\beta = -\mathrm{d}(\mathrm{d}f \wedge \beta) \quad \text{for some } \beta \in \Omega_0^{n-2}.$$

This is equivalent to $\mathrm{d}(\theta + \mathrm{d}f \wedge \beta) = 0$. Thus, there exists $\gamma \in \Omega_0^{n-2}$ such that $\theta + \mathrm{d}f \wedge \beta = \mathrm{d}\gamma$, that is,

$$\mathrm{d}f \wedge \theta = \mathrm{d}f \wedge \mathrm{d}\gamma .$$

Hence, D is a 1-1 connection.

Moreover, since every $\omega \in \Omega_0^n$ is exact, D is surjective.

If $t \neq 0$ is given, let $X_t = B_\epsilon \cap f^{-1}(t)$ be the Milnor fiber of the function f (cf. [5]), where B_ϵ is the ball in $\mathbb{C}^n$ with center in $(0,0)$ and radius $\epsilon > 0$. Also, let $0 < |t| < \delta \ll \epsilon$.

If $\omega \in \Omega_0^n$ and $p \in X_t$, there exists a $(n-1)$-form α, holomorphic in a neighborhood of p, such that $\omega = \mathrm{d}f \wedge \alpha$, in neighborhood of p, and that $\alpha\big|_{X_t}$ is well defined. We define $\omega/\mathrm{d}f$ this way. $\omega/\mathrm{d}f$ is a $(n-1)$-form over each fiber.

Let $\gamma_t \subset X_t$, $0 < |t| < \delta$, be a $(n-1)$ vanishing cycle.

If $N \subset G$ is the set

$$N = \left\{ [\omega] \in G \;\middle|\; \int_{\gamma_t} \frac{\omega}{\mathrm{d}f} = 0 \right\},$$

then $N \subset G$ as a $\mathbb{C}\{t\}$-submodule.

Observe that $\int_{\gamma_\epsilon} \omega/\mathrm{d}f$ depends only on the class $[\omega] \in G$.

Let $M = \mathrm{D}^{-1}(N)$. We want to show that if $[\mathrm{d}f \wedge \theta] \in M$, then $\int_{\gamma_t} \theta = 0$. First we are going to prove the following lemma.

LEMMA 2.1

$$\int_{\gamma_t} \frac{\mathrm{D}[\mathrm{d}f \wedge \theta]}{\mathrm{d}f} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta .$$

Proof.— Suppose $\mathrm{d}\theta = \mathrm{d}f \wedge \pi$, where $\pi \in \Omega_0^{n-1}$. By [3, § 4.3],

$$\int_{\gamma_t} \frac{\mathrm{d}\theta}{\mathrm{d}f} = \int_{\gamma_t} \pi = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta . \tag{3}$$

In the general case, if $\theta \in \Omega_0^{n-1}$ is given, then there exists an integer $\mathcal{N}$ and $\eta \in \Omega_0^{n-1}$ such that $f^{\mathcal{N}} \mathrm{d}\theta = \mathrm{d}f \wedge \eta$.

For $\omega = f^{\mathcal{N}}\theta$, we have

$$\mathrm{d}\omega = \mathcal{N} f^{\mathcal{N}-1} \mathrm{d}f \wedge \theta + f^{\mathcal{N}} \mathrm{d}\theta = \mathrm{d}f \wedge \alpha .$$

By (3),

$$\int_{\gamma_t} \frac{\mathrm{d}\omega}{\mathrm{d}f} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \omega .$$

On the other hand,

$$\int_{\gamma_t} \frac{\mathrm{d}\omega}{\mathrm{d}f} = \int_{\gamma_t} \mathcal{N} f^{\mathcal{N}-1}\theta + \int_{\gamma_t} f^{\mathcal{N}} \frac{\mathrm{d}\theta}{\mathrm{d}f} = \mathcal{N} t^{\mathcal{N}-1} \int_{\gamma_t} \theta + t^{\mathcal{N}} \int_{\gamma_t} \frac{\mathrm{d}\theta}{\mathrm{d}f}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \omega = \frac{\mathrm{d}}{\mathrm{d}t} t^{\mathcal{N}} \int_{\gamma_t} \theta = \mathcal{N} t^{\mathcal{N}-1} \int_{\gamma_t} \theta + t^{\mathcal{N}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta .$$

Since $t \neq 0$, it follows from the equations above that

$$\int_{\gamma_t} \frac{\mathrm{d}\theta}{\mathrm{d}f} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta .$$

This proves the lemma. □

PROPOSITION 2.1.— $M = \left\{ [\mathrm{d}f \wedge \theta] \in F \mid \int_{\gamma_t} \theta = 0 \right\}$.

Proof.— $M \subset F$, by definition. If $\int_{\gamma_t} \theta = 0$, then, by the lemma,

$$\int_{\gamma_t} \frac{\mathrm{D}[\mathrm{d}f \wedge \theta]}{\mathrm{d}f} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta = 0 .$$

In other words, $\mathrm{D}[\mathrm{d}f \wedge \theta] \in N$, i.e., $[\mathrm{d}f \wedge \theta] \in M$.

Suppose $[\mathrm{d}f \wedge \theta] \in M$. There exists $\omega \in \Omega_0^n$ such that $[\omega] \in N$ and $\mathrm{D}^{-1}[\omega] = [\mathrm{d}f \wedge \theta]$. Hence, by the definition of N and by the lemma,

$$0 = \int_{\gamma_t} \frac{\omega}{\mathrm{d}f} = \int_{\gamma_t} \frac{\mathrm{D}[\mathrm{d}f \wedge \theta]}{\mathrm{d}f} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \theta .$$

It follows that $\int_{\gamma_t} \theta$ is constant. Then, $\int_{\gamma_t} \theta = 0$ (cf. [3, § 4.5]). □

PROPOSITION 2.2.— $M = N \cap F$.

Proof.— It follows from Proposition 2.1 that M is a $\mathbb{C}\{t\}$-submodule of G and that $M \subset N$.

On the other hand, if $[\omega] \in N \cap F$, there exists $\theta \in \Omega_0^{n-1}$ such that $[\omega] = [\mathrm{d}f \wedge \theta]$. From this, it follows that $[\omega] \in M$, since

$$\int_{\gamma_t} \theta = \int_{\gamma_t} \frac{\mathrm{d}f \wedge \theta}{\mathrm{d}f} = \int_{\gamma_t} \frac{\omega}{\mathrm{d}f} = 0 \,. \square$$

PROPOSITION 2.3.— *N/M is a torsion module.*

Proof.— Since G/F is a torsion module, there exists an integer m such that $t^m \circ [\omega] \in F$, whatever $[\omega] \in N$. From Proposition 2.2 and from the fact that N is a $\mathbb{C}\{t\}$-module, it follows that $t^m \circ [\omega] \in M$. $\square$

PROPOSITION 2.4.— $\operatorname{rank} M = \dim_{\mathbb{C}} N/(N \cap F)$.

Proof.— We have already seen that

$$\mathrm{D} : M \xrightarrow{\approx} N$$

and that N/M is a torsion module (Prop. 2.3). By the Malgrange index formula [4],

$$0 = \chi(\mathrm{D}; M, N) = \operatorname{rank} M - \dim_{\mathbb{C}} N/M \,.$$

Hence, by Proposition 2.2,

$$\nu = \operatorname{rank} M = \dim_{\mathbb{C}} N/(N \cap F) \,. \square$$

Observation.— If γ_t is homologous to 0, then $N = G$. Consequently, $M = F$ and

$$\dim_{\mathbb{C}} N/(N \cap F) = \dim_{\mathbb{C}} \Omega_0^n / \mathrm{d}f \wedge \Omega_0^{n-1} = \mu \,.$$

In this particular case, we obtain again the formula

$$\operatorname{rank} G = \operatorname{rank} F = \mu \,.$$

3. The rank of the module T

We may consider f a restriction, to $\mathbb{R}^2$, of an analytic function in a neighborhood of $0 \in \mathbb{C}^2$, that will also be denoted by f. With the hypothesis on J_f, 0 is an isolated critical point of the extension.

We define $\mathbb{C}\{t\}$-modules G and F as in Section 2. The sub-index 0 will be suppressed for we are only interested in functions and differential forms holomorphic in a neighborhood of the origin. We write

$$G = \frac{\Omega^2}{\mathrm{d}f \wedge \mathrm{d}\Omega^0} \quad \text{and} \quad F = \frac{\mathrm{d}f \wedge \Omega^1}{\mathrm{d}f \wedge \mathrm{d}\Omega^0},$$

where Ω^0 is the space of germs of analytic functions in $0 \in \mathbb{C}^2$.

(x, y) will denote the coordinates in $\mathbb{R}^2$ and (z, w) the coordinates in $\mathbb{C}^2$.

Define $\Lambda = \mathbb{R}\{x, y\}$ and $S = \Lambda/\Gamma$.

It is clear that Λ and S are $\mathbb{R}\{t\}$-modules, that $\Gamma \subset \Sigma \subset \Lambda$ as $\mathbb{R}\{t\}$-submodules, and that $T \subset S$ as $\mathbb{R}\{t\}$-submodule.

Let

$$\sigma : \Lambda \longrightarrow G, \quad \sigma(g) = [g \, \mathrm{d}z \wedge \mathrm{d}w].$$

PROPOSITION.— *σ is $\mathbb{R}\{t\}$-linear and* $\ker \sigma = \Gamma$.

Proof.— $\Gamma \subset \ker \sigma$ trivially.

If $g \in \ker \sigma$,

$$g \, \mathrm{d}z \wedge \mathrm{d}w = -\mathrm{d}f \wedge \mathrm{d}u,$$

which means that

$$g = f_w u_z - f_z u_w \quad \text{for a certain } u \in \mathbb{C}\{z, w\}.$$

Considering the restriction of u to $\mathbb{R}^2$,

$$u(x, y) = u_R(x, y) + i u_I(x, y)$$

where u_R and u_I are real functions,

$$g = f_y (u_R)_x - f_x (u_R)_y + i\Big[f_y (u_I)_x - f_x (u_I)_y\Big].$$

Since g is a real function, $f_y (u_I)_x - f_x (u_I)_y = 0$ and $g(x, y) = f_y (u_R)_x - f_x (u_R)_y$, that is, $g \in \Gamma$. □

Let $S_c = S \bigotimes_{\mathbb{R}} \mathbb{C}$ and $T_c = T \bigotimes_{\mathbb{R}} \mathbb{C}$. It is clear that S_c is a $\mathbb{C}\{t\}$-module and that $T_c \subset S_c$ as a $\mathbb{C}\{t\}$-submodule.

THEOREM 3.1.— *σ passes to quotient and defines a $\mathbb{C}\{t\}$-module isomorphism*

$$\tau : S_c \xrightarrow{\approx} G .$$

Proof.— By the definitions of S_c and G, τ is a surjective homomorphism

Let $g_1, g_2, \ldots, g_h \in \Lambda$, be $\mathbb{R}$-independent $\operatorname{mod} \Gamma$, and let $[g_j]$ be the class of g_j in S, $j = 1, 2, \ldots, h$.

Suppose

$$\tau\left(\sum_{j=1}^{h} c_j [g_j]\right) = 0 , \quad c_1, c_2, \ldots, c_h \in \mathbb{C} .$$

Then,

$$\left(\sum_{j=1}^{h} c_j g_j\right) \mathrm{d}z \wedge \mathrm{d}w = \mathrm{d}f \wedge \mathrm{d}u$$

for a certain u analytic on a neighborhood of $0 \in \mathbb{C}^2$. Thus,

$$\sum_{j=1}^{h} c_j g_j = f_z u_w - f_w u_z .$$

Let $c_j = a_j + i b_j$, $a_j, b_j \in \mathbb{R}$ and $u\big|_{\mathbb{R}^2} = u_R + i u_I$ where u_R and u_I are, respectively, the real and imaginary parts of the function $u\big|_{\mathbb{R}^2}$. Hence,

$$\sum a_j g_j = f_x \left(u_R\right)_y - f_y \left(u_R\right)_x$$
$$\sum b_j g_j = f_x \left(u_I\right)_y - f_y \left(u_I\right)_x .$$

In other words, $\sum a_j g_j$, $\sum b_j g_j \in \Gamma$. Consequently, $\forall j$, $a_j, b_j = 0$, i.e., $c_j = 0$. Then, τ is an isomorphism. □

COROLLARY 3.1.— *S is a free $\mathbb{R}\{t\}$-module of* rank μ, *where μ is the Milnor number of the function f in* 0 *(cf. [3, Sect. 3] and [5]).*

COROLLARY 3.2.— *If, for some $\mathcal{N} \in \mathbb{N}$,*

$$f_y u_x - f_x u_y = f^{\mathcal{N}} g$$

has a regular solution. Then,

$$f_y u_x - f_x u_y = g$$

has a regular solution.

COROLLARY 3.3.— *T is a free $\mathbb{R}\{t\}$-module and*

$$\operatorname{rank}_{\mathbb{R}\{t\}} T = \operatorname{rank}_{\mathbb{C}\{t\}} T_c \leq \mu\,.$$

To compute the rank of T, we must compute the rank η of the $\mathbb{C}\{t\}$-free module T_c.

Let γ_ϵ be the curve $f(x, y) = \epsilon$, that is, $\gamma_\epsilon = X_\epsilon \cap \mathbb{R}^2$, where X_ϵ is the Milnor fiber of f over ϵ. γ_ϵ is prolonged to a vanishing cycle (Sect. 2).

Let us recall the definition of the $\mathbb{C}\{t\}$-submodule N of G, given in Section 2:

$$N = \left\{ \omega \in G \;\middle|\; \int_{\gamma_t} \frac{\omega}{\mathrm{d}f} = 0 \right\}$$

where $\gamma_t \subset X_t$, for $0 < |t|$ small enough, is the vanishing cycle defined above.

THEOREM 3.2.— $\tau(T_c) = N$.

Proof.— Let $g \in \Lambda$ whose class in S belongs to T. There exists $\eta \in \Omega^1$ such that $\tau(g) = [g \, \mathrm{d}z \wedge \mathrm{d}w] = [\mathrm{d}\eta]$.

$$\int_{\gamma_\epsilon} \eta = \int_{f \leq \epsilon} \mathrm{d}\eta = \int_{f \leq \epsilon} g \, \mathrm{d}x \wedge \mathrm{d}y = 0$$

by the Theorem 1.1 and $\int_{\gamma_t} \eta$ is a multiform analytic function of t. Since the equality above holds for all $\epsilon > 0$ small enough, and $\gamma_\epsilon = X_\epsilon \cap \mathbb{R}^2$, it results that $\int_{\gamma_t} \eta = 0$ for all $0 < |t|$ small enough.

By Proposition 2.1, $[\mathrm{d}f \wedge \eta] \in M = \mathrm{D}^{-1}(N)$. Hence, $\mathrm{D}[\mathrm{d}f \wedge \eta] = [\mathrm{d}\eta] \in N$, and then, $[g \, \mathrm{d}z \wedge \mathrm{d}w] \in N$.

On the other side, if $[\omega] \in N$, there exists $\theta \in \Omega^1$ such that $\omega = \mathrm{D}[\mathrm{d}f \wedge \theta]$ and $\int_{\gamma_\epsilon} \theta = 0$ (Prop. 2.1). Besides, by the definition of D, $[\omega] = [\mathrm{d}\theta]$. Thus,

$$\int_{f \leq \epsilon} \omega = \int_{\gamma_\epsilon} \theta = 0 .$$

Let $\omega = h \mathrm{d}z \wedge \mathrm{d}w$, $h \in \mathbb{C}\{z, w\}$. Considering the restriction of h to $\mathbb{R}^2$, let

$$h\big|_{\mathbb{R}^2} = h_R + i h_I .$$

Then, we have

$$\begin{aligned} 0 = \int_{f \leq \epsilon} \omega &= \int_{f \leq \epsilon} h \,\mathrm{d}x \wedge \mathrm{d}y \\ &= \int_{f \leq \epsilon} h_R \,\mathrm{d}x \wedge \mathrm{d}y + i \int_{f \leq \epsilon} h_I \,\mathrm{d}x \wedge \mathrm{d}y . \end{aligned}$$

Therefore,

$$\int_{f \leq \epsilon} h_R \,\mathrm{d}x \wedge \mathrm{d}y = \int_{f \leq \epsilon} h_I \,\mathrm{d}x \wedge \mathrm{d}y = 0 .$$

Since the classes of h_R and h_I belong to T (Sect. 1), the class of h belongs to T_c and $\tau\big([h]\big) = [\omega]$. □

COROLLARY 3.4.— $\operatorname{rank} T_c = \dim_{\mathbb{C}} N/(N \cap F)$.

Proof.— Propositions 2.3 and 2.4.

COROLLARY 3.5.— $\operatorname{rank} T = \dim_{\mathbb{R}} \Sigma/(\Sigma \cap J_f)$.

Proof.— By theorems 3.1 and 3.2, it is enough to show that

$$\dim_{\mathbb{R}} \Sigma/(\Sigma \cap J_f) = \dim_{\mathbb{C}} N/(N \cap F) .$$

If $g \in \Sigma$, $\big[g(z, w) \,\mathrm{d}z \wedge \mathrm{d}w\big] \in N$ (Theorem 3.2).

We define

$$\begin{aligned} &\lambda : \Sigma \longrightarrow N/(N \cap F) \\ &\lambda(g) = \text{class of } \big[g(z, w) \,\mathrm{d}z \wedge \mathrm{d}w\big] = \text{class of } \sigma(g) . \end{aligned}$$

λ is a homomorphism between $\mathbb{R}$-vector spaces whose kernel is J_f. In fact,

- if $g \in \Sigma$ is such that $g = af_x + bf_y$, for certain a, $b \in \mathbb{R}\{x, y\}$,

$$\begin{aligned}\lambda(g) &= \big[\big(a(z,w)f_z(z,w) + b(z,w)f_w(z,w)\big)\,\mathrm{d}z \wedge \mathrm{d}w\big] \\ &= \big[\mathrm{d}f \wedge \big(a(z,w)\,\mathrm{d}w - b(z,w)\,\mathrm{d}z\big)\big] \in N \cap F\,;\end{aligned}$$

- on the other side, if $g \in \ker\lambda$, there exists $\alpha = \alpha_1\,\mathrm{d}z + \alpha_2\,\mathrm{d}w \in \Omega^1$ such that

$$\begin{aligned}g\,\mathrm{d}z \wedge \mathrm{d}w = \mathrm{d}f \wedge \alpha &= (f_z\,\mathrm{d}z + f_w\,\mathrm{d}w) \wedge (\alpha_1\,\mathrm{d}z + \alpha_2\,\mathrm{d}w) \\ &= (f_z\alpha_2 - f_w\alpha_1)\,\mathrm{d}z \wedge \mathrm{d}w\,;\end{aligned}$$

  which means that $g(z,w) = (f_z\alpha_2 - f_w\alpha_1)(z,w)$.

If we make the restriction to real numbers,

$$\begin{aligned}g(x,y) &= (f_x\alpha_2 - f_y\alpha_1)(x,y) \\ &= f_x(x,y)\big(\alpha_{2_R}(x,y) + i\alpha_{2_I}(x,y)\big) + \\ &\quad - f_y(x,y)\big(\alpha_{1_R}(x,y) + i\alpha_{1_I}(x,y)\big)\,,\end{aligned}$$

where $\alpha_j\big|_{\mathbb{R}^2} = \alpha_{j_R} + i\alpha_{j_I}$ and α_{j_R}, α_{j_I} are real functions, $j = 1, 2$.

Since $g(x,y)$, $f_x(x,y)$ and $f_y(x,y)$ are real numbers,

$$(f_x\alpha_{2_I} - f_y\alpha_{1_I})(x,y) = 0\,.$$

Consequently,

$$g = \alpha_{2_R}f_x - \alpha_{1_R}f_y \in J_f\,.$$

λ passes to quotient and defines a one to one homomorphism

$$\overline{\lambda} : \Sigma/(\Sigma \cap J_f) \longrightarrow N/(N \cap F)\,.$$

If $[\omega] \in N$, by Theorem 3.2, there exist g_1, $g_2 \in \Sigma$ such that

$$[\omega] = \overline{\lambda}[g_1] + i\overline{\lambda}[g_2]\,.$$

Thus,

$$\overline{\lambda}\,(\Sigma/\Sigma \cap J_f) + i\overline{\lambda}\,(\Sigma/\Sigma \cap J_f) = N/(N \cap F)\,.$$

It remains to show that $\overline{\lambda}(\Sigma/\Sigma \cap J_f) \cap i\overline{\lambda}(\Sigma/\Sigma \cap J_f) = \{0\}$.

Suppose g, $h \in \Sigma$ and

$$[g(z,w)\,\mathrm{d}z \wedge \mathrm{d}w] = i[h(z,w)\,\mathrm{d}z \wedge \mathrm{d}w] \quad \text{in } N/(N \cap F).$$

Hence,

$$[(g(z,w) - ih(z,w))\,\mathrm{d}z \wedge \mathrm{d}z] \in F,$$

i.e., $g - ih = af_z + bf_w$ for some functions a, $b \in \mathbb{C}\{z,w\}$.

Restricting to $\mathbb{R}^2$,

$$\begin{aligned} g - ih &= (a_R + ia_I)f_x + (b_R + ib_I)f_y \\ &= (a_R f_x + b_R f_y) + i(a_I f_x + b_I f_y), \quad a_R, a_I, b_R, b_I \in \mathbb{R}\{x,y\}. \end{aligned}$$

Thus g, $h \in \Sigma \cap J_f$.

Then, $\operatorname{rank} T = \dim_{\mathbb{C}} N/(N \cap F) = \dim_{\mathbb{R}} \Sigma/(\Sigma \cap J_f)$. □

Now, let us prove theorems 1 and 2.

Proof of Theorem 1

Part of it is a corollary of theorems 3.1 and 3.2.

It just remains to show that

$$\operatorname{rank} T = \nu = \mu - k.$$

Since G/N is torsion free, we may write $G = N \bigoplus P$. Let $[\omega_0]$, $[\omega_1]$, …, $[\omega_{\mu-1}]$ be a basis of G as a $\mathbb{C}\{t\}$-module where $[\omega_0]$, $[\omega_1]$, …, $[\omega_{\nu-1}]$ is a basis of N and $[\omega_\nu]$, $[\omega_{\nu-1}]$, …, $[\omega_{\mu-1}]$ is a basis of P as $\mathbb{C}\{t\}$-modules.

Let $\{\delta_{0_t}, \delta_{1_t}, \ldots, \delta_{(k-1)_t}\}$, $0 < |t| < \epsilon$, $\epsilon > 0$ sufficiently small, be a basis of E, where $\delta_{s_t} = L^s_{\gamma_t}$, $0 \le s \le k-1$.

If $[\omega_i]$ belongs to N, $\int_{\gamma_t} \omega_i = 0$. Hence, $\int_{L^s_{\gamma_t}} \omega_i = 0$, for all s, $0 \le s \le k-1$.

Thus, if

$$\operatorname{rank}_{\mathbb{C}\{t\}} N = \nu > \mu - k,$$

the matrix

$$\left[\int_{\delta_{j_t}} \frac{\omega_\ell}{\mathrm{d}f}\right]_{0 \le \ell, j \le \mu-1}$$

has a vanishing $k \times \nu$ minor, with $\nu + k > \mu$. Which means that

$$\det \left(\left[\int_{\delta_{jt}} \frac{\omega_\ell}{\mathrm{d}f} \right]_{0 \le \ell, j \le \mu-1} \right)^2 = 0$$

on a neighborhood of $t = 0$. And this contradicts the fact that (cf. [1], 12):

$$\det \left(\left[\int_{\delta_{jt}} \frac{\omega_\ell}{\mathrm{d}f} \right]_{0 \le \ell, j \le \mu-1} \right)^2 \neq 0 \,.$$

Suppose now $\nu < \mu - k$. Let

$$A(t) = \left[\int_{\delta_{jt}} \frac{\omega_\ell}{\mathrm{d}f} \right]_{\substack{\nu \le \ell \le \mu-1 \\ 0 \le j \le k-1}} \,.$$

By [2],

$$A(t) = \widetilde{A}(t) \exp(C \ln t) \,,$$

where $\widetilde{A}(t)$ is meromorphic and C is a constant $(k \times k)$-matrix.

Since $\widetilde{A}(t)$ is a $((\mu - \nu) \times k)$-matrix with $\mu - \nu > k$, then there exist holomorphic functions $g_\nu(t)$, $g_{\nu+1}(t)$, $\ldots$, $g_{\mu-1}(t)$, not all of them identically zero in $|t| < \epsilon$, such that

$$\sum_{\ell=\nu}^{\mu-1} g_\ell \int_{\delta_{jt}} \frac{\omega_\ell}{\mathrm{d}f} = 0 \,, \quad j = 0, \ldots, k-1 \,.$$

Let $\alpha = \sum g_\ell(f) \omega_\ell$. Then we have $[\alpha] \in P$ and, since

$$\int_{\delta_{jt}} \frac{\alpha}{\mathrm{d}f} = 0 \,, \quad j = 0, \ldots, k-1 \,,$$

$[\alpha]$ also belongs to N. This means that $[\alpha] = 0$. Thus there exists a non trivial linear combination of $[\omega_0]$, $\ldots$, $[\omega_{\mu-1}]$ that vanishes. That is impossible, since it is a basis of G.

Therefore, $\nu = \mu - k$. $\square$

Proof of theorem 2

Theorem 3.2, Corollary 3.5.

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References

[1] ARNOLD (V.), VARCHENKO (A.) and GOUSSEIN-ZADE (S.) .— *Singularités des applications différentiables*, 2^e partie, Monodromie et comportement asymptotiques des intégrales, Éditions Mir - Moscou, 1986.

[2] BRIESKORN (E.) .— *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. **2** (1970), pp. 103-161.

[3] MALGRANGE (B.) .— *Intégrales asymptotiques et monodromie*, Ann. Scient. Éc. Sup., série 4, **7** (1974), pp. 405-430.

[4] MALGRANGE (B.) .— *Sur les points singuliers des équations différentielles*, L'Enseignement Math. **20** (1974), pp. 147-176.

[5] MILNOR (J.) .— *Singular points of Complex Hypersurfaces*, Annals of Mathematics Studies, Princeton University Press, Princeton, New Jersey, **61** (1968).