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The classification of curvilinear angles in the complex plane and the groups of $\pm$ holomorphic diffeomorphisms[(*)]

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RÉSUMÉ. — La classification des configurations des germes des courbes lisses et analytiques réelles dans le plan complexe est connue comme problème local de géométrie conforme de Poincaré ([5], [6], [10], [11]). Ce problème a été classiquement étudié en terme de groupe des germes de difféomorphismes $\pm$ holomorphes du plan complexe engendré par les reflexions antiholomorphes de Schwarz par rapport aux composants lisses. Dans cette note, nous étudions la structure du groupe en utilisant la méthode du cylindre sectoriel de Ecalle–Voronin, et nous déterminons l'espace de modules des couples de courbes lisses et tangentes ainsi que des fronces.

ABSTRACT. — The classification of configurations of germs of real analytic smooth curves in the complex plane is known as Poincaré's local problem of conformal geometry ([5], [6], [10], [11]). This was classically studied in terms of the group of germs of $\pm$ holomorphic diffeomorphisms of the complex plane generated by anti-holomorphic Schwarz reflections with respect to smooth components. In this note we investigate the structure of the group using the method of Ecalle–Voronin cylinder and determine the moduli space of the pairs of tangent smooth curves and also cusps.

KEY-WORDS : Anti-holomorphic involution, holomorphic diffeomorphism.

AMS Classification : 57S.

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1. Introduction

Let $C = (C_i)$, $D = (D_i)$ be m-tuples of germs C_i, D_i of real analytic curves in the complex plane $\mathbb{C}$ at 0. Let $s_i, t_i : \mathbb{C}, 0 \to \mathbb{C}, 0$ be germs of holomorphic maps such that $s_i(\mathbb{R}) = C_i$, $t_i(\mathbb{R}) = D_i$. We say C, D are *equivalent* (respectively *formally equivalent*) and denote $h(C) = D$ if there exist holomorphic germs of (resp. formal) diffeomorphisms h, k_i of the complex plane such that $h(0) = k_i(0) = 0$ and $h \circ s_i = t_i \circ k_i$ for $i = 1, \ldots, m$. The equivalence class is independent of the choice of the parametrizations s_i, t_i. The classification problem was historically studied, and known as Poincaré's local problem in conformal geometry [11].

We consider the case $m = 2$. So we denote $C_1 = K$, $C_2 = L$ and call $C = (K, L)$ a *curvilinear angle*. Let σ_K, σ_L denote Schwarz reflections respecting K, L. These reflections generate a solvable group G_C, of which the orientation preserving subgroup is generated by the composite $f = \sigma_L \circ \sigma_K$. Let $D = (K', L')$ be a curvilinear angle. By definition C, D are equivalent (respectively formally equivalent) if and only if there exists a holomorphic (resp. formal) diffeomorphism h, $h(0) = 0$ of the complex plane such that $h \circ \sigma_L \circ h^{(-1)} = \sigma_{L'}$ and $h \circ \sigma_K \circ h^{(-1)} = \sigma_{K'}$, where $h^{(-1)}$ denotes the inverse of h. So the classification problem under the equivalence relation reduces to finding the generator (σ_K, σ_K) of G_C. The involutions σ_K, σ_L satisfy the relation

$$\sigma \circ f \circ \sigma = f^{(-1)}, \quad \sigma \circ \sigma = \mathrm{id}\ . \tag{1.1}$$

Conservely a solution $\sigma = \sigma_K$ defines the other involution σ_L by $\sigma_L = f \circ \sigma_K$. Therefore the classification problem is equivalent to solving (1.1).

If K, L are transverse, then f has the linear term $e^{i\theta} z$, θ being twice the angle of tangents to K and L at 0. The structure of those pairs seems to be closely related to the linearization problem of f when θ is irrational, which is now well understood by the theory of complex dynamical systems (see [9]). On the other hand if θ is rational, the classification may be reduced to the case $\theta = 0$ as the images of K, L under an iteration of f by a certain number of times have the same tangent lines as K, L. In this case the problem is related to the classification of solvable groups of germs of $\pm$-holomorphic diffeomorphisms (Theorem 8.2).

If K, L have contact of order $k+1$ at 0 (C is a *curvilinear angle of order* $k+1$), then f is *k-flat*, i.e, f is the form $z+az^{k+1}+\cdots$. Kasner and Pfeiffer ([5], [6], [10], [11]) studied those curvilinear angles by the classification of the composite f and found unique formal invariant, which is the residue of f. It is seen in the proof of Theorem 2 that sole formal invariant of curvilinear angles of order 2 is the residue (normalized residue) of the composite f, while the formal equivalence of curvilinear angles with equal residue does not converge in general as a rule in the theory of the functional cochain [4]. One can now complete the classification problem with the method of Ecalle–Voronin cylinder ([4], [12]).

From now on we assume K, L are tangent. Since $K = \mathrm{fix}(\sigma_K)$, we obtain $\sigma_L(K) = \mathrm{fix}(\sigma_L \circ \sigma_K \circ \sigma_L)$ from which $\sigma_{\sigma_L(K)} = \sigma_L \circ \sigma_K \circ \sigma_L$ hence $\sigma_{\sigma_L(K)} \circ \sigma_L = \sigma_L \circ \sigma_K = f$. This tells that in terms of anti-holomorphic involutions one can not distinguish the pairs (K, L) and $\big(\sigma_L(K), L\big)$. Now consider formal equivalence of those pairs. Clearly $\sigma_K \circ f^{(-1)} \circ \sigma_K = f$ and by Proposition 2.6, $\sigma_K \circ f^{(-1/2)} \circ \sigma_K = f^{(1/2)}$ hence $\sigma_L = f \circ \sigma_K = f^{(1/2)} \circ \sigma_K \circ f^{(-1/2)}$, where $f^{(1/2)}$ is the 1/2-times iteration of f tangent to identity. By this relation

$$\sigma_{f(K)} = f \circ \sigma_K \circ f^{(-1)} = f^{(1/2)} \circ \sigma_L \circ f^{(-1/2)}$$

and

$$\sigma_L = f^{(1/2)} \circ \sigma_K \circ \sigma f^{(-1/2)}\,.$$

Therefore $C = (K, L)$ and $\big(L, \sigma_L(K)\big)$ are equivalent by the formal diffeomorphism $f^{(1/2)}$:

$$\big(L, f(K)\big) = \big(L, \sigma_L(K)\big) = f^{(1/2)}(K, L)\,.$$

DEFINITION.— *The associativity relation of curvilinear angles is generated by the holomorphic equivalence and the following formal equivalence relation:*

$$C \sim D \quad \textit{if } f = \sigma_L \circ \sigma_K = \sigma_{L'} \circ \sigma_{K'} \quad \textit{and} \quad D = h(C) = \big(h(K), h(L)\big)\,,$$

where $h = f^{(t)}$ is a formal complex t-times iteration of f (defined in Sect. 2) such that $h \circ h = f^{(2t)}$ is convergent.

Before stating our main result, we seek a geometric interpretation of associativity. By definition a curvilinear angle associative to (K, L) can be formally presented as $f^{(t)}(K, L)$ with a $t \in \mathbb{C}$. If $f^{(t)}$ is convergent, $f^{(t)}(K, L)$ is equivalent to (K, L). This is the case if f is a time one map of a holomorphic flow vanishing at $0 \in \mathbb{C}$ by Theorem 2.7. The other case is that $f^{(t)}$ is convergent if and only if $t \in 1/n\,\mathbb{Z}$. Then $f^{(2t)}$ is convergent if and only if $t = m/2n \in 1/2n\,\mathbb{Z}$ and then $f^{(t)}(K, L) = f^{(m/2n)}(K, L)$ is equivalent to (K, L) if m is even and $f^{(1/2n)}(K, L)$ otherwise. Therefore we obtain the following result.

PROPOSITION 1.— *Given an associative class of tangent curvilinear angle (K, L), there exist at most two equivalence classes, i.e, (K, L) and $f^{(1/2n)}(K, L)$, f, n being as above.*

Similarly $f^{(1/2)}(K, L)$ is equivalent to (K, L) if n is even and $f^{(1/2n)}(K, L)$ otherwise. By Proposition 2.5 (K, L) and $f^{(1/2)}(K, L) = \big(L, \sigma_L(K)\big)$ are equivalent if and only if unique formal conjugacy $f^{(1/2)}$ and hence $f^{(1/2n)}$ are convergent.

Now assume (K, L) and $\big(L, \sigma_L(K)\big)$ are not equivalent. Then by Proposition 1, (K, L), (K', L') are associative if and only if (K', L') is equivalent to either (K, L) or $\big(L, \sigma_L(K)\big)$.

THEOREM 2.— *Let $C = (K, L)$, $D = (K', L')$ be curvilinear angles of order $k + 1$. If k is odd, then C, D are associative if and only if $\sigma_L \circ \sigma_K$, $\sigma_{L'} \circ \sigma_{K'}$ are holomorphically equivalent. If k is even, then there exist at most two associative classes for a holomorphic equivalence class of $\sigma_L \circ \sigma_K$.*

The purpose of this note is to prove the following theorems.

THEOREM 3.— *The associative classes of curvilinear angles of order 2 are in one-to-one correspondence with the linear equivalence classes of triples (A, B, m) of germs of real analytic smooth curves A at 0, B at ∞ in $\mathbb{P}^1$ and a pure imaginary m with the condition*

$$2(\operatorname{angle} T_0 A - \operatorname{angle} T_\infty B) = -2\pi\sqrt{-1}\, m \mod 2\pi\,. \tag{2.1}$$

Here $\operatorname{angle} T_0 A = \arg w$, $w \in T_0 A$ *and* $\operatorname{angle} T_\infty B = \lim_{z\to\infty} \arg w$, $w \in T_z B$ *and the linear equivalence relation $\approx$ of triples is defined as follows:*

$$(A, B, m) \approx (A', B', m') \quad \text{if } m = m'$$

and there is a complex $c \neq 0$ such that $A' = cA$, $B' = cB$. Assume (K, L) corresponds to (A, B, m). Then the transposition (L, K) corresponds to $(s \circ \sigma(B),\ s \circ \sigma(A),\ -m)$, where $s(z) = 1/z$ and σ is the complex conjugation. And (K, L) is equivalent to (L, K) if and only if $m = 0$ and $B = s \circ \sigma(A)$, i.e, B is the reflection of A with respect to the unit circle $\{|z| = 1\} \subset \mathbb{C}$.

This theorem is roughly explained as follows. Assume K, L have contact of order 2. Let $\mathbb{P}^-$, $\mathbb{P}^+$ denote Ecalle–Voronin cylinders of the composite $f = \sigma_L \circ \sigma_K$ (see Sect. 4 for the definition). Denote $K^{\pm} = f^{(\pm n)}(K)$, $L^{\pm} = f^{(\pm n)}(L)$ with a sufficiently large n. Schwarz reflections of $K^{\pm n}$ $(L^{\pm n})$, carry $L^{\pm n-1}$ to $L^{\pm n}$ ($K^{\pm n}$ to $K^{\pm n+1}$). The components of $K^{\pm} - 0$, $L^{\pm} - 0$ project to smooth arcs in $\mathbb{P}^{\pm}$ at 0, ∞. Denote the closure of the union of those arcs at 0, ∞ in $\mathbb{P}^{\pm}$ by $A^{\pm}$, $B^{\pm}$ respectively. Schwarz reflections induce germs of anti-holomorphic involutions $\sigma_0^{\pm}$, $\sigma_\infty^{\pm}$ of $\mathbb{P}^{\pm}$ at 0, ∞, which respect $A^{\pm}$, $B^{\pm}$. Therefore $A^{\pm}$, $B^{\pm}$ are smooth real analytic curves. Schwarz reflections induce also an anti-holomorphic isomorphism of $\mathbb{P}^+$ and $\mathbb{P}^-$, which carries A^-, B^- to A^+, B^+ respectively. Define the pair (A, B) by (A^-, B^-). It is not difficult to see that the pair (A, B) is determined by the associative class of (K, L) up to linear equivalence. The number m is given by the residue of the composite of Schwarz reflections of K, L. In Section 3, we reconstruct K, L from the data (A, B, m) using Ecalle–Voronin method.

A $(2,3)$-cusp is an image of a real analytic map $x + \sqrt{-1}\,y : \mathbb{R}, 0 \to \mathbb{C}, 0$, where $x(t) = t^2 + \cdots$, $y(t) = t^3 + \cdots$ are real analytic functions.

THEOREM 4.— *The equivalence classes of germs of real analytic $(2,3)$-cusps are one-to-one correspondence with the equivalence classes of real analytic smooth curves at $0 \in \mathbb{P}$ under linear rotation.*

The formal normal form C_m of curvilinear angles of order 2 and C^c of $(2,3)$-cusps are defined as follows. Let $\chi = \big(z^2/(1 + m\sqrt{-1}\,z)\partial/\partial z\big)$. Define the anti-holomorphic involution σ_z of $\mathbb{C}$ at 0 by

$$\sigma_z\big(\exp t\chi(z)\big) = \exp \bar{t}\chi(z)$$

where $\bar{t}$ denotes the complex conjugate of t. Let K_a denote the fixed point set of the involution

$$\sigma_{\exp a\sqrt{-1}}\,\chi(1)\,.$$

Define the normal form C_m by $(K_{-1/4}, K_{1/4})$. (For the detailed argument, see Sect. 7).

If $m = 0$, then C_0 is symmetrie : $K_{-1/4} = -K_{1/4}$. Define the normal form C^c by the image of C_0 under the map $z \to z^2$. Let C be a curvilinear angle of order 2 and $\mathrm{Res}(f) = m$. Then C is formally equivalent to the normal form C_m as f is formally determined by its residue. For a $(2,3)$-cusp C, its preimage by the map $z \to z^2$ is a union of smooth curves K, $-K$ with contact of order 2 at 0. For such a symmetric curvilinear angle $(K, -K)$, the normalized residue vanishes, hence the $(2,3)$-cusp C as well as $(K, -K)$ is formally unique. But the formal equivalences are not convergent in general.

THEOREM 5

(1) A curvilinear angle of order 2 is equivalent to the normal form C_m if and only if it corresponds to a triple (A, B, m) of real lines A, B and m with condition (2.1).

(2) A $(2,3)$-cusp is equivalent to the normal form if and only if it corresponds to a real line.

2. Formal normal form and Normalized Residue

Germs of holomorphic diffeomorphims f, g, of $\mathbb{C}$ which fix 0 are *holomorphically* (resp. *formally*) *equivalent* if there exists a germ of holomorphic (resp. formal) diffeomorphism ϕ of $\mathbb{C}$ at 0 such that $\phi(0)$ and $\phi \circ f = g \circ \phi$. A k-flat (parabolic) diffeomorphism $f(z) = z + a_{k+1}z^{k+1} + \cdots$, $a_{k+1} \neq 0$ is equivalent to $z + z^{k+1} + bz^{2k+1} + \cdots$ and formaly equivalent to $z + z^{k+1} + bz^{2k+1}$. The number $-b \in \mathbb{C}$ is the unique formal invariant, which is the *residue* of f and denoted by $\mathrm{res}(f)$. Define the *normalized residue* by

$$\mathrm{Res}(f) = \mathrm{res}(f) + \frac{k+1}{2} = -b + \frac{k+1}{2}.$$

Clearly $\mathrm{res}(f)$ and $\mathrm{Res}(f)$ are invariant under formal equivalence. For the complex conjugation $\overline{\sigma}$, a direct calculation tells

$$\mathrm{Res}(\overline{\sigma} \circ f \circ \overline{\sigma}) = \overline{\mathrm{Res}(f)}.$$

By this and the invariance of the normalized residue under holomorphic equivalence, we obtain the following result.

PROPOSITION 2.1.— *For anti-holomorphic diffeomorphisms g*

$$\mathrm{Res}(g^{(-1)} \circ f \circ g) = \overline{\mathrm{Res}(f)}\,.$$

A holomorphic vector field χ' is holomorphically equivalent to the following normal form

$$\chi(z) = \frac{z^{k+1}}{1+mz^k}\frac{\partial}{\partial z}\,.$$

The m is the *residue* of χ' and denoted $\mathrm{res}(\chi')$. Clearly we obtain the following lemma.

LEMMA 2.2 [8].— $\mathrm{Res}(\exp\chi) = \mathrm{res}(\chi) = m$ *and* $\mathrm{res}(d\chi) = \mathrm{res}(\chi)/d$ *for complex d.*

The formal equivalence class of a germ of diffeomorphism f tangent to identity is determined by the residue $\mathrm{res}(f) = -b$. So there is a formal diffeomorphism ϕ of $\mathbb{C}$, 0 such that $f = \phi^{(-1)} \circ \exp\chi \circ \phi$ with $\mathrm{res}(\chi) = \mathrm{Res}(f)$. The *complex iteration* $f^{(t)}$, $t \in \mathbb{C}$, is defined by the formal power series $f^{(t)} = \phi^{(-1)} \circ \exp t\chi \circ \phi$. Clearly $f^{(t)}$ commutes with f.

From Lemma 2.2, we obtain the following proposition.

PROPOSITION 2.3.— *For complex t,* $\mathrm{Res}(f^{(t)}) = \mathrm{Res}(f)/t$.

The following theorem is well known (cf. [4]).

PROPOSITION 2.4.— *Assume f, $g \neq \mathrm{id}$ be tangent to identity and commuting. Then $g = f^{(t)}$ with unique complex t.*

By Proposition 2.4 given an $f \neq \mathrm{id}$ tangent to identity the commutativity relation $f \circ g = g \circ f$ admits a unique formal flat solution g with any $k+1$-st order term. So we obtain the following results.

PROPOSITION 2.5.— *Let K, L have contact of order $k+1$. Then the relation $g(K, L) = (K, L)$ admits a unique k-flat formal diffeomorphism g with a given $k+1$-st order term.*

PROPOSITION 2.6.— *Let g be a germ of $\pm$ holomorphic diffeomorphism. Assume f is flat and $g^{(-1)} \circ f \circ g = f^{(-1)}$. Then $g^{(-1)} \circ f^{(t)} \circ g = f^{(-t)}$ holds for all $t \in \mathbb{C}$.*

Proof.— The coefficients of Taylor expansions of $f^{(t)} = \exp t\chi$ in z are polynomials of t. Clearly the assumption implies the equality for integers t. Therefore the coefficients of both sides of $g^{(-1)} \circ f^{(t)} \circ g = f^{(-t)}$ are equal for all t.

The following theorem is due to Ecalle (cf. [3] and [4]).

THEOREM 2.7.— *Let f be a germ of flat diffeomorphism of $\mathbb{C}$ at 0. Let $\Lambda \subset \mathbb{C}$ be the subgroup of those $t \in \mathbb{C}$ for which $f^{(t)}$ is convergent. Assume Λ is not isomorphic to $\mathbb{Z}$. Then f is holomorphically equivalent to an* $\exp\chi$ *and $\Lambda = \mathbb{C}$, where χ is a holomorphic vector field of the normal form.*

3. Proof of Theorem 2

It is easy to see by definition that the equivalence class of the composite $f = \sigma_L \circ \sigma_K$ is determined by the associative class of C. So we discuss the converse. Namely given a composite f we discuss to find σ_K and σ_L.

Consider the equations so that $f \circ \sigma$ is involutive

$$\sigma \circ f \circ \sigma = f^{(-1)}, \quad \sigma \circ \sigma = \mathrm{id}\ . \tag{1.1}$$

Clearly (1.1) admits the solution $\sigma = \sigma_K$. So let $\sigma_{K'}$ be another solution. Then $\sigma_{K'} = h \circ \sigma_K$, h being a holomorphic diffeomorphism commuting with f.

First assume that $\sigma_{K'}$ has the same linear term as σ_K, in other words, K' is tangent to K. Then h is tangent to identity and $\sigma_{K'} = h \circ \sigma_K = h^{(1/2)} \circ \sigma_K \circ h^{(-1/2)}$ by Proposition 2.5, where $h^{(1/2)}$ is the half-iteration of h defined in the previous section. Define the involution $\sigma_{L'}$ by $f \circ \sigma_{K'}$ and let K', L' be their fixed point sets. Then the pair $D = (K', L')$ is associative with C by the formal diffeormorphism $h^{(1/2)}$. This argument tells that all curvilinear angles (K', L') with $\sigma_{L'} \circ \sigma_{K'} = f$ and equal tangent line at 0 are associative.

Secondly assume $f(z) = z + az^{k+1} + \cdots$ and $\sigma_{K'}(z) = b\overline{z} + \cdots$, $b\overline{b} + 1$, $\sigma_{K'}^{(2)} = \mathrm{id}$ and K', K are transverse. The $k+1$-order term of the equation (1.1) implies that $a + \overline{a}\overline{b}^k = 0$, so there exist at most k (if k is odd and $k/2$ otherwise) different tangent lines of the fixed point sets of the solutions. By conjugating as $\sigma_{K'} \circ \sigma_K \circ \sigma_{K'}$, the solutions of (1.1) generate other solutions. And the fixed point set of $\sigma_{K'} \circ \sigma_K \circ \sigma_{K'}$ is $\sigma_{K'}(\mathrm{fix}(\sigma_K))$. Therefore the set of tangents to the fixed point sets of the solutions is invariant under a linear rotation R of order $2n$, n being a divisor of k. Assume that the angle of K', K at 0 is π/n (generator of R). Then $h = \sigma_{K'} \circ \sigma_K$ has the linear term $\omega_n z$, ω_n being the n-th root of unity. Clearly $(K^i, L^i) = \left(h^{(i)}K\,,\, h^{(i)}L\right)$ is equivalent to C, $\sigma_{L^i} \circ \sigma_{K^i} = f$ and the tangent to (K^i, L^i) is that of (K, L) rotated by $i\omega_n$. If n is odd, the linear term $\omega_n z$ and $-z$ generate the linear rotation group $\mathbb{Z}_{2n}$ generated by R. Therefore a curvilinear angle (K', L') defined by $\sigma_{L'} \circ \sigma_{K'} = f$ has the same tangent line as (K^i, L^i) for an i, and (K', L') is associative with (K^i, L^i) by the previous argument.

If n is even, the curvilinear angles split into two associative classes which are determined by the difference of the angle of the tangent lines at 0 divided by π/n modulo 2.

This completes the proof of Theorem 2.

4. Functional cochain and moduli of diffeomorphisn tangent to identity: Ecalle–Voronin theory

We recall some results from the papers [1], [4] and [8]. The whole results are translated to classify the diffeomorphisms $f = \sigma_L \circ \sigma_K$ in the next section. In this section we assume that f is of the form

$$f(z) = z - \frac{2\pi\sqrt{-1}}{k}\, z^{k+1} + c z^{2k+1} + \cdots$$

On the k-sheet covering $\widetilde{C}_k$ of the punctured $\widetilde{z}$-plane $\mathbb{C} \setminus 0$, $\widetilde{z} = z^{-k}$, f lifts to a germ of diffeomorphism F defined at infinity

$$F(\widetilde{z}) = \widetilde{z} + 2\pi\sqrt{-1} + \frac{a'}{\widetilde{z}} + \cdots,$$

where $\operatorname{res}(f) = -ck^2/4\pi^2$ and $a' = -4\pi^2 \operatorname{Res}(f)/k$.

Let $\widetilde{S}_i^+$, $\widetilde{S}_i^-$ be the lifts of the attracting and repelling petals of f on the i-th sheet of the covering $\widetilde{\mathbb{C}}_k$. We call $\widetilde{S}_i^+$, $\widetilde{S}_i^-$ petals of F. Voronin ([4], [12]) proved that the quotient space $\mathbb{P}_i^+$ (respectively $\mathbb{P}_i^-$) of the petal $\widetilde{S}_i^+$ (resp. $\widetilde{S}_i^-$) by F (resp. $F^{(-1)}$) is conformally isomorphic to the punctured 2-sphere $\mathbb{P} \setminus 0 \cup \infty$, which is called the *cylinder*. We say a fundamental domain $D_i^{\pm}$ in the petal $\widetilde{S}_i^{\pm}$ is *rectangular* if the boundary projects to a real line in $\mathbb{P}_i^{\pm}$ joining 0 to ∞. Here 0 (resp. ∞) corresponds to the left (resp. right) end of the fundamental domain $D_i^{\pm}$. The isomorphism from $\mathbb{P}\setminus\{0,\infty\}$ to the quotient space $\mathbb{P}_i^{\varepsilon}$, $\varepsilon=\pm$ lifts to the isomorphism $\widetilde{\phi}_i^{\varepsilon}$ of the band

$$B_\varepsilon=\left\{0\le \varepsilon\Im z\le 2\pi\right\}\subset\mathbb{C}$$

to a rectangular fundamental domain in the petal $\widetilde{S}_i^{\varepsilon}$ which extends to the isomorphism of the upper (if $\varepsilon=+$, and lower if $\varepsilon=-$ respectively) half plane into the petal of F by the relation $\widetilde{\phi}_i^{\varepsilon}+2\pi\sqrt{-1}=\widetilde{\phi}_i^{\varepsilon}(F)$. The extension of $\widetilde{\phi}_i^{\varepsilon}$ normalizes $F^{(\varepsilon)}$ on the petal to the translation by $2\pi\sqrt{-1}\,\varepsilon$, $\widetilde{\phi}_i^{\varepsilon}/2\pi\sqrt{-1}\,\varepsilon$ si called Fatou–Leau coordinate [1]. Fatou-Leau coordinate is unique modulo constant. The k-tuple $(\widetilde{\phi}_i^{\varepsilon(-1)})$ is called the *functional cochain* in [4].

An iteration of F of a large number of times carries both ends to the fundamental domain D_i^- corresponding to 0, ∞ into the attracting petals $\widetilde{S}_{i+1}^+$, $\widetilde{S}_i^+$ respectively, and it induces germs of diffeomorphisms of the quotient spaces

$$\phi_{i,0}=\exp\circ\widetilde{\phi}_{i+1}^{+(-1)}\circ F^{(n)}\circ\widetilde{\phi}_i^-\circ\log\,:\,\mathbb{P}_i^-,0\longrightarrow\mathbb{P}_{i+1}^+,0$$

$$\phi_{i,\infty}=\exp\circ\widetilde{\phi}_{i}^{+(-1)}\circ F^{(n)}\circ\widetilde{\phi}_i^-\circ\log\,:\,\mathbb{P}_i^-,\infty\longrightarrow\mathbb{P}_{i}^+,\infty$$

n being sufficiently large. The $2k$-tuple $(\phi_{i,0},\phi_{i,\infty})$ is called the *iterative coboundary* [4]. By definition

$$\log d\phi_{i,0}(0)=\lim_{z\to 0}\log d\phi_{i,0}(z)=\lim_{w\to-\infty}\widetilde{\phi}_{i+1}^{+(-1)}F^{(n)}\widetilde{\phi}_i^-(w)$$

$$\log d\phi_{i,\infty}(\infty)=\lim_{z\to\infty}\log d\phi_{i,\infty}(z)=\lim_{w\to\infty}\widetilde{\phi}_{i}^{+(-1)}F^{(n)}\widetilde{\phi}_i^-(w)\,.$$

This number depends on the choice of the rectangular domains and Fatou–Leau coordinates, while we obtain the following proposition.

PROPOSITION 4.1 (cf. [4, p. 20 (2.10)])

$$\begin{aligned}\sum_{i=1}^{k}\log d\phi_{i,\infty}(\infty)-\log d\phi_{i,0}(0)&\equiv 2\pi\sqrt{-1}\,\mathrm{Res}(f)\\ &=\frac{a'k}{2\pi\sqrt{-1}},\quad \operatorname{mod}2\pi\sqrt{-1}\end{aligned}\tag{4.1}$$

where $d\phi_{i,\infty}(\infty)$ is defined as above by using the coordinate $z\in\mathbb{C}$ centered at 0.

This relation is seen also by (7.1) in Section 7. The pair $(\phi, \operatorname{Res}(f))$ of the iterative coboundary $\phi = (\phi_{i,j})$ and the residue characterizes f in the following manner.

DEFINITION.— *$(\phi, m) \sim (\phi', m')$ if $m = m'$ and there exist an integer r and constants $c_i^\varepsilon \neq 0$, $i = 1, \ldots, k$, $\varepsilon = \pm$ such that $\phi'_{i+r,\infty}(c_i^- z) = c_i^+ \phi_{i,\infty}(z)$ and $\phi'_{i+r,0}(c_i^- z) = c_{i+1}^+ \phi_{i+1,0}(z)$ for $i = 1, \ldots, k$.*

The equivalence class of $(\phi, \operatorname{Res}(f))$ is independent of the choice of Fatou–Leau coordinates. The following was proved by many authors (e.g. [4], [7] and [12]). Here the theorem is stated involving the normalized residue to complete the relation of the pairs (ϕ, m) to formal equivalence classes. The residue plays a central role to classify the composites $f = \sigma_L \circ \sigma_K$ of Schwarz reflections in the next section.

THEOREM 4.2.— (moduli space of diffeomorphisms tangent to identity: Ecalle, Kimura, Malgrange, Voronin, etc.) *There exists a one-to-one correspondence between the following sets.*

(1) The set of holomorphic equivalence classes of germs of k-flat diffeomorphisms f of $\mathbb{C}$, 0, which are holomorphically equivalent to $z + z^{k+1} + \mathrm{O}(z^{2k+1})$.

(2) The set of equivalence classes of pairs (ϕ, m) under the equivalence relation $\sim$, which satisfy the relation (4.1).

Sketch of the proof.— We begin explaining the synthesizing method of a germ of diffeomorphism tangent to identity with a prescribed data (ϕ, m) due to Voronin [12] (see also [4]). Let H_i^+ (H_i^+) be the upper (lower) half plane of $\mathbb{C}$ for $i = 1, \ldots, k$ and let $\widetilde{H}_i^-$, be the lower half plane with two handles

$$\widetilde{H}_i^- = H_i^- \cup \{r < |\Re \widetilde{z}|\}$$

and let

$$\phi'_{i,0} : \mathbb{P}_i^-, 0 \longrightarrow \mathbb{P}_{i+1}^+, 0, \qquad \phi'_{i,\infty} : \mathbb{P}_i^-, \infty \longrightarrow \mathbb{P}_{i+1}^+, \infty$$

be representatives of $\phi_{i,0}$, $\phi_{i,\infty}$ defined on some neighbourhoods of $j = 0$, $\infty \in \mathbb{P}^1$. Let $r > 0$ be sufficiently large so that

$$\widetilde{\phi}_{i,j} = \log \circ \phi'_{i,j} \circ \exp$$

is respectively defined on the half space $\{r < \Re\tilde{z}\}$ for $j = \infty$ and $\{\Re\tilde{z} < -r\}$ for $j = 0$. Choose the branch of $\tilde{\phi}_{i,j}$ so that it sends the real line in the domain of definition into the lower half plane, and glue the left (resp. right) handle of the half plane $\widetilde{H}_i^-$ with the upper half plane H_{i+1}^+ (resp. H_i^+) identifying $\tilde{z}$ with the image $\tilde{\phi}_{i,0}(\tilde{z}) \in H_{i+1}^+$ (resp. $\tilde{\phi}_{i,\infty}(\tilde{z}) \in H_i^+$). Denote by E^+ the gasket surface obtaind by glueing these half planes with handles. Construct the gasket surface E^- similarly by choosing the branch $\tilde{\phi}_{i,j} - 2\pi\sqrt{-1}$. By construction E^- is naturally regarded as a subset of E^+ and if r is sufficiently large these gasket surfaces E^+, E^- are quasiconformally homeomorphic hence isomorphic to the punctured unit disc $D \setminus 0 \subset \mathbb{C}$ ([4], [8], [12]). Since the maps $\tilde{\phi}_{i,j}$ commute with the translation by $2\pi\sqrt{-1}$, the translation on the half planes induces the shift map $F : E^- \to E^+$. Regarding as $E^- \subset E^+ = D \setminus 0$, the shift map F extends to a germ of holomorphic diffeomorphism f of $\mathbb{C}$ at 0 tangent to identity. By construction the extension has the iterative coboundary ϕ. The normalized residue $\mathrm{Res}(f)$ for the shift map is determined modulo an integer: it depends on the choice of the branch of $\phi_{i,j}$. Define the shift map F' replacing $\tilde{\phi}_{1,\infty}$ with $\tilde{\phi}'_{1,\infty} = \tilde{\phi}_{1,\infty} + 2\pi n\sqrt{-1}$. Then the resulting germ of diffeomorphism f' has the normalized residue $\mathrm{Res}(f) + n$. It is easy to see that the above method reconstructs a germ of k-flat diffeomorphism f from the functional moduli (ϕ, m) of f.

5. Moduli of composites of anti-holomorphic involutions and the proof of Theorem 3

Consider the equation (1.1). The purpose of this section is to classify the solutions of (1.1).

Assume f, σ satisfy (1.1). By Proposition 2.1,

$$\overline{\mathrm{Res}(f)} = -\mathrm{Res}(f)$$

hence $\mathrm{Res}(f)$ is pure imaginary. Now let f be 1-flat ($k = 1$). We use all notations in the previous section without reference to the index i. Let F, $\tilde{\sigma}$ be the lifts of f, σ to the punctured $\tilde{z}$-plane, $\tilde{z} = z^{-1}$. The relation (1.1) implies

$$\tilde{\sigma} \circ F \circ \tilde{\sigma} = F^{(-1)}. \tag{5.1}$$

The lift $\tilde{\sigma}$ induces an anti-holomorphic isomorphism of the cylinders $\mathbb{P}^+$, $\mathbb{P}^-$, which respects 0, ∞. We may assume $\tilde{\sigma}(D^-) = D^+$ and by suitable coordinates on the cylinders the induced isomorphism is the complex conjugation $\overline{\sigma}$. Then (5.1) implies

$$\overline{\sigma} \circ \phi_0 \circ \overline{\sigma} = \phi_0^{(-1)}, \quad \overline{\sigma} \circ \phi_\infty \circ \overline{\sigma} = \phi_\infty^{(-1)}. \tag{5.2}$$

Let f' be a 1-flat germ of diffeomorphism of $\mathbb{C}, 0$, which admits an anti-holomorphic involution σ' which satisfies (1.1). Assume f, f' are holomorphically equivalent. By Theorem 4.2, there exist c^+, $c^- \neq 0$ such that

$$\phi_0'(c^- z) = c^+ \phi_0(z), \quad \phi_\infty'(c^- z) = c^+ \phi_\infty(z).$$

Let F', $\widetilde{\sigma}'$ be the lifts of f', σ' to $\widetilde{\mathbb{C}}$ and let $D'^{\pm}$ be the rectangular fundamental domains in the petals $\widetilde{S}'^{\pm}$ of F' such that $\widetilde{\sigma}'(D'^-) = D'^+$. The holomorphic equivalence of f to f' sends the rectangular fundamental domains $D^{\pm}$ to $D'^{\pm}$ respectively. By the symmetry of the fundamental domains, c^- is the complex conjugate of $c^+ = c$.

Define the equivalence relation $\approx$ of pairs of an iterative coboundary $\phi = (\phi_0, \phi_\infty)$ with (5.2) and a pure imaginary m as follows.

DEFINITION.— *$(\phi, m) \approx (\phi', m')$ if $m = m'$ and there exists a $c \neq 0$ such that $c\phi_0(z) = \phi_0'(\overline{c}z)$ and $c\phi_\infty(z) = \phi_\infty'(\overline{c}z)$.*

The above argument shows that the equivalence of (ϕ, m) for f with (1.1) is determined by the equivalence of f.

Conversely let (ϕ_0, ϕ_∞) be an iterative coboundary with the above relation (5.2) in some coordinates of the cylinders, m a pure imaginary number with the relation (4.1), and let $E^{\pm}$ be the gasket surface constructed in the previous section by gluing the petals with handles. Define the anti-holomorphic isomorphism $\widetilde{\sigma}$ of D^- to D^+ (and D^+ to D^-) by the composite $\sigma' \circ T$ of the transposition T of rectangle fundamental domains D^-, D^+ and the anti-holomorphic involution σ' of D^+ (and D^+) transposing the boundaries and respecting the ends corresponding to 0, ∞, which induces the complex conjugation $\overline{\sigma}$ in (5.2). The relation (5.2) enables us to extend $\widetilde{\sigma}$ by the relation

$$\widetilde{\sigma}\big(z + 2n\pi\sqrt{-1}\big) = \widetilde{\sigma}(z) - 2n\pi\sqrt{-1}$$

to an anti-holomorphic diffeomorphism of the surface $E^{\pm}$ on a neighbourhood of infinity. Then the diffeomorphisms F and $\widetilde{\sigma} : E^- \leftrightarrow E^+$ satisfy the relation (5.1), from which the relation (1.1) follows.

Therefore we proved the following result.

THEOREM 5.1.— *There is a one-to-one correspondence between the following sets.*

(1) The set of holomorphic equivalence classes of germ of 1*-flat diffeomorphisms, which admit anti-holomorphic involutions σ such that $\sigma \circ f \circ \sigma = f^{(-1)}$.*

(2) The set of equivalence classes of the pairs (ϕ, m) of an iterative cochain $\phi = (\phi_0, \phi_\infty)$ under the equivalence relation $\approx$ and a pure imaginary m, which satisfies (5.2) and (4.1).

Proof of Theorem 3

Let $C = (K, L)$ be a curvilinear angle of order 2, $f = \sigma_L \circ \sigma_K$ the composite of the anti-holomorphic involutions of K, L and let (ϕ_0, ϕ_∞) be the iterative cochain of f. By (5.2), $\phi_0 \circ \overline{\sigma}$ and $\phi_\infty \circ \overline{\sigma}$ are involutive. Let A, B be the fixed point sets of these anti-holomorphic involutions, in other words, $\phi_0 = \sigma_A \circ \overline{\sigma}$, $\phi_\infty = \sigma_B \circ \overline{\sigma}$. The triple (A, B, m) corresponds to the C.

To prove that the equivalence class of the triple is well defined by the equivalence class of C, let $C' = (K', L')$ be a curvilinear angle of order 2, $f' = \sigma'_L \circ \sigma'_K$ and assume C, C' are equivalent. Then f, f' are holomorphically equivalent and by Theorem 5.1 $(\phi_0, \phi_\infty, \mathrm{Res}(f))$, $(\phi'_0, \phi'_\infty, \mathrm{Res}(f'))$ are equivalent:

$$c\phi_0(z) = \phi'_0(\bar{c}z)\,, \quad c\phi_\infty(z) = \phi'_\infty(\bar{c}z)\,, \qquad c \neq 0\,.$$

Let $\phi'_0 = \sigma_{A'} \circ \overline{\sigma}$ and $\phi'_\infty = \sigma_{B'} \circ \overline{\sigma}$. Then

$$c\sigma_A(z) = \sigma_{A'}(cz)\,, \quad c\sigma_B(z) = \sigma_{B'}(cz)\,,$$

from which $A' = cA$ and $B' = cB$. The relation (4.1) in Proposition 4.1 implies

$$2(\mathrm{angle}\, T_0 A - \mathrm{angle}\, T_\infty B) = -2\pi\sqrt{-1}\, \mathrm{Res}(f)\,, \quad \mathrm{mod}\, 2\pi\,. \tag{5.3}$$

This argument shows also how to reconstruct a curvilinear angle from a given triple (A, B, m). Therefore the correspondence is one-to-one.

This completes the proof of Theorem 3.

6. Proof of Theorem 4

Let $C \subset \mathbb{C}$ be a *cusp of type* (p, q), $p < q$; C is the image of a real analytic map $x + \sqrt{-1}\, y : \mathbb{R}, 0 \to \mathbb{C}, 0$, $x(t) = t^p + \cdots$, $y(t) = t^q + \cdots$, $t \in \mathbb{R}$ being real analytic. The lift $\widetilde{C} \subset \mathbb{C}$ of the cusp via the map $z \to z^p$ is the union of real analytic smooth curves, which is invariant under the linear rotation of order p. Let K denote one of those components.

Now assume $p = 2$. Then the lift is a union of K and $-K$, which have contact of order $q - 1$. Clearly $\sigma_{-K}(z) = -\sigma_K(-z)$. The orientation preserving subgroup of G_C is generated by $f = -\sigma_K(-\sigma_K) = \sigma_{-K} \circ \sigma_K$. By Theorem 2, the associative class of $(K, -K)$ is determined by the holomorphic equivalence class of f if q is odd. In this section, we discuss to classify $(K, -K)$ by $\mathbb{Z}_2$-equivariant diffeomorphism with the involution $z \to -z$ for the case $q = 3$.

Let K', $-K'$ be the preimages of another cusp C' of type $(2, 3)$ and let $f' = \sigma_{-K'} \circ \sigma_{K'}$. By definition f, f' are holomorphically equivalent if there exists a germ of holomorphic diffeomorphism h of $\mathbb{C}$ at 0 such that $h(0) = 0$ and $f' \circ h = h \circ f$.

THEOREM 6.1.— *Let C, C' be $(2,3)$-cusps. Then the following conditions are equivalent.*

(1) C, C' are diffeomorphic.

(2) f, f' are holomorphically equivalent.

(3) f, f' are holomorphically equivalent by a $\mathbb{Z}_2$-symmetric diffeomorphism h such that $-h(-z) = h^{(-1)}(z)$.

Proof of the implication (2)⇒(3).— Let (ϕ_0, ϕ_∞), (ϕ'_0, ϕ'_∞) be the iterative coboundaries characterizing $f = -\sigma_K(-\sigma_K)$, $f' = -\sigma_{K'}(-\sigma_{K'})$. The equivalence h induces isomorphisms h^-, h^+ of the cylinders P^-, P^+ such that

$$h^+ \circ \phi_0 = \phi'_0 \circ h^- , \quad h^+ \circ \phi_\infty = \phi'_\infty \circ h^- .$$

The linear involution $-z$ induces the map $1/z : \mathbb{P}^- \to \mathbb{P}^+$. The relations

$$-f(-z) = f^{(-1)}, \quad -f'(-z) = f'^{(-1)}$$

imply

$$\phi_\infty^{(-1)}(z) = \phi_0\left(\frac{1}{z}\right)^{-1}, \quad \phi_\infty'^{(-1)}(z) = \phi_0'\left(\frac{1}{z}\right)^{-1}$$

and $\mathrm{Res}(f) = \mathrm{Res}(f') = 0$. Since this iterative coboundary is symmetric with respect to $z \to -z$, the isomorphisms h^-, h^+ extend to a $\mathbb{Z}_2$-symmetric diffeomorphism of the gasket surface $E^\pm$ to $E'^\pm$ on a neighbourhood of ∞, which gives a $\mathbb{Z}_2$-symmetric equivalence of f to f' by the argument in the proof of Theorem 5.1.

Proof of the implication (3)⇒(1).— It suffices to prove the uniqueness of the anti-holomorphic involution σ_K such that $\sigma_K \circ f \circ \sigma_K = f^{(-1)}$ and $\sigma_{-K} \circ \sigma_K = f$. So let $\sigma_{K'}$ be another solution of the equations. Let $g = \sigma_{K'} \circ \sigma_K$. Then g, f commute and by Theorem 2.7, $g = f^{(\alpha)}$ with an $\alpha \in \mathbb{C}$. By Proposition 2.5

$$\begin{aligned}\sigma_{-K'} \circ \sigma_{K'} &= -\sigma_{K'}(-\sigma_{K'}) = -f^{(\alpha)} \circ \sigma_K(-f^{(\alpha)} \circ \sigma_K)\\ &= -f^{(\alpha)} \circ \sigma_K \circ f^{(-\alpha)}(-\sigma_K) = -f^{(2\alpha)} \circ \sigma_K(-\sigma_K)\\ &= f^{(-2\alpha)} \circ \sigma_{-K} \circ \sigma_K = f^{(1-2\alpha)}.\end{aligned}$$

Therefore $\alpha = 0$ and $\sigma_{K'} = \sigma_K$. The other implications are trivial by definition.

Proof of Theorem 4

By Theorem 6.1, an equivalence class of $(2,3)$-cusp corresponds to a $\mathbb{Z}_2$-symmetric equivalence class of 1-flat diffeomorphism f with $-f(-z) = f^{(-1)}(z)$. By Proposition 2.3, $\mathrm{Res}(f) = -\mathrm{Res}(f)$ hence $\mathrm{Res}(f) = 0$. By Theorem 5.1 and the above argument, such an f corresponds to an equivalence class of a $\mathbb{Z}_2$-symmetric iterative coboundary (ϕ_0, ϕ_∞) with $\phi_\infty^{(-1)}(z) = \phi_0(1/z)^{-1}$.

DEFINITION.— *$\mathbb{Z}_2$-symmetric coboundaries (ϕ_0, ϕ_∞), (ϕ_0', ϕ_∞') are equivalent if ϕ, ϕ' are linearly equivalent : there is a $c \neq 0$ such that $c\bar{c} = 1$ and*

$$c\phi_0(z) = \phi_0'(\bar{c}z), \quad c\phi_\infty(z) = \phi_\infty'(\bar{c}z). \tag{6.1}$$

Recall the relation $\overline{\sigma}\circ\phi_0\circ\overline{\sigma} = \phi_0^{(-1)}$ (5.2). By Theorem 5.1 and Theorem 6.1(3), an equivalence class of cusp corresponds to an equivalence class of ϕ_0 in the above relation. Define the anti-holomorphic involution $\sigma_A = \phi_0 \circ \overline{\sigma}$ and similarly $\sigma_{A'} = \phi_0' \circ \overline{\sigma}$. Then (6.1) implies $A' = cA$, where $|c| = 1$.

Conversely given an A, define $\phi_0 = \sigma_A \circ \overline{\sigma}$ and $\phi_\infty^{(-1)} = \phi_0(1/z)^{-1}$, and define a diffeomorphism f by the data $(\phi_0, \phi_\infty, 0)$. By Theorem 3, there exists a curvilinear angle $C = (K, L)$ of order 2 with the data. By construction, C is $\mathbb{Z}_2$-symmetric: $C = (K, -K)$ and the $\mathbb{Z}_2$-symmetric equivalence class is independent of the linear rotation of A. This completes the proof of Theorem 4.

Theorem 4 can be stated in a more detailed form as follows.

THEOREM 4bis.— *There is a one-to-one correspondence between the following sets.*

(1) The set of equivalence classes of $(2,3)$*-cusps.*

(2) The set of holomorphic equivalence classes of composites of anti-holomorphic involutions $f = \sigma_{-K}\circ\sigma_K$.

(3) The set of equivalence classes of germs of diffeomorphisms ϕ *of* $\mathbb{C}, 0$ *such that* $\overline{\sigma}\circ\phi\circ\overline{\sigma} = \phi^{(-1)}$ *by the equivalence relation as in (6.1), where* $\overline{\sigma}$ *denotes the complex conjugation.*

7. Proof of Theorem 5

Proof of Theorem 5(1).— Let m be pure imaginary and $C_m = (K_{-1/4}, K_{1/4})$ the normal form after Theorem 5 in the introduction. Let

$$\xi = -\frac{\partial}{\partial w}, \quad \chi = \frac{z^2}{1 + m\sqrt{-1}\,x}\,\frac{\partial}{\partial z}$$

and $\widetilde{\chi}$ the lift of χ to the $\widetilde{z}$-line, $\widetilde{z} = z^{-1}$. Define $\phi_m = \widetilde{z} - m\sqrt{-1}\log\widetilde{z}$, $\psi_m = \phi_m(1/z)$. It is easy to see that $d\psi_m(\chi) = d\phi_m(\widetilde{\chi}) = \xi$. Consider the analytic continuation of the complex flow $\exp t\xi(w_0)$ along a big anti-clockwise cycle $\circlearrowleft \subset \mathbb{C}$ starting at $t = 0$ such that the trajectory turns anti-clockwisely around the origin and returns to w_0. Since ϕ_m is infinitely many valued, the trajectory on the z-plane as well as $\tilde{z}$-plane is not closed if $m \neq 0$. The gap is caused by the residue m and presented by the functional equation

$$\exp \circlearrowleft \chi(z) = \exp(2\pi m)\chi(z) \tag{7.1}$$

(see also [8]). Here we may assume the complex conjugate at the cycle $\circlearrowleft$ is its inverse $\circlearrowright = -\circlearrowleft$. Define the germ of anti-holomorphic involution σ_{z_0} of the z-plane at 0 by

$$\sigma_{z_0}(\exp t\chi(z_0)) = \exp \bar{t}\chi(z_0))$$

where $\bar{t}$ is the complex conjugate of t. By the relation (7.1) and since m is pure imaginary, σ_{z_0} is well defined and clearly involutive. Define $K_{\pm 1/4}$ to be the fixed point set of the involution $\sigma_{\exp \pm 1/4\sqrt{-1}\chi(1)}$. It is not difficult to see the composite f for C_m is time $\sqrt{-1}$ map of χ, and by Lemma 2.2 $\operatorname{Res}(f) = m$.

Now regard $T = -\sqrt{-1}t$ as Fatou–Leau coordinate of f. By definition, the branches of $K_{\pm 1/4} - 0$ are real trajectories og the holomorphic flow χ. The litts of the branches to Fatou–Leau coordinate are all real lines with constant real part. Therefore, A, B are real linear and an iterative coboundary (ϕ_0, ϕ_∞) is linear, hence the involutions $\sigma_A = \phi_0 \circ \overline{\sigma}$ and $\sigma_B = \phi_\infty \circ \overline{\sigma}$ are complex linear. If a curvilinear angle C is equivalent to the normal form C_m, then the corresponding smooth curves A, B are equivalent by a complex linear map to those of the normal form. Conversely if A, B are real linear, then ϕ_0, ϕ_∞ are complex linear and the dynamics constructed in Section 5 is equivalent to the time $\sqrt{-1}$ map of the normal form χ with an m, m being the residue of f.

Proof of Theorem 5(2).— If $m = 0$, the union of the curves $I_{-1/4}$, $I_{1/4}$ as well as the mapping ψ_m admit the $\mathbb{Z}_2$-symmetry by $-\tilde{z}$. The symmetry implies $A = B$. Conversely if $A = B$ and $m = 0$, it corresponds to the normal form C_0.

8. Solvable groups of germs of $\pm$ holomorphic diffeomorphisms

In general the classification of (p, q)-cusps reduces to that of the groups G_C, which are solvable groups of length 2 for curvilinear angles and $(2, 3)$-cusps. We investigate the classification of solvable groups G of germs of $\pm$ holomorphic diffeomorphisms of $\mathbb{C}, 0$. The linear terms of the members of a G form a subgroup of the semidirect product $\mathbb{C}^* \times \mathbb{Z}_2$ consisting of the linear maps cz and $c\bar{z}$, $c \in \mathbb{C}^*$. Let G^+, $G^0 \subset G$ denote respectively the orientation preserving subgroup and the flat subgroup consisting of parabolic diffeomorphisms. The group G is an extension of the linear term group $L = G/G^0 \subset \mathbb{C}^*$ by G^0.

PROPOSITION 8.1.— *The following conditions are equivalent.*

(1) G^+ is solvable.

(2) G^0 is commutative.

(3) G^+ is metabelian, that is, the commutator subgroup $[G^+, G^+]$ is commutative.

(4) G is solvable.

(5) The second commutator subgroup G_2 of G is commutative.

If G is solvable, all diffeomorphisms f in G^0 different from the identity have the same order of flatness k and the projections L, Λ of G/G^0, G^0 respectively to the linear and the $(k+1)$-st order terms are injective homomorphisms into $\mathbb{C}^* \times \mathbb{Z}_2$, $\mathbb{C}$.

Proof.— The equivalence of (1), (2) and (3) is proved in [4] and [8]. The part (5)$\Rightarrow$(4)$\Rightarrow$(1) is clear. We prove the implication (1)$\Rightarrow$(5). The second commutator subgroup G_2 of G is a subgroup of G^0. Assume that G^+ is solvable. Then G^0 is commutative by (2), hence G^2 is commutative. The second statement is proved in [8].

We say two groups G, G' consisting of germs of $\pm$ holomorphic diffeomorphisms of $\mathbb{C}$ at 0 are *holomorphically* (resp. *formally*) *equivalent* if there exist a group isomorphism $\phi : G \to G'$ and a germ of holomorphic (resp. formal) diffeomorphism h of $\mathbb{C}$, $h(0) = 0$ such that $h \circ g = \phi(g) \circ h$ for $g \in G$.

THEOREM 8.2 (solvable groups).— *Assume G is a solvable group consisting of germs of $\pm$ holomorphic diffeomorphism of $\mathbb{C}, 0$.*

(1) G is formally equivalent to a subgroup of the semidirect product $(\mathbb{C}^* \times \mathbb{Z}_2) \times \mathbb{C}$. Here the multiplication in $(\mathbb{C}^* \times \mathbb{Z}_2) \times \mathbb{C}$ is defined by

$$(a, \varepsilon, b) * (c, \varepsilon', d) = \left(a\overline{\sigma}^{\varepsilon}(c)\,,\, \varepsilon + \varepsilon'\,,\, a\overline{\sigma}^{\varepsilon}(d) + b\overline{\sigma}^{\varepsilon}(c)^{k+1}\right).$$

The action of $(a, \varepsilon, b) \in G$ *on* $\mathbb{C}$ *is formally equivalent to*

$$a \exp \frac{b}{a}\, \chi(z) = a\overline{\sigma}^{\varepsilon}(z) + b\overline{\sigma}^{\varepsilon}(z)^{k+1} + \cdots ,$$

where χ *is a holomorphic vector field and* $\overline{\sigma}$ *is the complex conjugation.*

(2) *If* G^{+} *is non commutative, then* $\operatorname{res}(\chi) = 0$, $\chi = z^{k+1}\, \partial/\partial z$.

(3) *If* $G^{0} \neq \mathbb{Z}$, *then the action of* G *is holomorphically equivalent to the action of a subgroup of* $(\mathbb{C}^{*} \times \mathbb{Z}_2) \times \mathbb{C}$ *in (1).*

Proof of (1). — The flat subgroup G^{0} is commutative by Proposition 8.1, and G is an extension of the linear term group $L = G/G^{0}$ by G^{0}. By Proposition 2.4 there exists a formal diffeomorphism ϕ and a holomorphic vector field of the normal form χ such that G^{0} consists of convergent diffeomorphism $f^{(t)}(z) = \phi^{(-1)} \circ \exp t\chi \circ \phi(z)$ with some $t \in \mathbb{C}$. Since the statement in (1) is formal, we may assume $f = \exp \chi$, $f(z) = z + z^{k+1} + \cdots$ and G^{0} consists of the diffeomorphisms $f^{(t)}$ with t in a subgroup $\Lambda \subset \mathbb{C}$.

Since $g^{(-1)} f^{(t)} g \in G^{0}$ and

$$g^{(-1)} f^{(t)} g(z) = z + \overline{\sigma}^{\varepsilon}(a^{k} t) z^{k+1} + \cdots$$

for $(-1)^{\varepsilon}$-holomorphic diffeomorphisms $g \in G$, $g(z) = a\overline{\sigma}^{\varepsilon} z + \cdots$, the adjoint action μ of L on G^{0} is presented as

$$\mu(g, f^{(t)}) = g^{(-1)} f^{(t)} g = f^{(\overline{\sigma}^{\varepsilon}(a^{k} t))} \tag{8.1}$$

for $g \in G/G^{0}$. Let g, g' be orientation preserving diffeomorphisms with equal linear term, which satisfy the relation (8.1). Then $\mu(g, f) = \mu(g', f)$ and $g^{(-1)} \circ g'$ commutes with f, and by Proposition 2.4 $g^{(-1)} \circ g' = f^{(s)}$ hence $g' = g f^{(s)}$ for an s. Clearly all g' of this form satisfy (8.1). It is easy to see the linear maps $g' = az$ satisfy (8.1). Therefore all orientation preserving formal solutions of (8.1) are of the form $a f^{(s)}$, $a \in \mathbb{C}^{*}$, $s \in \mathbb{C}$. Similarly all orientation reversing formal solutions of (8.1) are of the form $a f^{(s)}(\overline{z})$, $a \in \mathbb{C}^{*}$, $s \in \mathbb{C}$.

The correspondence of $a f^{(s)} \circ \overline{\sigma}^{\varepsilon}$ $(= a\overline{\sigma}^{\varepsilon}(z) + as\overline{\sigma}^{\varepsilon}(z)^{k+1} + \cdots)$ to (a, ε, as) gives the isomorphism of the group of those diffeomorphisms $a f^{(s)} \circ \overline{\sigma}^{\varepsilon} \in G$ into the semidirect product $(\mathbb{C}^{*} \times \mathbb{Z}_2) \times \mathbb{C}$. The straightforword calculation

$$\left(a\overline{\sigma}^{\varepsilon}(z) + b\overline{\sigma}^{\varepsilon}(z)^{k+1} + \cdots\right) \circ \left(c\overline{\sigma}^{\varepsilon}(z) + d\overline{\sigma}^{\varepsilon}(z)^{k+1} + \cdots\right) =$$

$$= a\overline{\sigma}^{\varepsilon}(c)\overline{\sigma}^{\varepsilon+\varepsilon'} z + \left(a\overline{\sigma}^{\varepsilon}(d) + b\overline{\sigma}^{\varepsilon}(c)^{k+1}\right)\overline{\sigma}^{\varepsilon}(z)^{k+1} + \cdots$$

tells the multiplication $*$ on $(\mathbb{C}^* \times \mathbb{Z}_2) \times \mathbb{C}$ is defined by

$$(a, \varepsilon, b) * (c, \varepsilon', d) = \left(a\overline{\sigma}^{\varepsilon}(c)\, , \, \varepsilon + \varepsilon'\, , \, a\overline{\sigma}^{\varepsilon}(d) + b\overline{\sigma}^{\varepsilon}(c)^{k+1}\right).$$

The remaining part of (1) follows from (2).

Proof of (2).— If G^+ is non commutative, the action μ is not trivial: $c = dg(0)^k \neq 1$ for a $g \in G^+$. By the invariance of the normalized residue under holomorphic coordinate transformations, the relation (8.1) implies $\mathrm{Res}(f^{(t)}) = \mathrm{Res}(f^{(ct)})$. By the multiplicative formula in Proposition 2.3, $\mathrm{Res}(f^{(ct)}) = \mathrm{Res}(f^{(t)})/c$. Therefore $\mathrm{Res}(f^{(t)}) = 0$ hence $\chi = z^{z+1}\, \partial/\partial z$.

Proof of (3).— In this case the flat subgroup G^0 is holomorphically embedded into a 1-parameter group $\exp t\chi$ by Theorem 2.7. Therefore the above formal equivalence of G^0 to a subgroup to $\exp t\chi$ is convergent. □

9. A remark on formal equivalence

The classification theorem Theorem 8.2 tells that the groups G_C generated by curvilinear angles of order 2 are formally determined by the normalised residue $\mathrm{Res}(f)$ of the composites $f = \sigma_L \circ \sigma_K$, and for the symmetric curvilinear angles $(K, -K)$ the residue vanishes hence all such groups are formally equivalent. This tells that the formal equivalence class of curvilinear angles of order 2 is determined by their residue and the formal equivalence class of $(2,3)$-cusps is unique. In these cases, the formal classification of the group G_C is quite different from the classification by holomorphic equivalence by the presence of the exceptional group of anti-holomorphic diffeomorphisms, which is an extension of the exceptional group by an anti-holomorphic involution.

Here a subgroup of the group of germs of holomorphic diffeomorphisms is *exceptional* if it is formally equivalent to the solvable subgroup $G_{\omega,p}$, $p \in \mathbb{N}$, $\omega \in \mathbb{C}$ which is isomorphic to the semidirect product $\mathbb{Z} \times \mathbb{Z}_p$ and generated by

$$x \longrightarrow \omega x \quad \text{and} \quad h_p(x) = x\left(1 - px^p\right)^{-1/p} \quad \text{with} \quad \omega^p = -1\, , \; (-1)^{1/p} = 1\, .$$

Cerveau and Moussu [2] proved the following theorem.

THEOREM 9.1.— *Let G, G' be non-commutative groups consisting of germs of diffeomorphism of $\mathbb{C}$ at 0 which fix $0 \in \mathbb{C}$. Assume that G, G' are non-exceptional and formally equivalent. Then the formal equivalence is convergent to a germ of diffeomorphism linking G and G'.*

This theorem suggests that the configurations of smooth curves and cusps for which the group generated by the anti-holomorphic involutions is exceptional have a rich structure. The classification problem of those curves waits for another investigation.

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