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A closure operation on complex analytic cones and torsion[(*)]

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RÉSUMÉ. — Nous étudions l'opération qui consiste à fermer (analytiquement) la partie localement libre d'un espace linéaire complexe. Plus généralement nous démontrons : soit un cône analytique $\pi : X \to S$ défini par une $\mathcal{O}_S$-algèbre graduée $\mathcal{A}$ et soit A un sous-ensemble analytique de S. Alors la fermeture du sous-cône ouvert $X \setminus \pi^{-1}(A)$ dans X est un cône analytique sur la fermeture de $S \setminus A$ dans S et il est défini par l'algèbre graduée $\mathcal{A}/\mathcal{H}^0_A\mathcal{A}$. Dans le cas où S est réduit, A est d'intérieur vide dans S et $\mathcal{A}$ est sans torsion sur $S \setminus A$; on a $\mathcal{A}/\mathcal{H}^0_A\mathcal{A} = \mathcal{A}/\mathcal{T}(\mathcal{A})$ où $\mathcal{T}(\mathcal{A})$ désigne l'idéal gradué des éléments de torsion dans $\mathcal{A}$. Nous établissons des relations entre les éléments de torsion et les éléments nilpotents d'une algèbre symétrique d'un faisceau analytique cohérent. En appliquant nos résultats aux espaces tangents globaux de Zariski, nous obtenons, pour S réduit et localement intersection complète, une condition nécessaire et suffisante en termes de dimensions pour que le cône de Whitney $C_4(S)$ soit la réduction de l'espace tangent global de Zariski, et deux critères de régularité pour des courbes analytiques. Finalement, il y a des applications aux notions de positivité (amplitude).

ABSTRACT. — We study the operation of taking the (analytic) closure of the locally free part of a linear space in a more general setting: For a cone $\pi : X \to S$ defined by a graded $\mathcal{O}_S$-algebra sheaf $\mathcal{A}$ and a closed analytic subset A the closure in X of the open subcone $X \setminus \pi^{-1}(A)$ is a cone over the closure in S of $S \setminus A$ defined by the graded algebra sheaf $\mathcal{A}/\mathcal{H}^0_A\mathcal{A}$. In the case that S is reduced, A is analytically rare in S and $\mathcal{A}$ is torsion free on $S\setminus A$ this can also be described as the algebra $\mathcal{A}/\mathcal{T}(\mathcal{A})$, where $\mathcal{T}(\mathcal{A})$ is the graded ideal of torsion elements in $\mathcal{A}$. We prove a relation between the torsion ideal and the ideal of nilpotent elements in the symmetric algebra of a coherent analytic sheaf. Among applications to linear spaces, in particular tangent spaces, are a necessary and sufficient condition involving dimensions for the Whitney cone $C_4(S)$ to be the reduction of the global Zariski tangent space, valid for a reduced complex analytic space S that is locally a complete intersection; two regularity criteria

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for complex analytic curves and some applications involving notions of positivity.

Introduction

Certain natural operations that can be performed on linear spaces over a complex analytic space — the natural generalization of holomorphic vector bundles — produce not linear spaces, but *complex analytic cones.* In [2], where we studied complex analytic cones in general, we gave an important example of such an operation, namely the blowing-down of the zero section of a linear space. In this paper we study another operation of this kind: Taking the closure in a linear space of its locally free part. More generally, we study the operation of closing (analytically) the restriction of a complex analytic cone to a Zariski-open subset of the base space and describe it in terms of the connected graded sheaf of algebras defining the cone.

This operation has been studied earlier in special cases: In [23] Whitney defined several tangent cones to a reduced analytic space S. One of them, the tangent cone that Whitney denoted by $C_4(S)$, may be defined as the closure of the locally free part of the global Zariski tangent space of S. The operation was defined for general linear spaces by Rabinowitz in [17] (see also [16]), who used it to define a notion of *primary weakly positive sheaves.* Let S be a reduced compact complex analytic space and let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Denote by $\mathbf{V}(\mathcal{F})$ the linear space associated with $\mathcal{F}$ (see [10]) and let A be the analytic subset of S over which $\mathcal{F}$ is not locally free. Rabinowitz defines the *primary component* of $\mathbf{V}(\mathcal{F})$ as the closure $\mathbf{V}(\mathcal{F})_\sharp$ in $\mathbf{V}(\mathcal{F})_{\text{red}}$ of $\mathbf{V}(\mathcal{F})_{\text{red}}|S\setminus A$, and calls $\mathcal{F}$ *primary weakly positive* if the zero section of $\mathbf{V}(\mathcal{F})_\sharp$ is exceptional; here $\mathbf{V}(\mathcal{F})_{\text{red}}$ denotes the reduction of the space $\mathbf{V}(\mathcal{F})$. The space $\mathbf{V}(\mathcal{F})_\sharp$ is, in general, not a linear space over S. It is a complex analytic cone over S.

Since the definition of the "primary component" involves the reduction of a linear space, we also examine more generally the operation of reducing a complex analytic cone, another operation which, when performed on a linear space, produces in general not a linear space, but a complex analytic cone. When considering the duality between coherent analytic sheaves on a complex space S and linear spaces over S it is essential that the linear spaces are allowed to be non-reduced, also when the base space S is reduced.

It turns out that some very natural linear spaces associated with a complex space, such as its global tangent space, are in general non-reduced, even if the original space is reduced. Thus, if we want to work with linear spaces, we are forced to allow them to be non-reduced. There are however notions, similar to that of linear spaces, defined in the category of reduced spaces. Thus Grauert defined in [8] a class of reduced spaces that he called linear spaces over a (reduced) complex space; in this paper we shall call them *Grauert-linear spaces.* The reduction of a linear space is a Grauert-linear space, but not a linear space in general, and there are Grauert-linear spaces that are not obtained by reducing linear spaces. In the paper we clear up the relationship between Grauert-linear spaces and linear spaces and characterize the former as a certain type of complex analytic cones.

In Sections 1 and 2 of this paper, we consider the algebraic analogues of taking the reduction of a cone and the closure of an open subcone. We show (the non-surprising fact) that the reduction of a cone corresponds to the reduction of its algebra (Theorem 1.2). For a cone $\pi : X \to S$ defined by the graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a closed analytic subset A of S we show that the closure in X of the open subcone $X \setminus \pi^{-1}(A)$ is a cone defined by the algebra $\mathcal{A}/\mathcal{H}^0_A\mathcal{A}$ (Theorem 2.2). In the case that S is reduced, A is analytically rare in S and $\mathcal{A}$ is torsion free on $S \setminus A$ this can also be described as the algebra $\mathcal{A}_\sharp := \mathcal{A}/\mathcal{T}(\mathcal{A})$, where $\mathcal{T}(\mathcal{A})$ is the graded ideal of torsion elements in $\mathcal{A}$. We also prove a certain proposition concerning the fibre dimension of the closure (Prop. 2.8).

In Section 3, we consider these operations on linear spaces. In Proposition 3.4, we characterize Grauert-linear spaces. We prove a relation between the torsion ideal and the ideal of nilpotent elements in the symmetric algebra of a coherent analytic sheaf (Prop. 3.6) and draw some corollaries. We also show that the primary component (in the sense of Rabinowitz) of a coherent sheaf $\mathcal{F}$ is given by $\mathbf{V}(\mathcal{F})_\sharp = \mathrm{Specan}(\mathcal{S}(\mathcal{F})_\sharp)$, where $\mathcal{S}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$ (Theorem 3.10). In Proposition 3.11, we consider a dimension condition necessary for the equation $\mathbf{V}(\mathcal{F})_\sharp = \mathbf{V}(\mathcal{F})_{\mathrm{red}}$, where $\mathcal{F}$ is a coherent analytic sheaf over a reduced pure dimensional complex space, and prove that it is also sufficient in the case that $\mathcal{F}$ is everywhere of projective dimension ≤ 1. Finally, we give some examples; in particular we show that the reduction of a linear space need not be a linear space.

In Section 4, we apply the foregoing to tangent cones, in particular the tangent cone $C_4(S)$ of Whitney. In Proposition 4.6, we give a necessary and sufficient condition involving dimensions for $C_4(S)$ to be the reduction of the global Zariski tangent space, valid for a reduced complex analytic space S that is locally a complete intersection. Among other results of the section we point out two regularity criteria for complex analytic curves: We show that a reduced complex analytic curve S is regular if the symmetric algebra $\mathcal{S}(\Omega^1_S)$ is torsion free (Prop. 4.3); a stronger unproved conjecture by Berger (see [4]) states that a reduced complex analytic curve is regular if the sheaf Ω^1_S is torsion free. We also prove that a reduced complex curve S is regular at the point s if it is locally irreducible and its tangent cone $C_4(S, s)$ is reduced (Prop. 4.9).

In Section 5, we introduce the notion of cohomological positivity for a connected graded algebra $\mathcal{A}$, locally of finite presentation. We prove that a complex analytic cone is weakly negative if and only if its algebra is cohomologically positive (Theorem 5.2). We give a characterization of primary weak positivity (Theorem 5.4) and prove that a reduced compact complex analytic space is Moišezon if and only if it carries a primary weakly positive coherent sheaf (Theorem 5.5); in the case of normal irreducible spaces this result was proved by Rabinowitz in [17]. Finally, we give an example of a coherent analytic sheaf that is torsion free and primary weakly positive but not weakly positive; thus answering a question of Rabinowitz [17].

1. Reduction of complex analytic cones

1.1 Complex analytic cones

We recall that a *complex analytic cone* over a complex analytic base space S is a complex analytic space $\pi : X \to S$ over S together with an S-morphism $\mu : \mathbb{C} \times X \to X$, called *multiplication*, and a section $v : S \to X$ of π, called the *vertex* of the cone, satisfying the axioms:

(i) $\mu \circ (\mu^{\mathbb{C}} \times \mathrm{id}_X) = \mu \circ (\mathrm{id}_{\mathbb{C}} \times \mu)$,

(ii) $\mu \circ (1_X, \mathrm{id}_X) = \mathrm{id}_X$,

(iii) $\mu \circ (0_X, \mathrm{id}_X) = v \circ \pi$,

where $\mu^{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the usual multiplication in $\mathbb{C}$ and $1_X, 0_X : X \to \mathbb{C}$ are the constant mappings with values 1, 0 respectively.

By Theorem 1.4 in [2], every complex analytic cone X over S can be obtained as $X = \operatorname{Specan} \mathcal{A}$, where $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ is a (commutative) graded $\mathcal{O}_S$-algebra, locally of finite presentation, such that $\mathcal{A}_0 = \mathcal{O}_S$; we call such algebras *connected graded $\mathcal{O}_S$-algebras of finite presentation.* The algebra $\mathcal{A}$ can be obtained as a subalgebra of the direct image $\pi_*\mathcal{O}_X$: for an open set U in S the m-th component $\mathcal{A}_m(U)$ consists of the holomorphic functions $f : \pi^{-1}(U) \to \mathbb{C}$ that are homogeneous of degree m with respect to the multiplication μ, i.e. satisfy $f \circ \mu = \mu^{\mathbb{C}} \circ (z^m \times f)$ on $\mathbb{C} \times \pi^{-1}(U)$, where $z^m : \mathbb{C} \to \mathbb{C}$, $t \mapsto t^m$.

Let $\pi : X \to S$ be a complex analytic cone over S with multiplication $\mu : \mathbb{C} \times X \to X$ and vertex $v : S \to X$. Then clearly the reduction $\pi_{\mathrm{red}} : X_{\mathrm{red}} \to S_{\mathrm{red}}$ is a complex analytic cone over S_{red} with multiplication $\mu_{\mathrm{red}} : \mathbb{C} \times X_{\mathrm{red}} = (\mathbb{C} \times X)_{\mathrm{red}} \to X_{\mathrm{red}}$ and vertex $v_{\mathrm{red}} : S_{\mathrm{red}} \to X_{\mathrm{red}}$.

Let $\mathcal{A}$ be a connected graded $\mathcal{O}_S$-algebra of finite presentation. We denote by $\mathcal{N}(\mathcal{A})$ the graded ideal of nilpotent elements in $\mathcal{A}$. Clearly, $\mathcal{N}(\mathcal{A})_0 = \mathcal{N}_S =$ the ideal of nilpotent elements in $\mathcal{O}_S$. Thus we may consider $\mathcal{A}_{\mathrm{red}} := \mathcal{A}/\mathcal{N}(\mathcal{A})$ as a connected graded $\mathcal{O}_{S_{\mathrm{red}}}$-algebra.

THEOREM 1.2.— *Let X be a cone over the complex analytic space S corresponding to the connected graded $\mathcal{O}_S$-algebra $\mathcal{A}$. Then X_{red} is the cone over S_{red} corresponding to the algebra $\mathcal{A}_{\mathrm{red}}$. In particular, the algebra $\mathcal{A}_{\mathrm{red}}$ is a connected $\mathcal{O}_{S_{\mathrm{red}}}$-algebra of finite presentation, and the $\mathcal{A}$-ideal $\mathcal{N}(\mathcal{A})$ is locally finitely generated.*

Proof.— We identify S (resp. S_{red}) with a subspace of X (resp. X_{red}) via the vertex mapping and write $X_{\mathrm{red}} = \operatorname{Specan} \mathcal{B}$, where $\mathcal{B}$ is a connected graded $\mathcal{O}_{S_{\mathrm{red}}}$-algebra of finite presentation. The natural embedding $X_{\mathrm{red}} \to X$ is in an obvious sense a morphism of cones over the natural embedding $S_{\mathrm{red}} \to S$ and thus induced by a graded algebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$. We put $\mathcal{K} := \operatorname{Ker} \psi$ and obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{A} & \xrightarrow{\ \psi\ } & \mathcal{B} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_X|S & \longrightarrow & \mathcal{O}_X|S & \xrightarrow{\ \tau\ } & \mathcal{O}_{X_{\mathrm{red}}}|S & \longrightarrow & 0
\end{array}$$

of sheaves over S, where the vertical arrows are inclusions and τ is the natural projection. We have $\mathcal{K} = \mathcal{A} \cap (\mathcal{N}_X|S) = \mathcal{N}(\mathcal{A})$. It remains to show that the homomorphism ψ is surjective. Let $s \in S$ and $b \in \mathcal{B}_s$. Write $b = \sum_{m=0}^{M} b_m$ with $b_m \in \mathcal{B}_{m,s}$. Then there is an element $a \in \mathcal{O}_{X,s}$ such that $\tau(a) = b$. By [2, Lemma 1.9], a can be written uniquely as $a = \sum_{m=0}^{\infty} a_m$ in the natural topology of $\mathcal{O}_{X,s}$ with $a_m \in \mathcal{A}_{m,s}$. By the continuity of τ we have $\tau(a) = \sum_{m=0}^{\infty} \tau(a_m)$ and thus $\tau(a_m) = b_m$ for $m = 0, \ldots, M$ by the uniqueness of the representation. But then

$$\sum_{m=0}^{M} a_m \in \mathcal{A}_s \quad \text{and} \quad \psi\left(\sum_{m=0}^{M} a_m\right) = b \,. \square$$

COROLLARY 1.3.— *Let X be a cone over the complex analytic space S corresponding to the graded $\mathcal{O}_S$-algebra $\mathcal{A}$. The space X is reduced if and only if the algebra $\mathcal{A}$ is reduced; and then necessarily the space S is reduced.*

Remark 1.4.— Let X be a complex analytic cone over a complex space S such that the fibre X_s is reduced for every point s in S, and let $\mathcal{A}$ be the corresponding graded algebra. Then for every open set U in S and every $m \geq 1$ we have

$$\mathcal{N}(\mathcal{A})_m(U) := \left\{ f \in \mathcal{A}_m(U) : (f)_s \in \mathfrak{m}_s \mathcal{A}_{m,s} \text{ for every } s \text{ in } U \right\}.$$

In fact, $\mathcal{A}_m(U)$ consists of the holomorphic functions f in $\mathcal{O}_X(\pi^{-1}(U))$ that are homogeneous of degree m, where $\pi : X \to S$ is the projection, and $\mathcal{N}(\mathcal{A})_m(U)$ is the subset of $\mathcal{A}_m(U)$ consisting of those functions f that are nilpotent, locally with respect to S, i.e. for every s in S there is an open neighbourhood V of s such that $f|\pi^{-1}(V)$ is nilpotent. Because of the homogeneity, this just means that f is nilpotent, locally with respect to X, which again means that f induces the zero function on $\pi^{-1}(U)_{\text{red}}$. This is equivalent to saying that f induces the zero function on each (reduced) fibre.

2. Closure of open subcones

2.1 Zariski closure

We recall some basic facts about the closure of Zariski-open subspaces of a complex analytic space. (We refer, of course, to the analytic Zariski-topology.)

Let S be a complex analytic space, $\mathcal{F}$ be an $\mathcal{O}_S$-module and A be a closed analytic subset of S. Denote by $\mathcal{H}_A^0\mathcal{F}$ the subsheaf of $\mathcal{F}$ defined by

$$(\mathcal{H}_A^0\mathcal{F})(U) := H_{A\cap U}^0(U, \mathcal{F}) = \left\{ f \in \mathcal{F}(U) : f|U \setminus A = 0 \right\}$$

for every open set U of S.

Let $\mathcal{I}$ be a coherent $\mathcal{O}_S$-ideal such that $A = \mathrm{supp}(\mathcal{O}_S/\mathcal{I})$, let s be a point in S and $f \in \mathcal{F}_s$. If $\mathcal{F}$ is $\mathcal{O}_S$-coherent, then a necessary and sufficient condition for f to be in $\left(\mathcal{H}^0_A\mathcal{F}\right)_s$ is that $\mathcal{I}^n_s \cdot f = \{0\}$ for some natural number n; it follows that $\mathcal{H}^0_A\mathcal{F}$ is a coherent $\mathcal{O}_S$- submodule of $\mathcal{F}$ (for the details see [7], [21]).

In particular, $\mathcal{H}^0_A\mathcal{O}_S$ is a coherent $\mathcal{O}_S$-ideal, and it is easily seen that $\mathrm{supp}(\mathcal{O}_S/\mathcal{H}^0_A\mathcal{O}_S) = \overline{S \setminus A}$. The *closure of* $S \setminus A$ *in* S is by definition the closed subspace of S defined by $\mathcal{H}^0_A\mathcal{O}_S$, i.e., the subspace

$$\mathrm{cl}_S(S \setminus A) := \left(\overline{S \setminus A}\,,\, (\mathcal{O}_S/\mathcal{H}^0_A\mathcal{O}_S)|\overline{S \setminus A}\right).$$

If $\mathcal{I}$ is a coherent $\mathcal{O}_S$-ideal such that $\mathcal{I}|S \setminus A = 0$, then clearly $\mathcal{I} \subset \mathcal{H}^0_A\mathcal{O}_S$. This means that $\mathrm{cl}_S(S \setminus A)$ is the smallest closed subspace of S containing $S \setminus A$ as an open subspace. If $\mathcal{F}$ is an $\mathcal{O}_S$-module, then $\mathcal{F}/\mathcal{H}^0_A\mathcal{F}$ has a natural $\mathcal{O}_S/\mathcal{H}^0_A\mathcal{O}_S$-module structure.

We recall that the closed analytic subset A of S is said to be *analytically rare* in S if $\mathrm{cl}_S(S \setminus A) = S$, i.e., if $\mathcal{H}^0_A\mathcal{O}_S = 0$. This is equivalent to the condition that $\mathcal{I}_{A,s}$ contains a regular element (i.e., an element that is not a zero divisor) for every s in S, where $\mathcal{I}_A$ is the full sheaf of ideals defining A (see [7, Prop. 0.43]). For a reduced space S this condition simply means that A is nowhere dense in S.

Now let $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ be a connected graded $\mathcal{O}_S$-algebra of finite presentation. Then

$$\mathcal{H}^0_A\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{H}^0_A\mathcal{A}_m$$

is a graded $\mathcal{A}$-ideal, and the algebra $\mathcal{A}/\mathcal{H}^0_A\mathcal{A}$ is locally of finite presentation. This follows from our next theorem, but may also be derived directly from [2, Prop. 1.17].

THEOREM 2.2.— *Let* $\pi : X \longrightarrow S$ *be a cone over the complex analytic space* S, $X = \mathrm{Specan}\,\mathcal{A}$, *where* $\mathcal{A}$ *is a connected graded* $\mathcal{O}_S$*-algebra of finite presentation, and let* A *be a closed analytic subset of* S. *Then* $\mathrm{cl}_X(X \setminus \pi^{-1}(A))$ *is the closed subcone of the analytic restriction of* X *to* $\mathrm{cl}_S(S \setminus A)$ *corresponding to the graded algebra* $(\mathcal{A}/\mathcal{H}^0_A\mathcal{A})|\overline{S \setminus A}$. *In particular, if* A *is analytically rare in* S, *then* $\mathrm{cl}_X\left(X \setminus \pi^{-1}(A)\right)$ *is a cone over* S.

Proof.— Put $Y := \mathrm{cl}_X\big(X\setminus\pi^{-1}(A)\big)$, $T := \mathrm{cl}_S(S\setminus A)$ and let $i : Y \to X$, $j : T \to S$ be the canonical embeddings. Since clearly $\pi^{-1}(\mathcal{H}^0_A\mathcal{O}_S)\cdot\mathcal{O}_X \subset \mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X$, we obtain a projection $\overline{\pi} : Y \to T$ such that $j\circ\overline{\pi} = \pi\circ i$. Since the multiplication $\mu : \mathbb{C}\times X \to X$ is an S-morphism, we have

$$\mu^{-1}(\mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X)\cdot\mathcal{O}_{\mathbb{C}\times X} \subset \mathrm{pr}_2^{-1}(\mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X)\cdot\mathcal{O}_{\mathbb{C}\times X}\,,$$

and thus obtain a holomorphic mapping $\overline{\mu} : \mathbb{C}\times Y \to Y$ such that $i\circ\overline{\mu} = \mu\circ(\mathrm{id}\times i)$. Finally, since $\pi\circ v = \mathrm{id}_S$, we have $v^{-1}(\mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X) \subset \mathcal{H}^0_A\mathcal{O}_S$, and we obtain a holomorphic mapping $\overline{v} : T \to Y$ such that $i\circ\overline{v} = v\circ j$. It is now easily verified that $\overline{\mu}$ is a T-morphism, that $\overline{v}$ is a section of $\overline{\pi}$, and that Y satisfies the cone axioms.

We have an exact sequence

$$0 \longrightarrow \mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

of $\mathcal{O}_X$-modules. We obviously have $\pi_*\mathcal{H}^0_{\pi^{-1}(A)}\mathcal{O}_X = \mathcal{H}^0_A\pi_*\mathcal{O}_X$. Because $\mathrm{R}^1\pi_*\mathcal{F} = 0$ for every coherent $\mathcal{O}_X$-module $\mathcal{F}$ by [2, Lemma 4.1] and $j\circ\overline{\pi} = \pi\circ i$, we get an exact sequence

$$0 \longrightarrow \mathcal{H}^0_A\pi_*\mathcal{O}_X \xrightarrow{\ \phi\ } \pi_*\mathcal{O}_X \xrightarrow{\ \psi\ } j_*\overline{\pi}_*\mathcal{O}_Y \longrightarrow 0\,.$$

Since Y is a cone over T, we can write $Y = \mathrm{Specan}\,\mathcal{B}$, where $\mathcal{B} = \bigoplus_{m\geq 0}\mathcal{B}_m$ is a connected graded $\mathcal{O}_T$-algebra of finite presentation. We can identify $\mathcal{B}_m$ (resp. $\mathcal{A}_m$) with the subsheaf of $\overline{\pi}_*\mathcal{O}_Y$ (resp. $\pi_*\mathcal{O}_X$) consisting of the holomorphic functions that are homogeneous of degree m with respect to the cone multiplication. The homomorphism $\psi : \pi_*\mathcal{O}_X \to j_*\overline{\pi}_*\mathcal{O}_Y$ clearly maps $\mathcal{A}$ to $j_*\mathcal{B}$. We will now show that ψ induces an isomorphism $\mathcal{A}/\mathcal{H}^0_A\mathcal{A} \to j_*\mathcal{B}$. Let $s\in S$, $b\in(j_*\mathcal{B})_s$ and write $b = \sum_{m=0}^M b_m$, where $b_m \in \big(j_*\mathcal{B}_m\big)_s$. Then $b = \psi(a)$ for some $a \in \big(\pi_*\mathcal{O}_X\big)_s$, and a is the germ at s of some section $f\in(\pi_*\mathcal{O}_X)(U) = \mathcal{O}_X\big(\pi^{-1}(U)\big)$, where U is an open neighbourhood of s in S. By [2, Lemma 3.1] we may uniquely write $f = \sum_{m=0}^\infty f_m$ in the canonical Fréchet topology of $\mathcal{O}_X\big(\pi^{-1}(U)\big)$ with $f_m\in\mathcal{A}_m(U)$. By continuity of the restriction homomorphism we have

$$b = \psi(a) = \sum_{m=0}^{\infty}\psi\big((f_m)_s\big)$$

with $\psi\big((f_m)_s\big) \in \big(j_*\mathcal{B}_m\big)_s$. By the uniqueness statement we have $\psi\big((f_m)_s\big) = b_m$ for $m = 0, \ldots, M$ and thus $b = \psi(a')$, where

$$a' := \sum_{m=0}^{M}\big(f_m\big)_s \in \mathcal{A}_s\,.$$

Thus ψ induces a surjective algebra homomorphism $\overline{\psi} : \mathcal{A} \to j_*\mathcal{B}$, and we have

$$\operatorname{Ker} \overline{\psi} = \mathcal{A} \cap \mathcal{H}^0_A \pi_* \mathcal{O}_X = \mathcal{H}^0_A \mathcal{A} \,. \ \square$$

2.3 Torsion of a graded algebra

Let S be a *reduced* complex analytic space and $\mathcal{F}$ be an $\mathcal{O}_S$-module. Recall that we can define the *torsion submodule* $\mathcal{T}(\mathcal{F})$ of $\mathcal{F}$ by the condition

$$\mathcal{T}(\mathcal{F})_s = \{f \in \mathcal{F}_s : \text{there is a regular element } a \text{ in } \mathcal{O}_{S,s} \text{ such that } a \cdot f = 0\}.$$

If $\mathcal{F}$ is coherent, then $\mathcal{T}(\mathcal{F})$ is the kernel of the canonical homomorphism of $\mathcal{F}$ to its bidual and thus coherent. It is well known (e.g. [9, Chap. 4, Sect. 4.2]) that the subset A of S where $\mathcal{F}$ is not locally free is an analytically rare closed analytic subset of S. We clearly have $\mathcal{T}(\mathcal{F})|S \setminus A = 0$.

For a family $(\mathcal{F}_i)_{i \in I}$ of $\mathcal{O}_S$-modules one clearly has

$$\mathcal{T}\left(\bigoplus_{i \in I} \mathcal{F}_i\right) = \bigoplus_{i \in I} \mathcal{T}(\mathcal{F}_i)\,.$$

We put

$$\mathcal{F}_\sharp := \mathcal{F}/\mathcal{T}(\mathcal{F})\,.$$

If $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ is a connected graded $\mathcal{O}_S$-algebra of finite presentation, then $\mathcal{T}(\mathcal{A})$ is a graded $\mathcal{A}$-ideal, and $\mathcal{A}_\sharp = \mathcal{A}/\mathcal{T}(\mathcal{A})$ is a connected graded algebra of finite presentation, by [2, Prop. 1.17].

LEMMA 2.4.— *Let S be a reduced complex analytic space and let A be an analytically rare closed analytic subset of S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module that is torsion free on $S \setminus A$. Then $\mathcal{T}(\mathcal{F}) = \mathcal{H}^0_A \mathcal{F}$.*

Proof.— Since $\mathcal{F}$ is torsion free on $S \setminus A$ we have $\mathcal{T}(\mathcal{F})|S \setminus A = 0$ and thus $\mathcal{T}(\mathcal{F}) \subset \mathcal{H}^0_A \mathcal{F}$. Let $s \in S$, $f \in (\mathcal{H}^0_A \mathcal{F})_s$. Then for some natural number n we have $\mathcal{I}^n_{A,s} \cdot \mathcal{F}_s = \{0\}$, where $\mathcal{I}_A$ is the full sheaf of ideals defining A. Since A is analytically rare, $\mathcal{I}_{A,s}$ contains a regular element; hence $f \in \mathcal{T}(\mathcal{F})_s$, and we conclude that $\mathcal{H}^0_A \mathcal{F} = \mathcal{T}(\mathcal{F})$. $\square$

As an immediate consequence we get the following proposition.

PROPOSITION 2.5.— *Let $\pi : X \to S$ be a complex analytic cone over a reduced complex analytic space S, $X = \operatorname{Specan}\mathcal{A}$, where $\mathcal{A}$ is a connected graded $\mathcal{O}_S$-algebra of finite presentation, and let A be an analytically rare closed analytic subset of S such that $\mathcal{A}$ is torsion free on $S \setminus A$. Then*

$$\operatorname{cl}_X(X \setminus \pi^{-1}(A)) = \operatorname{Specan}(\mathcal{A}_\sharp).$$

LEMMA 2.6.— *Let X be a complex analytic cone over the complex analytic space S and let k be a positive integer. Then the set*

$$\Sigma_k(X) := \{s \in S : \dim X_s \geq k\}$$

is an analytic subset of S.

Proof.— Let $\mathcal{A}$ be the graded algebra corresponding to X, put $Z := \operatorname{Projan}(\mathcal{A})$ and let $\varpi : Z \to S$ be the canonical projection. The set $\Sigma_k(X)$ is the image of the set $B_k := \{z \in Z : \dim_z(Z_{\varpi(z)}) \geq k-1\}$ under the projection ϖ. But ϖ is proper, and it is a well known result of Cartan and Remmert that the set B_k is analytic (e.g. [7, Prop. 3.6]). □

LEMMA 2.7.— *Let X be a complex analytic cone over the complex analytic space S of dimension n such that $\dim X_s = r$ for every point s in S. Then $\dim X = n + r$.*

Proof.— This is a simple consequence of the fact that for every point s in S there is an open neighbourhood U of S and an embedding over U of $X|U$ onto a subcone of $U \times \mathbb{C}^N$ for some $\mathbb{C}^N$ with weighted multiplication [2, Corollary 1.13]. □

PROPOSITION 2.8.— *Let X be a complex analytic cone over the complex analytic space S and let A be a nowhere dense analytic subset of S. We suppose that S is of pure dimension n and that for every point s in $S \setminus A$ we have $\dim X_s = r$. Put $Y := \operatorname{cl}_X(X \setminus \pi^{-1}(A))$. Then, using the notation of Lemma 2.6, we have $\dim \Sigma_{r+k}(Y) < n - k$ for every $k \geq 1$.*

Proof.— Obviously $Y|A$ is a nowhere dense subspace of Y. By Lemma 2.7, $\dim(Y|S \setminus A) = n + r$ and thus $\dim Y = n + r$, since closure does not increase the dimension. Hence $\dim(Y|A) < n+r$ and consequently $\dim(Y|\Sigma_{r+k}(Y)) < n+r$ for every $k \geq 1$. But Lemma 2.7 also implies that

$$\dim(Y|\Sigma_{r+k}(Y)) \geq \dim \Sigma_{r+k}(Y) + r + k \quad \text{if } \Sigma_{r+k}(Y) \neq \emptyset\,;$$

hence the result. □

3. Linear spaces and related notions

3.1 Linear spaces

Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Recall that the *linear space associated to* $\mathcal{F}$ is given by

$$\mathbf{V}(\mathcal{F}) := \operatorname{Specan} \mathcal{S}(\mathcal{F}),$$

where $\mathcal{S}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$. By a theorem conjectured by Fischer [6] and proved by Prill [15] every linear space over S arises in this manner. (We recall that a *linear space over* S is by definition a module over the ring object $\mathbf{V}(\mathcal{O}_S) = \mathbb{C} \times S$ in the category of complex spaces over the complex space S.)

In [2], we gave a new proof of this theorem along the following lines: The scalar multiplication of a linear space L over a complex analytic space S clearly defines L as a cone over S. Thus we have $L = \operatorname{Specan}(\mathcal{A})$, where $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ is a connected graded $\mathcal{O}_S$-algebra of finite presentation. We showed that for a linear space L, the canonical morphism $L \to \mathbf{V}(\mathcal{A}_1)$ of cones is an isomorphism of linear spaces over S. In addition to the theorem of Fischer-Prill this implies that the linear space structure of L is already determined by the cone structure: *For a complex analytic cone X over S there is at most one structure on X as a linear space over S compatible with the given cone structure.* This interesting fact holds in the category of complex spaces over a complex base space, but not in an arbitrary category; see however [3]. (We see, for instance, that there is only one *holomorphic* addition on $\mathbb{C}^n$ making it into a linear space over $\mathbb{C}$ together with the usual multiplication; but for $n \geq 2$ it is easy to find different non-holomorphic additions with the same property.)

It therefore makes sense to ask whether a given cone $X = \operatorname{Specan}(\mathcal{A})$ *is* a linear space; this means that the canonical morphism $X \to \mathbf{V}(\mathcal{A}_1)$ of cones over S is an isomorphism.

3.2 Reduction of a linear space

Now let $L = \mathbf{V}(\mathcal{F})$ be a linear space over S. Then L_{red} is a cone over S_{red}. In view of paragraph 3.1 it is natural to ask (see Fischer [7, § 1.6]) whether L_{red} is a linear space over S_{red}. Theorem 1.2 implies that L_{red} is a linear space if and only if the canonical $\mathcal{O}_S$-algebra homomorphism

$\mathcal{S}(\mathcal{F}/\mathcal{N}_1) \to \mathcal{S}(\mathcal{F})/\mathcal{N}$ is an isomorphism, where $\mathcal{N} := \mathcal{N}(\mathcal{S}(\mathcal{F}))$ is the graded ideal of nilpotent elements in $\mathcal{S}(\mathcal{F})$, or equivalently if $\mathcal{N}$ is generated as an $\mathcal{S}(\mathcal{F})$-ideal by $\mathcal{N}_1$. This is not always the case, as will be shown in Proposition 3.11(3).

We remark that as a consequence of paragraph 1.4, we have

$$\mathcal{N}_m(U) = \left\{ f \in \mathcal{S}_m(\mathcal{F})(U) : (f)_s \in \mathfrak{m}_s \mathcal{S}_m(\mathcal{F})_s \text{ for every } s \text{ in } U \right\} ;$$

in particular

$$\mathcal{N}_1(U) = \left\{ f \in \mathcal{F}(U) : (f)_s \in \mathfrak{m}_s \mathcal{F}_s \text{ for every } s \text{ in } U \right\} .$$

3.3 Grauert-linear spaces

In [8] Grauert introduced a notion of quasilinear and linear spaces in the category of *reduced* complex spaces as follows: Let S be a reduced complex space. A *quasilinear space* over S is a reduced complex space L over S together with holomorphic mappings $\alpha : (L \times_S L)_{\text{red}} \to L$ and $\mu : \mathbb{C} \times L \to L$ over S such that for every point s in S the induced mappings

$$\begin{aligned} (\alpha_s)_{\text{red}} &: (L_s)_{\text{red}} \times (L_s)_{\text{red}} \longrightarrow (L_s)_{\text{red}} , \\ (\mu_s)_{\text{red}} &: \mathbb{C} \times (L_s)_{\text{red}} \longrightarrow (L_s)_{\text{red}} \end{aligned}$$

define the structure of a vector space on the reduced fibre $(L_s)_{\text{red}}$.

A *morphism* $\phi : L_1 \to L_2$ of quasilinear spaces over S is a holomorphic mapping over S such that for every point s in S the induced mapping $(\phi_s)_{\text{red}} : (L_{1,s})_{\text{red}} \to (L_{2,s})_{\text{red}}$ of the reduced fibres is linear.

Let L be a quasilinear space over S. A *quasilinear subspace* of L is a reduced closed analytic subspace L' of L such that for every point s in S the reduced fibre $(L'_s)_{\text{red}}$ is a linear subspace of $(L_s)_{\text{red}}$. The restriction of the " quasiaddition " of L then induces a quasiaddition on L' making L' a quasilinear space, and the inclusion $L' \to L$ a morphism of quasilinear spaces.

For a quasilinear space L over S we get a mapping (of sets) $v : S \to L$, called the *zero section* of L, by putting $v(s) := 0_s$, where 0_s is the zero element of $(L_s)_{\text{red}}$.

The concept of a quasilinear space is very weak, as the following example shows: Let S be a non-discrete complex space, e.g. $S = \mathbb{C}$, and put $L := S^d \times \mathbb{C}^n$, where S^d is the set S with discrete topology, considered as a zero-dimensional reduced complex space. Clearly the projection $L \to S^d \hookrightarrow S$ is holomorphic, and the vector space structure on $\mathbb{C}^n$ induces the structure of a quasilinear space over S on L. In this example, the zero section of L is not continuous.

A *Grauert-linear space* over S is a quasilinear space L over S such that for every point s in S there is an open neighbourhood U of s in S such that the restriction $L|U$ is isomorphic to a quasilinear subspace of $U \times \mathbb{C}^n$ for some natural number n.

The reduction of a linear space over S is a clearly Grauert-linear space over S. More generally a Grauert-linear space over S is the same as a quasi-linear subspace of the reduction of a linear space over S. Clearly the zero section of a Grauert-linear space is holomorphic. It follows that a Grauert-linear space is a cone whose vertex is the zero section.

PROPOSITION 3.4.— *Let S be a reduced complex space.*

(1) A reduced cone over S is a Grauert-linear space if and only if the reduction of each fibre is a linear space.

(2) A quasilinear space over S is a Grauert-linear space if and only if its zero section is holomorphic.

(3) A Grauert-linear space over S is the reduction of a linear space if and only if each fibre is reduced.

For the proof we need a simple lemma. A complex analytic cone $X = \mathrm{Specan}(\mathcal{A})$ over S is a subcone of a linear space over S if and only if the algebra $\mathcal{A}$ is generated by $\mathcal{A}_1$. We call such cones *straight*. This property is clearly local with respect to the base space.

LEMMA 3.5.— *Let X be a reduced cone over the reduced complex space S. Suppose that for every s in S the reduction $(X_s)_{\mathrm{red}}$ of the fibre is a straight cone (over the reduced point). Then X is a straight cone.*

Proof.— Let $X = \mathrm{Specan}\,\mathcal{A}$ and let $s \in S$. There is an open neighbourhood U of s in S such that $\mathcal{A}$ is generated over U by homogeneous elements $a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}$, where $a_1, \ldots, a_n$ have degree 1 and $a_{n+1}, \ldots, a_{n+m}$ have degree greater than 1. These elements define a cone embedding over U of $X|U$ into $U \times \mathbb{C}^n \times \mathbb{C}^m$, where the multiplication in the first factor $\mathbb{C}^n$ is the usual vector space multiplication, but the multiplication in each coordinate of the second factor $\mathbb{C}^m$ is weighted of degree greater than 1. The condition that the reduction of each fibre is straight means that the embedding maps $(X_s)_{\text{red}}$ into $\{s\} \times \mathbb{C}^n \times \{0\}$. This means that $X|U$ gets mapped set-theoretically into $U \times \mathbb{C}^n \times \{0\}$. Since X is reduced, we obtain an embedding of $X|U$ into $U \times \mathbb{C}^n$. □

Proof of Proposition 3.4

(1) Let L be a reduced cone over S such that the reduction of each fibre of L is a linear space. By Lemma 3.5, L is a straight cone, hence a subcone of a linear space L_1. Since the linear space structure of each reduced fibre is uniquely determined by the cone structure, L is a quasilinear subspace of the reduction of L_1 and thus a Grauert-linear space.

(2) Let L be a quasilinear space over S with holomorphic zero section. Then L is a cone over S whose reduced fibres are linear spaces, hence a Grauert-linear space by (1).

(3) Let L be a Grauert-linear space over S, $L = \operatorname{Specan} \mathcal{A}$, and put $L_1 := \operatorname{Specan} \mathcal{S}(\mathcal{A}_1)$. For $s \in S$ the fibre L_s is the cone over the reduced point given by the algebra $\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s$, where $\mathfrak{m}_s$ is the maximal ideal of $\mathcal{O}_{S,s}$. Since the fibre is a reduced linear space, $\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s$ is a symmetric algebra, and then necessarily the symmetric algebra of $\mathcal{A}_{1,s}/\mathfrak{m}_s\mathcal{A}_{1,s}$. Hence $L_s = L_{1,s}$ for every s in S. Thus L and L_1 have the same underlying topological space. Since L is reduced, L is the reduction of L_1. □

PROPOSITION 3.6.— *Let S be a reduced complex analytic space, let A be an analytically rare closed analytic subset of S and $\mathcal{I}_A$ be the full sheaf of ideals defining A. Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module that is locally free on $S \setminus A$ and put $\mathcal{N} := \mathcal{N}(\mathcal{S}(\mathcal{F}))$ and $\mathcal{T} := \mathcal{T}(\mathcal{S}(\mathcal{F}))$. Then*

$$\mathcal{I}_A \cdot \mathcal{T} \subset (\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})) \cap \mathcal{T} \subset \mathcal{N} \subset \mathcal{T}.$$

In particular, if the linear space $\mathbf{V}(\mathcal{F})$ is reduced, then the sheaf $\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})$ is torsion free, and $\mathcal{I}_A \cdot \mathcal{T} = 0$.

Proof.— The first inclusion is obvious. Since $\mathcal{F}$ is locally free on $S \setminus A$, $\mathcal{S}(\mathcal{F})$ is torsion free on $S \setminus A$, and we have $\mathcal{T} = \mathcal{H}^0_A\mathcal{S}(\mathcal{F})$. Hence for every $f \in \mathcal{T}_s$ there is an n such that $\mathcal{I}^n_{A,s} \cdot f = 0$; if also $f \in (\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F}))_s$, then

$f^{n+1} \in (\mathcal{I}_{A,s}^n \cdot f) \cdot \mathcal{S}(\mathcal{F})_s = 0$ and thus $f \in \mathcal{N}_s$. Finally, let $g \in \mathcal{N}(U)$, where U is an open set in S, and suppose that $g^n = 0$. Since $\mathcal{F}$ is locally free on $U \setminus A$, we have $\operatorname{supp} g \subset U \cap A$ and hence

$$g \in \mathcal{H}_A^0 \mathcal{S}(\mathcal{F})(U) = \mathcal{T}(U) . \square$$

In the case that the $\mathcal{O}_S$-module $\mathcal{F}$ is locally free outside of a discrete set, one of the inclusions in Proposition 3.6 is an equation: Let us more generally suppose, in addition to the hypotheses in Proposition 3.6, that the restriction $\mathbf{V}(\mathcal{F}) \times_S A$ of the linear space $\mathbf{V}(\mathcal{F})$ to the reduced subspace A is reduced. This means that the symmetric algebra of the $\mathcal{O}_S/\mathcal{I}_A$-module $\mathcal{F}/\mathcal{I}_A\mathcal{F}$ is reduced. Then $\mathcal{N}$ is a subsheaf of $\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})$. Hence Proposition 3.6 implies that $\mathcal{N} = \big(\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})\big) \cap \mathcal{T}$. We have proved the following corollary.

COROLLARY 3.7.— *With the same hypotheses and notations as in Proposition 3.6, let us suppose in addition that the restriction* $\mathbf{V}(\mathcal{F}) \times_S A$ *of the linear space* $\mathbf{V}(\mathcal{F})$ *to* A *is reduced; this is in particular the case when* A *is discrete. Then*

$$\mathcal{N} = \big(\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})\big) \cap \mathcal{T} .$$

In particular we have $\mathcal{T} = \mathcal{N}$ *if and only if* $\mathcal{T} \subset \mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})$.

As another corollary of Proposition 3.6, we get the following result.

COROLLARY 3.8.— *Let* S *be a reduced complex analytic space, let* $\mathcal{F}$ *be a coherent* $\mathcal{O}_S$*-module that is locally free outside a discrete set. If the corresponding linear space* $\mathbf{V}(\mathcal{F})$ *is reduced, then the torsion* $\mathcal{T}\big(\mathcal{S}(\mathcal{F})\big)$ *is a direct summand of* $\mathcal{S}(\mathcal{F})$ *as a graded* $\mathcal{O}_S$*-module.*

Proof.— Let A be a discrete set such that $\mathcal{F}$ is locally free outside A and retain the notations of Proposition 3.6. We have a short exact sequence $0 \to \mathcal{I}_A\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F}) \to \mathcal{S}(F)/\mathcal{I}_A\mathcal{S}(\mathcal{F}) \to 0$. The hypothesis that $\mathbf{V}(\mathcal{F})$ is reduced means that $\mathcal{N} = 0$ and hence that $\big(\mathcal{I}_A \cdot \mathcal{S}(\mathcal{F})\big) \cap \mathcal{T} = 0$. This implies that $\mathcal{T}$ is mapped injectively into $\mathcal{S}(\mathcal{F})/\mathcal{I}_A\mathcal{S}(\mathcal{F})$. Since the latter sheaf is supported on a discrete set, the image of $\mathcal{T}$ has a direct complement as a graded $\mathbb{C}$-vector space at every point, and this is in a natural way a direct complement of the image of $\mathcal{T}$ as a graded $\mathcal{O}_S$-module. Its preimage in $\mathcal{S}(\mathcal{F})$ is then a direct complement of $\mathcal{T}$ as a graded $\mathcal{O}_S$-module. $\square$

3.9 The primary component

Let S be a reduced complex analytic space and let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Let $\mathbf{V}(\mathcal{F}) := \operatorname{Specan} \mathcal{S}(\mathcal{F})$ be the linear space associated with $\mathcal{F}$ and let A be the (nowhere dense) closed subset of S where $\mathcal{F}$ is not locally free. We put

$$\mathbf{V}(\mathcal{F})_\sharp := \mathrm{cl}_{\mathbf{V}(\mathcal{F})_{\mathrm{red}}}\Big(\mathbf{V}(\mathcal{F})_{\mathrm{red}} \setminus \pi^{-1}(A)\Big),$$

where $\pi : \mathbf{V}(\mathcal{F}) \to S$ is the projection. This is what Rabinowitz [17] calls the *primary component* of $\mathbf{V}(\mathcal{F})$. Clearly $\mathbf{V}(\mathcal{F})_\sharp$ is a reduced space.

THEOREM 3.10.— *Let S be a reduced complex analytic space and let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Then $\mathbf{V}(\mathcal{F})_\sharp$ is the cone over S defined by the graded $\mathcal{O}_S$-algebra $\mathcal{S}(\mathcal{F})_\sharp$.*

Proof.— By Proposition 2.5, we have

$$\operatorname{Specan}\Big(\mathcal{S}(\mathcal{F})_\sharp\Big) = \mathrm{cl}_{\mathbf{V}(\mathcal{F})}\big(\mathbf{V}(\mathcal{F}) \setminus \pi^{-1}(A)\big).$$

We have to show that

$$\mathbf{V}(\mathcal{F})_\sharp = \mathrm{cl}_{\mathbf{V}(\mathcal{F})}\big(\mathbf{V}(\mathcal{F}) \setminus \pi^{-1}(A)\big)$$

or, equivalently, that $\mathrm{cl}_{\mathbf{V}(\mathcal{F})}\big(\mathbf{V}(\mathcal{F}) \setminus \pi^{-1}(A)\big)$ is a reduced space. This follows from Corollary 1.3 and the fact that $\mathcal{N}\big(\mathcal{S}(\mathcal{F})\big) \subset \mathcal{H}^0_A\mathcal{S}(\mathcal{F}) = \mathcal{T}\big(\mathcal{S}(\mathcal{F})\big)$. □

Let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module over a reduced complex space S. We say that $\mathcal{F}$ has *free rank* r if it is free of rank r outside a nowhere dense analytic subset of S.

PROPOSITION 3.11.— *Let S be a reduced complex space of pure dimension n and let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module of free rank r. Put*

$$\Sigma_k(\mathcal{F}) := \Sigma_k\big(\mathbf{V}(\mathcal{F})\big) = \Big\{s \in S : \dim \mathbf{V}(\mathcal{F})_s \geq k\Big\}$$

and consider the conditions:

(i) $\mathbf{V}(\mathcal{F})_\sharp = \mathbf{V}(\mathcal{F})_{\mathrm{red}}$;

(ii) $\dim \Sigma_{r+k}(\mathcal{F}) < n - k$ *for every* $k \geq 1$.

Then condition (i) implies condition (ii). If $\mathcal{F}$ is everywhere of projective dimension ≤ 1, then the conditions (i) and (ii) are equivalent.

Proof. — That (i) implies (ii) is a special case of (2.8). Now suppose that $\mathcal{F}$ is everywhere of projective dimension ≤ 1. The conditions being local in S, we may assume that we have an exact sequence $0 \to \mathcal{O}_S^p \to \mathcal{O}_S^q \to \mathcal{F} \to 0$ of sheaves or, equivalently, an exact sequence

$$0 \longrightarrow L \longrightarrow S \times \mathbb{C}^q \xrightarrow{\ \phi\ } S \times \mathbb{C}^p$$

of linear spaces, where $L := \mathbf{V}(\mathcal{F})$ and ϕ is an epimorphism in the categorical sense; this means in our context that the linear mapping ϕ_s of fibres is surjective outside a nowhere dense set in S. Since $\mathcal{F}$ has free rank r we have $\dim L_s = r$ outside a nowhere dense set A in S. Hence $q = p + r$. We have $L = f^{-1}(0)$, where $f = \mathrm{pr}_2 \circ \phi : S \times \mathbb{C}^{p+r} \to \mathbb{C}^p$. Hence L is defined by p equations in a space of pure dimension $n + p + r$, and thus every irreducible component of L_{red} has dimension at least $n + r$. If $L_\sharp \neq L_{\mathrm{red}}$, then there must be an irreducible component of L_{red} lying over the nowhere dense set A in S. But condition (ii) guarantees that every component of $L \times_S A$ has dimension less than $n + r$. □

Examples 3.12

(1) Let S be a reduced complex space, let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}_S$ and A be the set of points where $\mathcal{I}$ is not locally free. The canonical homomorphism of algebras

$$\mathcal{S}(\mathcal{I}) \to \bigoplus_{m \geq 0} \mathcal{I}^m$$

of $\mathcal{S}(\mathcal{I})$ to the *Rees-algebra* $\bigoplus_{m\geq 0} \mathcal{I}^m$ of the ideal $\mathcal{I}$ is an isomorphism outside of A. Since the Rees-algebra is torsion free it induces an isomorphism

$$\mathrm{Specan}\left(\bigoplus_{m\geq 0} \mathcal{I}^m\right) \to \mathbf{V}(\mathcal{I})_\sharp\,.$$

(2) Let S be a complex space and Z be a subspace of S defined by an $\mathcal{O}_S$-ideal $\mathcal{I}$. The *normal cone* of Z in S is by definition the cone

$$C_{Z/S} := \mathrm{Specan}\left(\left(\bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}\right)|Z\right)\,.$$

It is a subcone of the *normal space* $N_{Z/S} := \mathrm{Specan}\,(\mathcal{S}(\mathcal{I}/\mathcal{I}^2)|Z)$ of Z in S.

Now suppose that the space S is reduced. Since the algebra

$$\bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}$$

is the analytic restriction to Z of the Rees-algebra of $\mathcal{I}$, our first example shows that

$$C_{Z/S} = \mathbf{V}(\mathcal{I})_\sharp \times_S Z\,.$$

(3) Let $z = (z_1, \ldots, z_n)$ be the coordinates in $\mathbb{C}^n$, $n \geq 2$, and let $\mathbb{C}\langle z\rangle$ be the algebra of convergent power series in z. For a holomorphic function g defined in a neighbourhood of the origin in $\mathbb{C}^n$ we denote the order of its Taylor series at the origin by $\operatorname{ord}(g)$.

Let f be a non-zero holomorphic function in an open neighbourhood U of the origin in $\mathbb{C}^n$ defining a *reduced* hypersurface S of U, and suppose that $\operatorname{ord}(f) \geq 3$. Let $\mathcal{I}$ be the full $\mathcal{O}_S$-ideal sheaf defining the origin. We shall show that the reduction of the linear space $\mathbf{V}(\mathcal{I})$ over S is not a linear space.

Looking at the germs at the origin, we consider $\mathcal{O}_{S,0} = \mathbb{C}\langle z\rangle/(f)$ and $\mathfrak{m} := \mathcal{I}_0$ as $\mathbb{C}\langle z\rangle$-modules. Let $Z = (Z_1, \ldots, Z_n)$ be indeterminates. We obtain a graded algebra homomorphism $\alpha : \mathbb{C}\langle z\rangle[Z] \to \mathcal{S}(\mathfrak{m})$ determined by $\alpha(Z_k) := z_k$ for $k = 1, \ldots, n$. Let

$$\beta : \mathcal{S}(\mathfrak{m}) \to \bigoplus_{m\geq 0} \mathfrak{m}^m$$

be the canonical homomorphism, and put $\gamma := \beta \circ \alpha$. Then

$$J = \bigoplus_{m\geq 0} J_m := \operatorname{Ker}\gamma$$

is the graded ideal of $\mathbb{C}\langle z\rangle[Z]$ consisting of all elements $P = P(z, Z)$ such that $P_m(z, z)$ is in the $\mathbb{C}\langle z\rangle$-ideal generated by f for every homogenoeus component P_m of P. Also $\operatorname{Ker}\alpha$ is the graded ideal generated by $J_0 + J_1$.

Put $\ell := \operatorname{ord}(f)$. One can choose an element $F = F(z, Z)$ in $\mathbb{C}\langle z\rangle[Z]$ such that $F(z, Z)$ is a homogeneous polynomial of degree ℓ in Z and $F(z, z) = f(z)$; this can be done by replacing ℓ of the z_k's with corresponding Z_k's in each term of the power series of f at the origin. Clearly $F \in J$, and thus $\alpha(F) \in \operatorname{Ker}\beta$. By example (1), $\alpha(F)$ is a torsion element of $\mathcal{S}(\mathfrak{m})$.

By Proposition 3.6, $\alpha(z_1F)$ is a nilpotent element of $\mathcal{S}(\mathfrak{m})$. Since $\mathcal{S}(\mathfrak{m})$ has no torsion elements and hence no nilpotent elements in degree 1, it suffices to show that $\alpha(z_1F) \neq 0$, for then we have shown that $\mathcal{N}(\mathcal{S}(\mathcal{I}))$ is not generated by $\mathcal{N}_1(\mathcal{S}(\mathcal{I}))$. Let $L_1(z,Z), \dots, L_r(z,Z)$ be generators of J_1. Suppose that z_1F were in $\operatorname{Ker}\alpha$, then we could write a relation

$$z_1F(z,Z) = R(z,Z)f(z) + \sum_{k=1}^{r} Q_k(z,Z)L_k(z,Z),$$

where R and $Q_1, \dots, Q_r$ are homogeneous of degree ℓ and $\ell - 1$ in Z respectively. Replacing Z by z, we obtain a relation

$$z_1f(z) = z_1F(z,z) = g(z)f(z),$$

with $\operatorname{ord}(g) \geq \ell - 1$; a contradiction, since $\ell \geq 3$.

(4) Let X be the cone over $\mathbb{C}$ defined by the graded algebra $\mathcal{O}_{\mathbb{C}}[X, Y]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the elements $zY - z^2X$ and $Y^2 - z^2X^2$ and z is the coordinate function of $\mathbb{C}$. Clearly X is reduced, the fibre X_z is a reduced line for $z \neq 0$, and the fibre X_0 is a double line. Thus the reduction of each fibre is a linear space, but one fibre is not reduced. This shows that X is a Grauert-linear space that is not the reduction of a linear space. For other examples, see Paragraph 4.1.

4. Tangent cones

4.1 Whitney's tangent cones

In [23] Whitney defined several notions of tangent cones to an analytic subset of some $\mathbb{C}^n$. In fact, for every analytic subset S of $\mathbb{C}^n$ and every point s of S he defined six subcones $C_k(S,s)$ of $\mathbb{C}^n$ for $k = 1, \dots, 6$. For $k = 4, 5, 6$ the cone $C_k(S,s)$ is the fibre of a globally defined cone $C_k(S)$ over S (or rather the reduction of the fibre, since Whitney only defines his cones as analytic subsets of $\mathbb{C}^n$). This is not the case for $k = 1, 2, 3$, as Whitney showed in [23].

For $k = 3, 4, 5, 6$, the Whitney cones can be defined in a natural way for (not necessarily embedded) complex spaces S, though S must be reduced in the case $k = 4$. These natural definitions, which we shall now describe briefly, yield in general non-reduced cones; to obtain Whitney's original definition one has to form the reduction of these cones and their fibres.

It is well known that $C_3(S, s)$ is the normal cone of the reduced one-point subspace $\{s\}$ of S and thus only lives naturally over the space $\{s\}$.

The globally defined cones $C_k(S)$ for $k = 4, 5, 6$ have the following definitions.

- The cone $C_6(S)$ is just the *global Zariski tangent space* $T_S := \mathbf{V}(\Omega^1_S)$, where Ω^1_S is the sheaf of holomorphic 1-forms on S. This can also be interpreted as the analytic restriction $\mathbf{V}(\mathcal{I}) \times_{S \times S} S$ of the linear space $\mathbf{V}(\mathcal{I})$ on $S \times S$ to the diagonal, where $\mathcal{I}$ is the ideal of the diagonal of $S \times S$ and S has been naturally identified with the diagonal.
- The cone $C_5(S)$ is the so-called *tangent star cone* of the space S, defined as the normal cone of S embedded as the diagonal of $S \times S$, i.e.,

$$\operatorname{Specan}\left(\bigoplus_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}\right),$$

  where $\mathcal{I}$ is the ideal sheaf of the diagonal in $S \times S$ (see e.g., [11], [12]). For a reduced space S this can by Example 3.12(2) in be interpreted as $\mathbf{V}(\mathcal{I})_\sharp \times_{S\times S} S$, i.e., the analytic restriction of the cone $\mathbf{V}(\mathcal{I})_\sharp$ on $S \times S$ to the diagonal.
- Finally, the cone $C_4(S)$, which according to Whitney is the closure of the part of the global tangent space lying over the regular points of S, can now for a reduced base space S easily be interpreted as the cone $(T_S)_\sharp$. If we again consider S embedded as the diagonal of $S \times S$ with ideal $\mathcal{I}$, then $C_4(S) = \left(\mathbf{V}(\mathcal{I}) \times_{S\times S} S\right)_\sharp$.

From Theorem 3.10 we get an algebraic description of $C_4(S)$ as

$$C_4(S) = \operatorname{Specan}\left(\mathcal{S}(\Omega^1_S)_\sharp\right) = \operatorname{Specan}\left(\mathcal{S}(\Omega^1_S)/\mathcal{T}\mathcal{S}(\Omega^1_S)\right).$$

We emphasize that the cone $C_4(S)$ is only defined for a reduced base space S, and it is necessarily a reduced space. However we note that the *fibres* $C_4(S, s) = C_4(S)_s$ of $C_4(S)$ often carry a natural non-reduced structure.

Thus it is easily seen that for the cubic parabola $S := \{(z,w) \in \mathbb{C}^2 : w^2 = z^3\}$ the fibre $C_4(S)_0$ is a double line. The reduction of the fibre is a line and in particular a linear space. Thus $C_4(S)$ is a natural example of a Grauert-linear space that is not a linear space. This is more generally the case for every singular, reduced and locally irreducible complex analytic curve (see Corollary 4.9).

From Proposition 2.8 and the description above of the cones $C_4(S)$ and $C_5(S)$, we immediately obtain the following result (see also Stutz [22]):

PROPOSITION 4.2.— *Let S be a reduced complex space of pure dimension n. Then*

$$\dim \Sigma_{n+k}\big(C_4(S)\big) < n - k$$

and

$$\dim \Sigma_{n+k}\big(C_5(S)\big) < n - k + 1$$

for every $k \geq 1$.

PROPOSITION 4.3.— *Let S be a reduced complex analytic curve. If $\mathcal{TS}(\Omega^1_S) = \mathcal{NS}(\Omega^1_S)$, then S is regular. In particular, S is regular if the symmetric algebra $\mathcal{S}(\Omega^1_S)$ is torsion free.*

Proof.— By Proposition 4.2, the cone $C_4(S)_s$ is 1-dimensional for every s in S. If $\mathcal{TS}(\Omega^1_S) = \mathcal{NS}(\Omega^1_S)$, then $C_4(S) = (T_S)_{\mathrm{red}}$; hence $\dim T_{S,s} = 1$ for every s. Hence S is regular. □

Remark 4.4.— In [4], Berger conjectured the much stronger result that a reduced curve with a torsion free differential module is regular. We note that an analogous result cannot hold in higher dimensions; see e.g. the proof of the next proposition.

Applying Proposition 3.11 to the Zariski tangent space we obtain the following result.

PROPOSITION 4.5.— *Let S be a reduced complex space of pure dimension n and put*

$$\Sigma_k := \Sigma_k(T_S) = \{s \in S : \operatorname{emdim}_s S \geq k\} ;$$

in particular Σ_{n+1} is the singular locus of the space S. Consider the conditions:

(i) $C_4(S) = (T_S)_{\text{red}}$;

(ii) $\dim \Sigma_{n+k} < n - k$ *for every* $k \geq 1$.

Then condition (i) implies condition (ii). If S is a locally complete intersection, then the conditions (i) and (ii) are equivalent.

Proof.— We just have to note that for a locally complete intersection S the sheaf of differentials Ω^1_S is everywhere of projective dimension ≤ 1: Embed S locally as a subspace of some $\mathbb{C}^N$ defined by a coherent ideal sheaf $\mathcal{I}$. It is well known that we have an exact sequence $(\mathcal{I}/\mathcal{I}^2)|S \rightarrow (\Omega^1_{\mathbb{C}^N}/\mathcal{I}\Omega^1_{\mathbb{C}^N})|S \rightarrow \Omega^1_S \rightarrow 0$ inducing a short exact sequence on the regular part of S, hence on the whole of S, since the condition that S is locally a complete intersection implies that the sheaf $(\mathcal{I}/\mathcal{I}^2)|S$ is locally free. □

COROLLARY 4.6.— *Let S be a reduced subspace of some $\mathbb{C}^N$ with singular locus Σ and suppose that S is locally a complete intersection satisfying the condition* $\operatorname{codim}_S \Sigma > \operatorname{codim}_{\mathbb{C}^N} S$. *Then* $C_4(S) = (T_S)_{\text{red}}$. *This is in particular the case if S is a hypersurface with singular locus of codimension* ≥ 2.

4.7 Inclusions of tangent cones

In [23], Whitney proved the set-theoretic inclusions $C_3(S,s) \subset C_4(S,s) \subset C_5(S,s) \subset C_6(S,s)$ for the tangent cones. The last two inclusions also hold when we give the cones their natural, possibly non-reduced, structures. The question when the first inclusion holds is more complicated; it is intimately related to the so-called torsion problem of Reiffen and Vetter.

In [18], Reiffen and Vetter discussed four possible definitions of holomorphic 1-forms on a reduced complex space S. In addition to the standard sheaf of holomorphic 1-forms Ω^1_S, which they denoted by Ω_a, they considered the sheaf Ω_g of holomorphic 1-forms defined by Rossi [19], the sheaf Ω_h defined as the dual sheaf of the sheaf of vector fields as defined by Rossi, and finally the sheaf $\Omega_b := i_*\Omega^1_U$, where U is the regular part of S and $i : U \rightarrow S$ is the inclusion. There are canonical mappings $\Omega_a \rightarrow \Omega_g \rightarrow \Omega_h \rightarrow \Omega_b$, and in the paper the question of the injectivity of the mapping $\Omega_g \rightarrow \Omega_h$ was raised.

Rossi's definitions in [19] of the sheaves of holomorphic 1-forms and holomorphic vector fields on a reduced complex space are like the usual definitions — holomorphic 1-forms are the linear forms on the tangent space and vector fields are the sections of the tangent space — *except* that Rossi is working in the reduced category, and his tangent space is the *reduction* of the Zariski tangent space. This has no effect on the definition of vector fields (since a section of a linear space over a reduced base space factors uniquely through the reduction of the linear space), but by the results of Section 1 the sheaf Ω_g of 1-forms according to Rossi is the sheaf $\Omega^1_S/\mathcal{N}\Omega^1_S$, where $\mathcal{N}\Omega^1_S$ is the submodule of Ω^1_S consisting of the elements that are nilpotent in the symmetric algebra; this also has the description

$$\mathcal{N}\Omega^1_S(U) = \left\{\omega \in \Omega^1_S(U) : \omega_s \in \mathfrak{m}_s\Omega^1_S \text{ for every } s \text{ in } U\right\}.$$

It also follows that the dual sheaf of the sections of vector fields is the bidual of Ω^1_S. The injectivity of the mapping $\Omega_g \to \Omega_h$ is thus equivalent to the equality $\mathcal{N}\Omega^1_S = \mathcal{T}\Omega^1_S$, or in other words to the inclusion $\left(\mathcal{T}\Omega^1_S\right)_s \subset \mathfrak{m}_s\Omega^1_{S,s}$ for every s in S. In this form the question was posed by Scheja in [20], where he proved the inclusion for quasi-homogeneous singularities. In [14], Platte gave an example showing that the inclusion does not hold in general in any dimension.

Now let S be a reduced space and $\mathcal{T} := \mathcal{TS}(\Omega^1_S)$. Then $C_3(S,s)$ is the subcone of the Zariski tangent space $T_{S,s} = \operatorname{Specan}\left(\mathcal{S}(\Omega^1_S)_s \otimes \mathbb{C}\right)$ defined by the graded algebra $\bigoplus_{m\geq 0} \mathfrak{m}_s^m/\mathfrak{m}_s^{m+1}$, where $\mathfrak{m}_s$ is the maximal ideal of $\mathcal{O}_{S,s}$. By the description of $C_4(S)$ given in Paragraph 4.1, we see that $C_3(S,s)$ is a subspace of $C_4(S,s)$, with their possibly non-reduced structures, if and only if

$$\mathcal{T}_s \subset \operatorname{Ker}\left[\mathcal{S}\left(\Omega^1_S\right)_s \to \bigoplus_{m\geq 0} \mathfrak{m}_s^m/\mathfrak{m}_s^{m+1}\right],$$

where $\mathcal{S}\left(\Omega^1_S\right)_s \to \bigoplus_{m\geq 0} \mathfrak{m}_s^m/\mathfrak{m}_s^{m+1}$ is the natural mapping. In particular the inclusion $\left(\mathcal{T}_1\right)_s \subset \mathfrak{m}_s\Omega^1_{S,s}$ is a *necessary* condition for the inclusion. Hence the examples of Platte show that the inclusion does not hold in general.

We finally remark, for use in the proof of the next result, that one of Rossi's theorems of his paper [19] can be reformulated in terms of the cone $C_4(S)$. It is the following regularity criterion: If S is a reduced complex space, $s \in S$, $\dim_s S = n$ and there are n holomorphic vector fields defined on a neighbourhood of s and linearly independent at s, then the space S

is non-singular at the point s. We note that such vector fields span an n-dimensional vector bundle that is a subcone of $C_4(S)$, and then necessarily equal to $C_4(S)$, over the given neighbourhood of s. Conversely, if $C_4(S)$ is a vector bundle with fibre dimension n in a neighbourhood of s, then such vector fields clearly exist. The Rossi regularity criterion can thus be formulated as follows: *If S is a reduced complex space, $s \in S$, $\dim_s S = n$ and $C_4(S)$ is a vector bundle of fibre dimension n in a neighbourhood of s, then S is non-singular at s.*

We use this to prove the following proposition.

PROPOSITION 4.8.— *Let S be a pure-dimensional reduced complex space and let $s \in S$. Suppose that the tangent cone $C_4(S, s)$ is a vector space of dimension $\dim_s S$; in particular $C_4(S, s)$ is reduced. Then S is regular at s.*

Proof.— Put $n := \dim_s S$. As is easily seen, a straight cone over a reduced point is a vector space if and only if it is non-singular, which again means that its dimension and embedding dimension at the vertex coincide. Since the embedding dimension of the fibre is upper semicontinuous [7, Prop. 3.6, p. 137] and S is pure-dimensional, it follows that $C_4(S, t)$ is an n-dimensional vector space for every t in an open neighbourhood U of s in S. It follows from Proposition 3.4 that the restriction of $C_4(S)$ to U is the reduction of a linear space L over U. The fibres of L are (reduced) vector spaces and thus identical to the fibres of $C_4(S)$ over U. By a theorem of Fischer [5], L is a vector bundle. A vector bundle over a reduced space is reduced, hence L is the restriction of $C_4(S)$ to U and so $C_4(S)$ is a vector bundle in a neighbourhood of s. By the theorem of Rossi mentioned above, S is non-singular at s. □

As a corollary we get.

COROLLARY 4.9.— *Let S be a reduced and locally irreducible complex analytic curve and let s be a point of S such that the cone $C_4(S, s)$ is reduced. Then S is regular at s.*

Proof.— Since S is irreducible at s, the reduction of $C_4(S, s)$ is clearly a complex line. □

We finally note the following corollary of Paragraph 3.8.

PROPOSITION 4.10.— *Let S be a reduced complex analytic space with isolated singularities. If the tangent space T_S is reduced, then the torsion ideal of $\mathcal{S}(\Omega^1_S)$ is a direct summand of $\mathcal{S}(\Omega^1_S)$ as a graded $\mathcal{O}_S$-module. In particular the torsion submodule of Ω^1_S is a direct summand.*

In this connection we recall the conjecture of Scheja ([20, p. 157]; see also Platte [14]), a weakened version of the Reiffen-Vetter torsion problem: If the torsion submodule of Ω^1_S is a direct summand, then it is the trivial submodule.

5. Notions of positivity

5.1 Wealey and cohomologically positive graded algebras

Let S be a compact complex analytic space, and let X be a cone over S defined by the connected graded $\mathcal{O}_S$-algebra $\mathcal{A}$. We say that the cone X is *weakly negative* and that the algebra $\mathcal{A}$ is *weakly positive* if the vertex of X is exceptional in X (over the reduced point). We say that $\mathcal{A}$ is *cohomologically positive* if for every coherent $\mathcal{O}_S$-module $\mathcal{G}$ there is an integer n_0 such that

$$H^p(X, \mathcal{A}_n \otimes \mathcal{G}) = 0 \quad \text{for all } n \geq n_0 \text{ and all } p \geq 1\,.$$

A coherent $\mathcal{O}_S$-module $\mathcal{F}$ is, by definition, weakly positive (resp. cohomologically positive) if and only if its symmetric algebra is. As an easy consequence of our results in [2], we obtain the following theorem.

THEOREM 5.2.— *Let S be a compact complex space and $\mathcal{A}$ be a connected graded algebra of finite presentation. Then $\mathcal{A}$ is weakly positive if and only if it is cohomologically positive.*

Proof.— The fact that weak positivity implies cohomological positivity is a special case of our [2, Theorem 4.4]. Now suppose that $\mathcal{A}$ is cohomologically positive and let $\mathcal{I}_{\{s\}}$ be the ideal sheaf of the point s in S. By [2, Theorem 3.2] we have to show that for all s, t in S the canonical homomorphism

$$\Gamma(S, \mathcal{A}_n) \longrightarrow \Gamma\big(S\,, \mathcal{A}_n/\mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}}\mathcal{A}_n\big)$$

is surjective for all large enough n. We have an exact sequence

$$0 \longrightarrow \mathcal{K}_n \longrightarrow \mathcal{A}_n \otimes \mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}} \longrightarrow \mathcal{A}_n \xrightarrow{\;\rho\;} \mathcal{A}_n/\mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}}\mathcal{A}_n \longrightarrow 0$$

of graded $\mathcal{O}_S$-modules. For every n we obtain an exact sequence

$$H^1(S,\mathcal{K}_n) \longrightarrow H^1(S,\mathcal{A}_n \otimes \mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}}) \longrightarrow H^1(S,\mathrm{Ker}\ \rho_n) \longrightarrow H^2(S,\mathcal{K}_n)\,.$$

Since $\mathcal{K}_n$ is supported on $\{s,t\}$ we have $H^1(S,\mathcal{K}_n) = H^2(S,\mathcal{K}_n) = 0$ and thus

$$H^1\big(S\,,\,\mathcal{A}_n \otimes \mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}}\big) = H^1(S,\mathrm{Ker}\ \rho_n) \quad \text{for all } n\,.$$

By hypothesis there is an integer n_0 such that the left side of the equation is zero for all $n \geq n_0$. It follows that $\Gamma(S,\mathcal{A}_n) \longrightarrow \Gamma\big(S\,,\,\mathcal{A}_n/\mathcal{I}_{\{s\}}\mathcal{I}_{\{t\}}\mathcal{A}_n\big)$ is surjective for all $n \geq n_0$. □

Remark.— In the special case that $\mathcal{A}$ is the symmetric algebra of a coherent $\mathcal{O}_S$-module $\mathcal{F}$ the previous theorem was proved by Ancona [1, corollaire 2.11] and earlier, in the case of a reduced space S, by Rabinowitz [17, Theorem 1].

5.3 Primary weakly positive sheaves

Let S be a reduced compact complex analytic space and let $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Following Rabinowitz [17] we say that $\mathcal{F}$ is *primary weakly positive* if the cone $\mathbf{V}(\mathcal{F})_\sharp$ is weakly negative. From Theorems 5.2 and 3.10, we immediately get the following result.

THEOREM 5.4.— *Let S be a reduced compact complex analytic space. A coherent $\mathcal{O}_S$-module $\mathcal{F}$ is primary weakly positive if and only if for every coherent $\mathcal{O}_S$-module $\mathcal{G}$ there exists an integer n_0 such that*

$$H^p\Big(S\,,\,\mathcal{S}_n(\mathcal{F})_\sharp \otimes \mathcal{G}\Big) = 0$$

for all $n \geq n_0$ and all $p \geq 1$.

Remark.— A slightly different cohomological characterization of primary weak positivity was obtained by Rabinowitz in [17].

From [2, Theorem 2.14 and Corollary 3.4], we immediately obtain the following result, which was proved by Rabinowitz in the case of normal irreducible spaces (see [17, Theorem 2]):

THEOREM 5.5.— *A reduced compact complex analytic space is Moišezon if and only if it carries a primary weakly positive coherent sheaf.*

From Theorem 3.10 and [2, Corollary 3.3], we obtain the following strengthened version of a theorem of Ancona [1, théorème 5.1]:

THEOREM 5.6.— *Let S be a reduced compact complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. If $\mathcal{F}$ is primary weakly positive, then there exists an integer n_0 such that $\mathcal{S}_n(\mathcal{F})_\sharp$ is weakly positive for all $n \geq n_0$.*

Example 5.7.— We construct an example of a torsion free primary weakly positive sheaf that is not weakly positive, thus answering a question of Rabinowitz [17].

Let z_0, z_1, z_2, z_3 be homogeneous coordinates of the three-dimensional projective space $\mathbb{P}_3$ of lines in $\mathbb{C}^4$ and let S be the union of the projective planes

$$H_1 := \{z \in \mathbb{P}_3 : z_2 = 0\} \quad \text{and} \quad H_2 := \{z \in \mathbb{P}_3 : z_3 = 0\}$$

Let α be the involutory automorphism of $\mathbb{C}^4$ defined by $\alpha(v_0, v_1, v_2, v_3) := (-v_0, v_1, v_2, v_3)$ and let E_1, E_2 be the line bundles on H_1, H_2 respectively defined by

$$E_1 := \{(z, v) \in H_1 \times \mathbb{C}^4 : v \in z\} \text{ and } E_2 := \{(z, v) \in H_2 \times \mathbb{C}^4 : \alpha(v) \in z\}.$$

Let L_k be the trivial extension of E_k to S for $k = 1, 2$. Then L_1, L_2 are reduced linear subspaces of the trivial bundle $S \times \mathbb{C}^4$, and their (reduced) union $X := L_1 \cup L_2$ is a subcone of $S \times \mathbb{C}^4$. Hence $X = \operatorname{Specan} \mathcal{A}$, where $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}_m$ is a connected $\mathcal{O}_S$-algebra, locally of finite presentation, and generated by $\mathcal{A}_1$. We show that the coherent $\mathcal{O}_S$-module $\mathcal{A}_1$ is torsion free and primary weakly positive, but not weakly positive:

Let $\pi : X \to S$ be the projection. Outside the projective line $T := H_1 \cap H_2$ the cone X is a line bundle, and since X is reduced, $\mathrm{cl}_X(X \setminus \pi^{-1}(T)) = X$, and thus $\mathcal{H}^0_T\mathcal{A} = 0$ by Theorem 2.2. Since T is analytically rare in S and $\mathcal{A}$ is torsion free on $S \setminus T$, Lemma 2.4 implies that $\mathcal{T}(\mathcal{A}) = \mathcal{H}^0_T(\mathcal{A}) = 0$. In particular $\mathcal{A}_1$ is torsion free.

The space $\mathbf{V}(\mathcal{A}_1)$ is a linear subspace of $S \times C^4$ containing X as a subcone and $\mathbf{V}(\mathcal{A}_1)|S \setminus T = X|S \setminus T$. Thus $\mathbf{V}(\mathcal{A}_1)_\sharp = X$. To show that $\mathbf{V}(\mathcal{A}_1)$ is primary weakly positive we must prove that the vertex of X is exceptional in X. But the restriction to X of the projection $S \times \mathbb{C}^4 \to \mathbb{C}^4$ maps the vertex S of X to zero and induces a finite holomorphic mapping $X \setminus S \to \mathbb{C}^4 \setminus \{0\}$. Hence it induces a finite holomorphic mapping $\psi : X \amalg_S P \to \mathbb{C}^4$, where P

is the reduced point and $X \amalg_S P$ is the push-out in the category of ringed spaces; in fact ψ has at most two points in each fibre. By [13, Korollar 1.3] the vertex of X is exceptional in X.

In order to show that $\mathcal{A}_1$ is not weakly positive we note that $\mathbf{V}(\mathcal{A}_1)$ is a linear subspace of $T \times \mathbb{C}^4$ containing the restriction of X to T, and for all points z in T except the points $(1:0:0:0)$ and $(0:1:0:0)$ we have

$$L_{1,z} + L_{2,z} = \{z\} \times \{v \in \mathbb{C}^4 : v_2 = v_3 = 0\} \cong \mathbb{C}^2 .$$

Thus the restriction of $\mathbf{V}(\mathcal{A}_1)$ to T contains the trivial bundle $T \times \mathbb{C}^2$ as a linear subspace; hence it cannot be weakly negative, and *a fortiori* $\mathbf{V}(\mathcal{A}_1)$ is not weakly negative.

Remark 5.8.— One can ask whether such an example may be found over a locally irreducible base space or even a manifold.

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