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Existence results for quasilinear problems via ordered sub and supersolutions(*)

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RÉSUMÉ. — On démontre l'existence de solutions maximales et minimales dans l'intervalle formé par une paire $\alpha \leq \beta$ de sous et sursolutions faibles du problème

$$\begin{cases} -\operatorname{div}\left(\left|\nabla u\right|^{p-1} \nabla u\right) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Ici $p > 1$, Ω est un domaine borné régulier de $\mathbb{R}^N$ et $f(x, s, t)$ est une fonction de Carathéodory dont la croissance en ∇u est inférieure à $p - 1 + \min\{p/N, 1\}$.

Notre démonstration utilise une généralisation pour le p-laplacien de l'inégalité de Kato. On étudie également les systèmes non coopératifs de deux équations quasi linéaires du même type que (P).

ABSTRACT. — We prove the existence of maximal and minimal solutions between an ordered pair of weak sub and supersolutions of the quasilinear problem

$$\begin{cases} -\operatorname{div}\left(\left|\nabla u\right|^{p-1} \nabla u\right) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

where $p > 1$, Ω is a smooth bounded domain of $\mathbb{R}^N$ and $f(x, s, t)$ is a Carathéodory function whose growth in ∇u is less than $p - 1 + \min\{p/N, 1\}$.

Our proof relies on a generalization for the p-laplacian of Kato's inequality. We also study non-cooperative systems of two quasilinear equations of the same type as (P).

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KEY-WORDS : Maximal and minimal solutions; p-Laplacian; Zorn's lemma; Kato's inequality; quasimonotone systems.

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1. Introduction

This paper is concerned with the existence of maximal and minimal solutions between an ordered pair of weak sub and supersolutions of the quasilinear problem:

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, is the well-known p-laplacian operator. We will assume throughout the paper that Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary and $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is a Carathéodory function, i.e., f is mesurable in $x \in \Omega$ and continuous in $(s,t) \in \mathbb{R} \times \mathbb{R}^N$.

Maximal and minimal solutions can be easily found by using the method of *monotone iterations* provided f satisfies a *monotonicity condition* like condition (H3) in Theorem 2.1. This method is the classical approach to the problem of the existence of solutions of (P) via sub and supersolutions. The case $p = 2$, i.e., $\Delta_p = \Delta$, has been extensively studied, starting from Keller, Amann [1], Sattinger, Amann–Crandall [2] (see references in [1]) and more recently by Clement–Sweers [4] and Dancer–Sweers [6].

The literature for the case $p \neq 2$ is less extensive. In the setting of weak solutions, existence results via sub and supersolutions can be found in Deuel–Hess [7] (see Theorem 2.2 in section 2), Hess [8], Boccardo–Murat–Puel [3] and, when f is monotone and independent of ∇u in Diaz [5] (see Theorem 2.1). A more systematic study of the subsolution-supersolution method for quasilinear operators of divergence form has been done by Kura [10]. Among other results, Kura proves the existence of maximal and minimal solutions when the given subsolution and supersolution are respectively locally bounded [10, Theorem 3.2] or bounded [10, Theorem 3.5] allowing, in this last case, the nonlinearity f to grow in ∇u up to the power $p - \varepsilon$ as in [8].

Later Dancer–Sweers [6] proved for $p = 2$ the existence of maximal and minimal solutions of weak type when the sub and supersolutions are not necessarily bounded and f grows in ∇u at most linearly. The approach of [6] is shorter than in Kura's paper.

We present in this paper a generalization to the p-laplacian of the result of Dancer–Sweers [6] for weak solutions (cf. Theorem 3.1 in Section 3). Our result improves slightly a similar result of Kura [10, Theorem 3.2] since, as in [6], we do not assume the pair of subsolution and supersolution to be locally bounded. Moreover we prove as a consequence of Proposition 3.2 and corollary 3.3 that, according to the notation of [10], W-subsolutions (respectively L-subsolutions), are actually weak subsolutions (respectively locally bounded subsolutions) and similarly for supersolutions. The proof of Theorem 3.1 uses Zorn's Lemma and an inequality of Kato type which is stated in Proposition 3.2. This generalization of Kato's inequality will be the key point of the proof. We also give the analogue of Theorem 3.1 for quasimonotone systems of two quasilinear equations. We have mainly followed here the results of Mitidieri–Sweers [11] for the case $p = 2$.

This note is organised as follows. In Section 2, we briefly recall two classical existence results using sub and supersolutions. The first theorem, Theorem 2.1, uses monotone iterations in a p-laplacian setting. The second theorem, Theorem 2.2, is close to the result of [7] already mentioned. In Section 3, we prove our main result. In Section 4 we generalize our result for a class of quasimonotone systems.

2. Two classical existence results using sub and supersolutions

Let $p > 1$ and $W_0^{1,p}(\Omega)$ the usual Sobolev space. We denote by $\|\cdot\|_p$ the norm of $L^p(\Omega)$ and by $\|u\| = \|\nabla u\|_p$ the norm of $W_0^{1,p}(\Omega)$. We denote

$$p' = \frac{p}{p-1}\,, \quad p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ \infty & \text{if } p \geq N. \end{cases}$$

We define $\psi(s) = |s|^{p-2}$, $s \in \mathbb{R}$.

We recall that the operator $\Delta_p u$ satisfies the well known *condition* (S_+):

$$\forall\ u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ with } \limsup_{n\to\infty}(-\Delta_p u_n\,,\, u_n - u) \leq 0$$
$$\text{one has } \|u_n - u\| \longrightarrow 0\,.$$

We will also use the following useful inequality:

$$\forall\ x \neq y \in \mathbb{R}^n, \quad \Big(|x|^{p-2}x - |y|^{p-2}y\Big)\cdot(x-y) > 0\,. \tag{2.1}$$

DEFINITION. — *A function $u \in W^{1,p}(\Omega)$ is called a solution (subsolution, supersolution) of (P) if $f(\cdot, u, \nabla u) \in L^{p'}(\Omega)$,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx - \int_\Omega f(x, u, \nabla u) \varphi \, dx = 0 \ (\leq 0 \,,\, \geq 0) \tag{2.2}$$

$\forall\ \varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and

$$u = 0 \ (\leq 0 \,,\, \geq 0) \quad \text{on } \partial\Omega \,. \tag{2.3}$$

The condition on $\partial\Omega$ is understood in $W^{1-1/p,p}(\partial\Omega)$ i.e, in the sense of traces (see Remark 5.1).

By an *ordered pair* of sub and supersolutions, we mean a subsolution α and a supersolution β such that $\alpha \leq \beta$ a.e.

The next theorem is only stated for functions $f = f(x, s)$. A similar result can be found in [5, Theorem 4.14].

THEOREM 2.1. — *Let us assume the following conditions:*

(H1) there exists an ordered pair $\alpha \leq \beta$ of sub and supersolutions of (P);

(H2') $|f(x, s)| \leq K(x)$

$$\forall\ s : \alpha(x) \leq s \leq \beta(x) \text{ for some } K \in L^p(\Omega) \,, \quad q > (p^*)' \,;$$

(H3) $\exists\ M > 0$ such that $\forall\ s_1, s_2 : \alpha(x) \leq s_1 \leq s_2 \leq \beta(x)$, we have

$$f(x, s_1) - f(x, s_2) \leq -M\big(\psi(s_1) - \psi(s_2)\big) \,.$$

Then there exist U, V solutions of (P) such that:

(i) $\alpha \leq U \leq V \leq \beta$ a.e. in Ω;

(ii) U is minimal and V is maximal in the following sense:

$$\forall\ u \in W_0^{1,p}(\Omega) \text{ solution of (P) with } \alpha \leq u \leq \beta \text{ a.e.}$$
$$\text{then } \alpha \leq U \leq u \leq V \leq \beta \text{ a.e.}$$

Proof. — Let $I = \{u \in W^{1,p}(\Omega) \mid \alpha \leq u \leq \beta \text{ a.e.}\}$. Define the map $T : I \mapsto W_0^{1,p}(\Omega)$ by $T(u) = v$ where v is the unique solution of

$$\begin{cases} -\Delta_p v + M\psi(v) = g(x, u) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

with $g(x,u) = f(x,u) + M\psi(u)$. It is easy to check that T is well defined since $\forall\ u \in I$ we have that $g(x,u) \in W^{-1,p'}(\Omega)$.

Let us prove that T is increasing. Take $u,\ v \in I$ with $u \leq v$. By (H3), $f(x,u) + M\psi(u) \leq f(x,v) + M\psi(v)$ and since the operator $-\Delta_p + M\psi$ satisfies the *weak comparison principle* (see [11, Lemma 3.1]), we conclude that $T(u) \leq T(v)$. The same principle proves that $\alpha \leq T(\alpha) \leq T(\beta) \leq \beta$, thus $T : I \mapsto I$.

We construct the following *monotone* sequences:

$$\begin{cases} u_1 = \alpha & u_n = T(u_{n-1}) \\ v_1 = \beta & v_n = T(v_{n-1}). \end{cases}$$

We claim that the sequences (u_n), (v_n) are convergent. In order to prove that, let us check that (u_n) is bounded in $W_0^{1,p}(\Omega)$ (the proof for (v_n) will be similar). Multiplying the equation (2.4) by $u_n = T(u_{n-1})$ and using (H2') we find

$$\begin{aligned} &\left\|u_n\right\|^p + M\left\|u_n\right\|_p^p = \\ &\quad = \int_\Omega g(x,u_{n-1})u_n \,\mathrm{d}x \leq \left\|u_n\right\|_{q'}\left\|K\right\|_q + M\left\|u_{n-1}\right\|_p^{p-1}\left\|u_n\right\|_p, \end{aligned}$$

and, since $\alpha \leq u \leq \beta$ a.e., then

$$\left\|u_n\right\| \leq C \tag{2.5}$$

where C depends only on M, K, α, β.

By Rellich's compact imbedding theorem we deduce the existence of a subsequence (still denoted u_n) and $U \in W_0^{1,p}(\Omega)$ such that (u_n) converges to U weakly in $W_0^{1,p}(\Omega)$, a.e. in Ω, and strongly in both $L^{q'}(\Omega)$ and $L^p(\Omega)$.

Using again that $u_n = T(u_{n-1})$ is a solution of (2.4) we obtain, after multiplying (2.4) by $v = u_n - U$,

$$\begin{aligned} &(-\Delta_p u_n\,,\, u_n - U) \leq \\ &\quad \leq \left\|K\right\|_q - \left\|u_n - U\right\|_{q'} + M\left(\left\|u_{n-1}\right\|_p^{p-1} + \left\|u_n\right\|_p^{p-1}\right)\left\|u_n - U\right\|_p \end{aligned}$$

and by (2.4) and the strong convergence in $L^{q'(\Omega)}$ and $L^p(\Omega)$, we have

$$(-\Delta_p u_n\,,\, u_n - U) = \mathrm{o}(1)\,.$$

Using condition (S_+) we finally get that $\lim_{n\to\infty} u_n = U$ in $W_0^{1,p}(\Omega)$. Therefore U is a solution of (P) which clearly satisfies $\alpha \le U \le \beta$.

Let us check that U is a minimal solution in the interval $[\alpha, \beta]$. Let u be a solution of (P). Then $T(u) = u$. If moreover $\alpha \le u \le \beta$ then, by the monotonicity of T, $\alpha \le u_n \le u$, $\forall\, n$. Hence $U = \lim_{u\to\infty} u_n \le u$ and U is minimal.

Similarly we construct the maximal function V from the sequence v_n. □

The next theorem and corollary are originally due to [7]. We have slightly changed the growth condition on f with respect to $|\nabla u|$: in [7] it is at most $p-1$ which is strictly less than our growth $p/(p^*)'$ in (H2). The proof of Theorem 2.2 in this case follows from [7] using Sobolev's imbedding theorems.

We point out that there are other existence results between sub and supersolutions that allow f to grow in ∇u up to the power $p-\varepsilon$ or even up to p. The hypothesis on the sub and supersolutions are however more restrictive than ours. See Remark 5.3 at the end of this paper.

Note that in the next theorem the monotonicity condition (H3) has been removed and that we consider functions f which may depend on ∇u.

THEOREM 2.2.— *Assume (H1) and*

(H2) $\left|f(x,s,t)\right| \le K(x) + a\left|t\right|^r$ *a.e.,* $x \in \Omega$, $\forall\, s : \alpha(x) \le s \le \beta(x)$ *and* $\forall\, t \in \mathbb{R}^N$, *where*

$$K \in L^q(\Omega)\,, \quad q > (p^*)' \quad \text{and} \quad 0 \le r < \frac{p}{(p^*)'}\,.$$

Then (P) has at least one solution $W_0^{1,p}(\Omega)$ *between* α *and* β.

Proof.— We introduce the function g defined by:

$$g(x,s,t) = \begin{cases} f\big(x,\alpha(x),\nabla\alpha(x)\big)\,, & s < \alpha(x) \\ f(x,s,t)\,, & \alpha(x) \le s \le \beta(x) \\ f\big(x,\beta(x),\nabla\beta(x)\big)\,, & s > \beta(x). \end{cases} \tag{2.6}$$

A lemma in [7] proves that the map $u \to g(x,u,\nabla u)$ from $W^{1,p}(\Omega)$ to itself is bounded and continuous. If we set $\ell = \max\{q', p/(p-r)\} - 1$ then $1 \le \ell + 1 < p^*$ and Holder's inequality gives

$$\left|\int_\Omega g(x,u,\nabla u)v\,\mathrm{d}x\right| \le c_1\|v\|_{\ell+1} + c_2\|\nabla u\|_p^r\,\|v\|_{\ell+1}\,, \quad \forall\; u,\, v \in W_0^{1,p}(\Omega)\,, \tag{2.7}$$

for some $c_1,\, c_2 > 0$.

Let us define the following *penalty* term (different from the one used in [7]):

$$\gamma(x,s) = -\left(\left(\alpha(x) - s\right)_+\right)^\ell + \left(\left(s - \beta(x)\right)_+\right)^\ell\,. \tag{2.8}$$

for $x \in \Omega$, $\forall\; s \in \mathbb{R}$. Then for any $u,\, v \in W_0^{1,p}(\Omega)$, we have

$$\left|\int_\Omega \gamma(x,u)v\,\mathrm{d}x\right| \le c_3\|v\|_{\ell+1} + c_4\|u\|_{\ell+1}^\ell\,\|v\|_{\ell+1} \tag{2.9}$$

and

$$\int_\Omega \gamma(x,u)u\,\mathrm{d}x \ge c_5\|u\|_{\ell+1}^{\ell+1} - c_6\,, \tag{2.10}$$

$\forall\; u \in W_0^{1,p}(\Omega)$, for some $c_3,\, c_4,\, c_5,\, c_6 > 0$.

Consider now the map $B : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ defined by

$$(B(u),v) = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x - \int_\Omega g(x,u,\nabla u)v\,\mathrm{d}x + M\int_\Omega \gamma(x,u)v\,\mathrm{d}x$$

for some $M > 0$ that will be fixed later on.

The map B is well defined by (2.7) and (2.9). It is also bounded, i.e., the image by B of a bounded set of $W_0^{1,p}(\Omega)$ is a bounded set in $W^{-1,p'}(\Omega)$. Moreover the inequality (2.1) implies that B is *pseudomonotone* [12, Theoreme 3.3.42]. Let us prove that, for M large enough B is *coercive*, that is $B(u,u)/\|u\| \to 0$ as $\|u\|$ goes to $+\infty$. Using (2.7) and (2.10) we have

$$(B(u),u) \ge \|\nabla u\|_p^p - \left(c_1\|u\|_{\ell+1} + c_2\|\nabla u\|_p^r\,\|u\|_{\ell+1}\right) + M\left(c_5\|u\|_{\ell+1}^{\ell+1} - c_6\right). \tag{2.11}$$

We pick any $0 < \varepsilon < 1/c_2$ and we use now Young's inequality to evaluate the term $\|\nabla u\|_p^r\,\|u\|_{\ell+1}$ in the inequality above to obtain

$$\|\nabla u\|_p^r\,\|u\|_{\ell+1} \le \varepsilon\|\nabla u\|_p^p + d(\varepsilon)\|u\|_{\ell+1}^{p/(p-r)}$$

and since $p/(p-r) \leq \ell + 1$ we get

$$\|\nabla u\|_p^r \|u\|_{\ell+1} \leq \varepsilon \|\nabla u\|_p^p + c(\varepsilon)\|u\|_{\ell+1}^{\ell+1},$$

where $d(\varepsilon)$ and $c(\varepsilon)$ are some constants depending on ε. Similarly $\|u\|_{\ell+1} \leq \|u\|_{\ell+1}^{\ell+1} + c_\gamma$. Replacing in (2.11) we get

$$\begin{aligned}(B(u), u) \geq (1 - c_2\varepsilon)\|\nabla u\|_p^p - \left(c_1\|u\|_{\ell+1}^{\ell+1} + c_2 c(\varepsilon)\|u\|_{\ell+1}^{\ell+1}\right) \\ + M c_5 \|u\|_{\ell+1}^{\ell+1} - M c_6 - c_1 c_7 .\end{aligned}$$

Choose now $M > 0$ large enough to have $M c_5 > c_2 c(\varepsilon) + c_1$. We conclude that

$$(B(u), u) \geq (1 - c_2\varepsilon)\|\nabla u\|_p^p - M c_6 - c_7 .$$

Whence B is coercive.

All these properties imply [12, Theorem 3.3.42] the surjectivity of B. In particular, there exists $W_0^{1,p}(\Omega)$ such that $B(u,v) = 0$, $\forall\ v \in W_0^{1,p}(\Omega)$. Thus u is a weak solution of the following problem:

$$\begin{cases} -\Delta_p u = g(x, u, \nabla u) - M\gamma(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Let us prove that u is a solution of problem (P) i.e., that $\alpha \leq u \leq \beta$. Consider the test function $v = (u - \beta)_+ \in W_0^{1,p}(\Omega)$. After multiplying in (2.12) by v we have

$$\begin{aligned}\left(-\Delta_p u\,,\ (u-\beta)_+\right) &= \int_\Omega \big(g(x,u,\nabla u) - M\gamma(x,u)\big)(u-\beta)_+\,\mathrm{d}x \\ &= \int_\Omega f(x,\beta,\nabla\beta)(u-\beta)_+\,\mathrm{d}x - M\int_\Omega (u-\beta)_+^{\ell+1}\,\mathrm{d}x .\end{aligned}$$

Since β is a supersolution of (P) and $(u-\beta)_+$ is positive it follows that

$$\int_\Omega f(x,\beta,\nabla\beta)(u-\beta)_+\,\mathrm{d}x \leq \left(-\Delta_p\beta\,,\ (u-\beta)_+\right) .$$

By combining these two last results we have

$$0 \leq \left(-\Delta_p u + \Delta_p\beta\,,\ (u-\beta)_+\right) \leq -M\int_\Omega (u-\beta)_+^{\ell+1}\,\mathrm{d}x \leq 0$$

which implies that $(u-\beta)_+ = 0$, i.e., $u \leq \beta$. The proof of $u \geq \alpha$ runs similarly. □

COROLLARY 2.3.— *There exists some constant $C = C(\alpha, \beta, \Omega, K, a)$ such that $\|u\| \leq C$ for any solution of (P) with $\alpha \leq u \leq \beta$ a.e.*

Proof.— The conclusion of the corollary follows easily after multiplying the equation of (P) by u and using (H2). □

3. Maximal and minimal solutions of weak type

The main result of this paper is the following theorem.

THEOREM 3.1.— *Assume (H1) and (H2). Then there exist U, V solutions of (P) such that:*

(i) $\alpha \leq U \leq V \leq \beta$ a.e.;

(ii) U is minimal and V is maximal in $[\alpha, \beta]$.

The proof of [6] of this theorem in the case $p = 2$ uses the following Kato's inequality:

$$\Delta|u| \geq \operatorname{sign}(u)\,\Delta u$$

where the inequality has to be interpreted in one of the following senses:

(i) for $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ it means

$$\int_\Omega |u|\,\Delta\phi\,\mathrm{d}x \geq \int_\Omega \phi\operatorname{sign}(u)\,\Delta u\,\mathrm{d}x\,, \quad \forall\,\phi \in C_0^\infty(\Omega)\,,\ \phi \geq 0\,;$$

(ii) (weak version) for $u \in H^1(\Omega)$ such that $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ it means

$$\int_\Omega \nabla|u| \cdot \nabla\phi\,\mathrm{d}x \leq -\int_\Omega \phi\operatorname{sign}(u)\,\Delta u\,\mathrm{d}x\,, \quad \forall\,\phi \in C_0^\infty(\Omega)\,,\ \phi \geq 0\,.$$

For $p \neq 2$, we vill prove in the next proposition a generalization of the *weak* version of Kato's inequality. We don't know if any such result exists in the literature.

PROPOSITION 3.2.— *Let u_1, $u_2 \in W^{1,p}(\Omega)$ such that there exist f_1, $f_2 \in L^1_{\mathrm{loc}}(\Omega)$ satisfying*

$$-\Delta_p u_i \leq f_i \quad \textit{for } i = 1, 2\,. \tag{3.1}$$

Let us define

$$g(x)=\begin{cases} f_1(x) & \text{if } u_1(x)>u_2(x) \\ f_2(x) & \text{if } u_1(x)\le u_2(x)\end{cases} \qquad a.e.\ x\in\Omega\,.$$

Set $u=\max\{u_1,u_2\}$. *Then*

$$-\Delta_p u\le g\,. \tag{3.2}$$

Proof.— Fix any $\phi\in C_0^\infty(\Omega)$, $\phi\ge 0$. We write

$$\begin{aligned} I&=(-\Delta_p u,\phi)\\ &=\int_{\Omega_1}|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla\phi\,\mathrm{d}x+\int_{\Omega_2}|\nabla u_2|^{p-2}\nabla u_2\cdot\nabla\phi\,\mathrm{d}x=I_1+I_2\end{aligned}$$

where

$$\Omega_1=\left\{x\in\Omega\mid u_1(x)>u_2(x)\right\},\quad \Omega_2=\Omega\setminus\Omega_1\,.$$

Let us take a sequence of functions $\xi_n:\mathbb{R}\to\mathbb{R}$ such that

$$\xi_n\in C_1(\mathbb{R})\,,\qquad \xi_n(t)=\begin{cases}1 & \text{if } t\ge 1/n\\ 0 & \text{if } t\le 0\,,\end{cases}\qquad \xi_n'>0\quad\text{on }(0,1/n)\,.$$

We now introduce the following sequence of functions:

$$q_n(x)=\xi_n\big((u_1-u_2)(x)\big)$$

for $x\in\Omega$. The function $q_n\in W^{1,p}(\Omega)$ since $\xi_n\in C^1(\mathbb{R}),\xi_n'\in C_0(\mathbb{R})$ and u_1, $u_2\in W^{1,p}(\Omega)$. It is clear that q_n converges pointwise to the characteristic function of Ω_1:

$$1_{\Omega_1}(x)=\begin{cases}1 & x\in\Omega_1\\ 0 & x\notin\Omega_1.\end{cases}$$

Moreover since $\|q_n\|_\infty\le 1$ by Lebesgue Theorem of dominated convergence we have

$$I_1=\lim_{n\to\infty}\int_\Omega q_n|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla\phi\,\mathrm{d}x\,. \tag{3.3}$$

Besides the function $0\le q_n\phi\in W_0^{1,p}(\Omega)$ because $q_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $\phi\in C_0^\infty(\Omega)$. Integrating by parts the above integral we obtain

$$\begin{aligned}&\int_\Omega q_n|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla\phi\,\mathrm{d}x=\\ &\qquad=\int_\Omega|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla(q_n\phi)\,\mathrm{d}x-\int_{\Omega_n}\phi|\nabla u_1|^{p-2}\nabla u_1\cdot\nabla q_n\,\mathrm{d}x\,.\end{aligned}$$

Hence

$$\int_\Omega q_n |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla\phi \, \mathrm{d}x \le \int_\Omega q_n \phi f_1 \, \mathrm{d}x - \int_{\Omega_n} \phi |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla q_n \, \mathrm{d}x \tag{3.4}$$

since $\nabla q_n = 0$ on $\Omega \setminus \Omega_n$ where

$$\Omega_n = \left\{ x \in \Omega \;\middle|\; u_2(x) < u_1(x) < u_2(x) + \frac{1}{n} \right\} .$$

Similarly, for I_2 we have

$$I_2 = \lim_{n\to\infty} \int_\Omega (1 - q_n) |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla\phi \, \mathrm{d}x \tag{3.5}$$

and after integrating by parts it becomes

$$\begin{aligned} &\int_\Omega (1 - q_n) |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla\phi \, \mathrm{d}x \le \\ &\qquad \le \int_\Omega (1 - q_n) \phi f_2 \, \mathrm{d}x + \int_{\Omega_n} \phi |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla q_n \, \mathrm{d}x . \end{aligned} \tag{3.6}$$

We have that:

(a) the first two terms on the left hand side of (3.4) (3.6) give at infinity,

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega q_n \phi f_1 \, \mathrm{d}x &= \int_{\Omega_1} \phi f_1 \, \mathrm{d}x \\ \lim_{n\to\infty} \int_\Omega (1 - q_n) \phi f_2 \, \mathrm{d}x &= \int_{\Omega_2} \phi f_2 \, \mathrm{d}x \end{aligned} \tag{3.7}$$

by Lebesgue Theorem of dominated convergence;

(b) the sum of the second terms on the left hand side of (3.4)-(3.6) can be estimated as follows. We replace $\nabla q_n = \xi_n'(u_1 - u_2) \nabla(u_1 - u_2)$ and we use that $\xi_n' \ge 0$, $\phi \ge 0$; then it follows from condition (2.1) that

$$\int_{\Omega_n} \phi \xi_n'(u_1 - u_2) \left(|\nabla u_2|^{p-2} \nabla u_2 - |\nabla u_1|^{p-2} \nabla u_1 \right) \cdot \nabla(u_1 - u_2) \, \mathrm{d}x \le 0 . \tag{3.8}$$

Finally, adding (3.3), (3.5) and using (3.4)-(3.8) we get, after going to infinity,

$$(-\Delta_p u, \phi) \le \int_{\Omega_1} \phi f_1 \, \mathrm{d}x + \int_{\Omega_2} \phi f_2 \, \mathrm{d}x = \int_\Omega \phi g \, \mathrm{d}x$$

and the proof is completed. □

COROLLARY 3.3.— *Let u_1, $u_2 \in W^{1,p}(\Omega)$ be two subsolutions of (P) and define $u = \max\{u_1, u_2\}$. Then u is a subsolution of (P).*

A similar statement holds for the minimum of two supersolutions.

Proof of Corollary 3.3

Condition 3.1 is automatically satisfied. Moreover we have that $g(x) = f\big(x, u(x), \nabla u(x)\big)$ a.e. $x \in \Omega$ and the corollary follows from (3.2). □

Proof of Theorem 3.1

Let us consider the following set

$$N = \left\{u \in W^{1,p}(\Omega) \;\middle|\; \alpha(x) \le u(x) \le \beta(x) \text{ a.e. and } u \text{ is a solution of (P)}\right\}.$$

We first prove the existence of the maximal solution V. In order to do that, we show that N satisfies the hypothesis of Zorn's Lemma. Let $\{u_i\}_{i \in I}$ be a completely ordered family of N. Let $\{u_n\}$ be a sequence that is cofinal with respect to the ordering $\le$. By corollary 2.3, $\|u_n\| \le C$. Then there exists a subsequence (still denoted u_n) converging to some u weakly in $W_0^{1,p}(\Omega)$ a.e. and strongly in $L^s(\Omega)$ where s is any fixed real number such that

$$\max\left\{q', \frac{p}{p-r}\right\} \le s < p^* .$$

Observe that, since u_n is increasing, we have

$$\alpha \le u_n \le u \le \beta \quad \text{a.e. in } \Omega .$$

We claim that u is a solution of (P), that is, $u \in N$. In order to prove the claim it will be enough to show that the sequence u_n converges strongly to u in $W_0^{1,p}(\Omega)$. We first observe that the sequence $\|f(x, u_n, \nabla u_n)\|_{s'}$ is bounded because of the growth condition (H2) and that $\|u_n\|$ is bounded as well. Using the L^s convergence we deduce that

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\,\mathrm{d}x \longrightarrow 0 . \tag{3.9}$$

Then testing (P) (for the solution u_n) by $v = u_n - u$ and using (3.9) we get

$$(-\Delta_p u_n\,,\, u_n - u) \longrightarrow 0 .$$

By (S_+) we conclude

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega)\,.$$

By Zorn's Lemma there exists a maximal element $V \in N$. We want to prove now that V is maximal in the sense of Theorem 2.1. For this purpose let u_1 be any function of N and put $u_2 = V$ in Proposition 3.2. Since

$$-\Delta_p u_i = f(x, u_i, \nabla u_i) \quad \text{for } i = 1,\, 2$$

then condition (3.1) is immediately satisfied. By corollary 3.3, $u = \max\{u_1, u_2\}$ is a subsolution of (P). We now apply Theorem 2.2 between $[\, u\,,\, \beta\,]$. Hence there exists a solution z of (P) such that

$$\alpha \leq u \leq z \leq \beta\,.$$

Hence $z \in N$. From the inequalities

$$V = u_2 \leq \max\{u_1, u_2\} = u \leq z$$

and the fact that V is maximal in N, we conclude that $V = z$. Therefore $u_1 \leq V$ as claimed.

The existence of U can be proved in a similar way. □

4. Quasimonotone systems

It is not very hard to extend the corresponding result of Theorem 3.1 for *quasimonotone systems*. For sake of simplicity we consider systems of two equations of the form:

$$\begin{cases} -\Delta_p u = g(x, u, v, \nabla u) & \text{in } \Omega \\ -\Delta_q v = h(x, u, v, \nabla v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

where $h,\, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions, that is:

- measurable in $x \in \Omega$,
- continuous in (s_1, s_2, t_i).

We recall that system (S) is quasimonotone if
$g(x, s_1, s_2, t_1)$ is increasing in $s_2 \in \mathbb{R}$ for all fixed $x \in \Omega$, $s_1 \in \mathbb{R}$, $t_1 \in \mathbb{R}^N$,
$h(x, s_1, s_2, t_2)$ is increasing in $s_1 \in \mathbb{R}$ for all fixed $x \in \Omega$, $s_2 \in \mathbb{R}$, $t_2 \in \mathbb{R}^N$.

When g (resp. h) is a function satisfying a (H3) type condition in u (resp. v), is increasing in v (resp. u) and it does not depend on ∇u (resp. ∇v), it is possible to prove the existence of maximal and minimal solutions between a pair of sub and supersolutions using monotone iterations as in Theorem 2.1

For $p = q = 2$ maximal and minimal solutions of weak type for quasimonotone systems has been obtained in [11]. See [11] for references on this subjet. We give here a generalization of this result to the system (S).

Let us recall some definitions. For simplicity we write $X = W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, $X_0 = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $\|(u,v)\| = \|u\| + \|v\|$, $\forall\ (u,v) \in X_0$.

DEFINITION.— *The pair* $(u_0, v_0) \in X$ *is called a subsolution of (S) if* $g(x, u_0, v_0, \nabla u_0) \in L^{p'}(\Omega)$, $h(x, u_0, v_0, \nabla v_0) \in L^{q'}(\Omega)$ *and*

$$\begin{cases} -\Delta_p u_0 \leq g(x, u_0, v_0, \nabla u_0) & \text{in } \Omega \\ -\Delta_q v_0 \leq h(x, u_0, v_0, \nabla v_0) & \text{in } \Omega \\ u_0\,,\ v_0 \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly we will define supersolutions of (S) reversing all the inequalities above. By $(u_0, v_0) \leq (u^0, v^0)$ *we will mean* $u_0 \leq u^0$ *and* $v_0 \leq v^0$ *a.e. in* Ω.

THEOREM 4.1.— *Assume that:*

(H1) there exists a subsolution $(u_0, v_0) \in X$ *and a supersolution* $(u^0, v^0) \in X$ *of (S) with* $(u_0, v_0) \leq (u^0, v^0)$;

(H2) $|g(x, s_1, s_2, t_1)| \leq K_1(x) + a_1 |t_1|^{r_1}$, $|h(x, s_1, s_2, t_1)| \leq K_2(x) + a_2 |t_2|^{r_2}$, $\forall\ s_1$, $s_2 \in \mathbb{R}$ *such that*

$$u_0(x) \leq s_1 \leq u^0(x)\,, \quad v_0(x) \leq s_2 \leq v^0(x) \quad \text{for all } t_1,\ t_2 \in \mathbb{R}^N$$

and for some

$$K_1 \in L^{n_1}(\Omega)\,, \quad K_2 \in L^{n_2}(\Omega)\,, \qquad n_1 > (p^*)'\,,\ n_2 > (q^*)'\,,$$
$$0 \leq r_1 < \frac{p}{(p^*)'}\,, \quad 0 \leq r_2 < \frac{q}{(q^*)'}\,.$$

Then there exists a maximal solution $(U, V) \in X_0$ *and a minimal solution* $(Z, W) \in X_0$ *between* (u_0, v_0) *and* (u^0, v^0).

Proof.— As in [11] we start proving the following result.

1) *There exists $M > 0$ depending on u_0, v_0, u^0, v^0, Ω, K_i, a_i such that for every subsolution $(\widetilde{u}, \widetilde{v})$ with $(u_0, v_0) \leq (\widetilde{u}, \widetilde{v}) \leq (u^0, v^0)$, then there exists a subsolution (u^*, v^*) satisfying $(\widetilde{u}, \widetilde{v}) \leq (u^*, v^*) \leq (u^0, v^0)$ and $\|(u^*, v^*)\| \leq M$.*

In order to prove **1)**. consider the following problems:

$$\begin{cases} -\Delta_p u = g(x, u, \widetilde{v}, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} -\Delta_q v = h(x, \widetilde{u}, v, \nabla v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We then apply Theorem 2.2 to (4.1) taking $\widetilde{u} \leq u^0$ as the ordered pair of sub and supersolutions and to (4.2) with the pair $\widetilde{v} \leq v^0$. It turns out that there exists $M = M(u_0, v_0, u^0, v^0, \Omega, K_i, a_i)$, $u^* \in W_0^{1,p}(\Omega)$ and $v^* \in W_0^{1,q}(\Omega)$ solutions of (4.1), (4.2), respectively, with

$$\widetilde{u} \leq u^* \leq u^0 \,, \quad \widetilde{v} \leq v^* \leq v^0$$

and

$$\|(u^*, v^*)\| \leq M \,.$$

Since $\widetilde{u} \leq u^*$ and $\widetilde{v} \leq v^*$ one finds $g(x, u^*, \widetilde{v}, \nabla u^*) \leq g(x, u^*, v^*, \nabla u^*)$ and $h(x, \widetilde{u}, v^*, \nabla v^*) \leq h(x, u^*, v^*, \nabla v^*)$ and hence (u^*, v^*) is a subsolution of (S).

2) *Zorn's lemma.*

Consider the set N of $(u^*, v^*) \in X_0$ such that there exists a subsolution $(\widetilde{u}, \widetilde{v})$ of (S) satisfying

$$\begin{cases} u_0 \leq \widetilde{u} \leq u^* \leq u^0 \,, \quad v_0 \leq \widetilde{v} \leq v^* \leq v^0 \,, \\ u^* \,,\ v^* \text{ are solutions respectively of (4.1), (4.2) for the pair } \widetilde{u} \,,\ \widehat{v} \,. \end{cases}$$

We now apply Zorn's Lemma to N. Take an ordered sequence $\{(u_n, v_n)\}_{n \in N}$ of subsolutions in N. Since $\|(u_n, v_n)\| \leq M$ there exists a subsequence (still denoted (u_n, v_n)) such that the subsequence converges to some $(u, v) \in X_0$ weakly in X_0, strongly in $L^p(\Omega) \times L^q(\Omega)$ and pointwise a.e. In fact we can prove that the sequence (u_n, v_n) converges strongly to (u, v) in X_0 arguing as in the proof of Theorem 3.1. One has to write u_n (resp. v_n) as a solution of the auxiliary problem (4.1) (resp. (4.2)) and repeat the proof. Finally we choose a (sub)sequence such that $(\nabla u_n, \nabla v_n)$ converges pointwise a.e. in Ω.

It is easy to see that (u, v) is a subsolution of (S). This is a straightforward consequence of the inequality

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi \,\mathrm{d}x \le \int_\Omega g(x, u_n, v_n, \nabla v_n)\phi \,\mathrm{d}x$$

for any $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$ and for any $n \in N$. We can pass to the limit inside both integrals using the strong convergence in the right hand side and the continuity of g (jointly with (H2)) in the left hand side. By the result of **1)** there exist $(u^*, v^*) \in N$ with $u_0 \le u \le u^* \le u^0$, $v_0 \le v \le v^* \le v^0$. Hence the sequence has an upper bound in N. By Zorn's Lemma, N has a maximal element (U, V) in the sense of the ordering.

3) *The maximum of two subsolutions in N is a subsolution.*

Let (u_1, v_1), $(u_2, v_2) \in N$. Call $u = \max\{u_1, u_2\}$, $v = \max\{v_1, v_2\}$. Apply Proposition 3.2 to the functions $f_1 = g(\,\cdot\,, u_1, v, \nabla u_1)$ and $f_2 = g(\,\cdot\,, u_2, v, \nabla u_2)$ (resp. $f_1 = h(\,\cdot\,, u, v_1, \nabla v_1)$ and $f_2 = h(\,\cdot\,, u, v_2, \nabla v_2)$). It follows that $-\Delta_p u \le g(x, u, v, \nabla u)$ and $-\Delta_q v \le h(x, u, v, \nabla v)$ that is, (u, v) is a subsolution.

4) *A maximal element in N is a solution of (S) and it is a maximum.*

Let (U, V) be a maximal element in N. First we prove that it is a solution of (S). If this was not the case then, by the result of **1)** there would exist (u^*, v^*) subsolution of (S) such that:

(i) $(U, V) \le (u^*, v^*)$

(ii) u^*, v^* are solutions respectively of (4.1), (4.2), for the pair (U, V).

Since (U, V) is maximal then necessarily $(U, V) = (u^*, v^*)$ and therefore (U, V) is a solution of (S).

Now we show that (U, V) is a maximum. Let (u, v) be any solution of (S) between (u_0, v_0) and (u^0, v^0). Then trivially (u, v) is a subsolution of (S) and u, v are solutions respectively of (4.1) and (4.2) for $\widetilde{u} = u$ and $\widetilde{v} = v$. Hence $(u, v) \in N$. Using the result of **3)** $\big(\max\{u, U\}, \max\{v, V\}\big)$ is a subsolution of (S), and by **1)** there exists an element $(u^*, v^*) \in N$ such that $\big(\max\{u, U\}, \max\{v, V\}\big) \le (u^*, v^*)$. Then $(U, V) \le (u^*, v^*)$ and hence $U = u^*$, $V = v^*$ which implies $(u, v) \le (U, V)$. The proof is now completed. □

5. Final comments and remarks

Remark 5.1.— We impose the regularity of the domain Ω only to give a sense to the inequalities defined on the boundary of Ω. Inequalities on $\partial\Omega$ can be defined in a different way as in Kinderlehrer and Stampacchia (see [6]). This entails only minor changes in the proofs.

Remark 5.2.— The results of Proposition 3.2 and Theorem 3.1 are also true if we replace Δ_p by any other operator

$$A_p(x,u,\nabla u) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,u,\nabla u)$$

satisfying conditions (A1)-(A3) as in [7] and [8]:

(A1) each A_i is a Carathéodory function and there exist $c_0 \geq 0$, $K_0 \in L^{p'}(\Omega)$ such that

$$\left\|A_i(x,s,t)\right\| \leq K_0(x) + c_0\Big(\left\|s\right\|^{p-1} + \left\|t\right\|^{p-1}\Big) \ ;$$

(A2) $\sum_{i=1}^{N} \big(A_i(x,s,t) - A_i(x,s,t')\big)(t_i - t_i') > 0$, $\forall\ s \in \mathbb{R}, \forall\ t \neq t' \in \mathbb{R}^N$, a.e. x;

(A3) $\sum_{i=1}^{N} A_i(x,s,t)t_i > \alpha\left\|t\right\|^p$, $\forall\ s \in \mathbb{R}, t \in \mathbb{R}^N$, a.e. x.

Notice that the key point in the proof in Proposition 3.2 is inequality (3.8) which will follow from hypothesis (A2).

Remark 5.3.— In Remark 1 of [6] the authors prove that their result on maximal weak solution for problem (P) with $p = 2$ can be extended to nonlinearities of the form $f(x,u,\nabla u)$ with:

$$\left|f(x,s,t)\right| \leq C\Big(1 + \left|t\right|^2\Big) \quad \text{for } x \in \Omega\,,\ \alpha(x) \leq s \leq \beta\,,$$

provided α, β belong to $W^{1,\infty}(\Omega)$ and Ω has a C^2-boundary.

For quasilinear problems, the existence of solutions between an ordered pair of sub and supersolutions α, $\beta \in W^{1,\infty}(\Omega)$ for nonlinearities of the form

$$\left|f(x,s,t)\right| \leq K(x) + C\left|t\right|^p$$

for a.e. $x \in \Omega$, $\alpha(x) \leq s \leq \beta(x)$ and $K \in L^{p'}(\Omega)$ has been established by [3]. However, we can not still assert the existence of maximal and minimal solutions (see [10, Remark 3.1]).

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