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On the geodesic connectedness of simply connected Lorentz surfaces[*]

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RÉSUMÉ. — Nous montrons que les surfaces lorentziennes simplement connexes globalement hyperboliques sont géodésiquement connexes. On a l'équivalence pour ces surfaces simplement connexes cocompactes.

ABSTRACT. — We discuss geodesic connectedness for simply connected Lorentz surfaces, and show that it holds for those which are globally hyperbolic. In the compact case, we establish the equivalence for the universal covering.

1. Introduction

It is well known that, according to Hopf–Rinow's theorem, any complete, connected Riemannian manifold is geodesically connected; that is, any two of its points can be joined by a (minimizing) geodesic.

By contrast, for Lorentz manifolds, geodesic completeness does not imply geodesic connectedness (Sect. 2).

Throughout this paper S denotes a smooth, connected and simply connected surface. If g is an indefinite metric on S, then we say that (S, g) is a Lorentzian surface.

Two Lorentzian surfaces (S, g) and (S', g') are said to be conformal if there is a diffeomorphism $\phi : S \longrightarrow S'$ such that $\phi^* g' = f \cdot g$, where f is a non-vanishing smooth function on S. So, a conformally flat surface will, in this paper, mean a Lorentzian surface which is globally conformal to a flat one.

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Globally hyperbolic spaces

Let denote by $J^+(p)$ the set of all points $q \in M$ which are related to p by future-directed nonspacelike curves. Similarly we define $J^-(p)$ with past-directed instead future-directed. If $q \in J^+(p)$ we say that q is causally related to p and we write $p < q$.

DEFINITION .— *Given a Lorentzian manifold (M, g), we say that (M, g) is globally hyperbolic provided that it is strongly causal and for each $p < q$ the set $J^+(p) \cap J^-(q)$ is compact.*

Recall that (M, g) is said to be strongly causal if for each $p \in M$ nonspacelike curves that start arbitrarily close to p and leave some fixed neighborhood cannot return arbitrarily close to p.

We point out that Claim 1 in Section 3 shows that all simply connected Lorentzian surfaces are strongly causal, and therefore every Lorentzian surface which is conformal to the Minkowski plane is globally hyperbolic. The converse is not true as can be seen on the universel covering of $(T^2, \mathrm{d}x\,\mathrm{d}y + f(x)\,\mathrm{d}y^2)$, where f is a smooth function which vanishes somewhere without being identically null. (For more about globally hyperbolic spaces, we refer to [1] and [4]).

The main purpose of this work is to study the geodesic connectedness of simply connected surfaces. Our first and main result states that a simply connected Lorentzian surface is geodesically connected if it is globally hyperbolic, and specially when it is conformal to the Minkowski 2-plane E_1^2, where this latter space denotes $\mathbb{R}^2$ with the metric $\mathrm{d}x\,\mathrm{d}y$.

We remark that simply connected, geodesically connected surfaces are not all globally hyperbolic (and so are not all conformal to E_1^2).

Consider the strip shown in diagram 2 with the metric $\mathrm{d}x\,\mathrm{d}y$. Of course, this region is geodesically connected since geodesics are exactly straight lines. However, it is neither globally hyperbolic nor conformal to E_1^2 by the fact that each boundary point is approached by two null geodesics.

In the compact case, we prove that the universal covering of a Lorentzian 2-torus $\mathbb{T}^2$ is geodesically connected if and only if it is globally hyperbolic.

Completeness is also obtained for Lorentzian, conformally flat 2-torus (a proof is given in [2]).

2. Generalities

2.1 Example. A non geodesically connected, complete Lorentzian manifold

Let $\mathbb{R}_1^{n+1}$ denote the usual vector space $\mathbb{R}^{n+1}$ with the indefinite metric $g = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$.

In $\mathbb{R}_1^{n+1}$, we consider the hyperquadric

$$\mathbf{S}_1^n = \left\{(x_0, x_1, \cdots, x_n) \in \mathbb{R}_1^{n+1} \mid -x_0^2 + x_1^2 + \cdots + x_n^2 = 1\right\}.$$

Clearly, g induces a Lorentzian metric on $\mathbf{S}_1^n$. Also, it is well known [4] that, with this metric, $\mathbf{S}_1^n$ is a geodesically complete Lorentzian manifold of constant curvature 1.

The geodesics of $\mathbf{S}_1^n$ are just the intersections of $\mathbf{S}_1^n$ with planes through the origin of $\mathbb{R}_1^{n+1}$ which meet $\mathbf{S}_1^n$.

The hyperquadric $\mathbf{S}_1^n$ is called either the de Sitter space or the unit pseudo-sphere.

By contrast with Riemannian case, the de Sitter space $\mathbf{S}_1^n$ is never geodesically connected since it always contains non-antipodal points p, q for which there exists no geodesic joining p and q (see [4, Prop. 5.38]).

2.2. The two null foliations of a Lorentz surface

Let (S, g) be a simply connected Lorentzian surface. Then the inextendible null geodesics of (S, g) may be partitioned into two transversal families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that each of them covers S simply; that is, each point $p \in S$ lies on exactly one null geodesic of each family. Thus, $\mathcal{F}_1$ and $\mathcal{F}_2$ determine two transversal foliations by the null geodesics of (S, g). We call them *the null foliations* of the Lorentzian surface (S, g).

For example, we remark that $\mathbf{S}_1^2$ is a ruled surface generated by each of null foliations. Also, each leaf of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) meets any leaf of $\mathcal{F}_2$ (resp. $\mathcal{F}_1$) except one. This fact can be well understood if we work with the universal covering of $\mathbf{S}_1^2$. This space is represented by the strip

$$\widetilde{\mathbf{S}_1^2} = \left\{(s, \theta) \;\middle|\; -\frac{\pi}{2} < s < \frac{\pi}{2}\right\}$$

in $\mathbb{R}^2$ with the Lorentzian metric $\mathrm{d}s^2 = \sec^2 s(-\mathrm{d}\theta^2 + \mathrm{d}s^2)$.

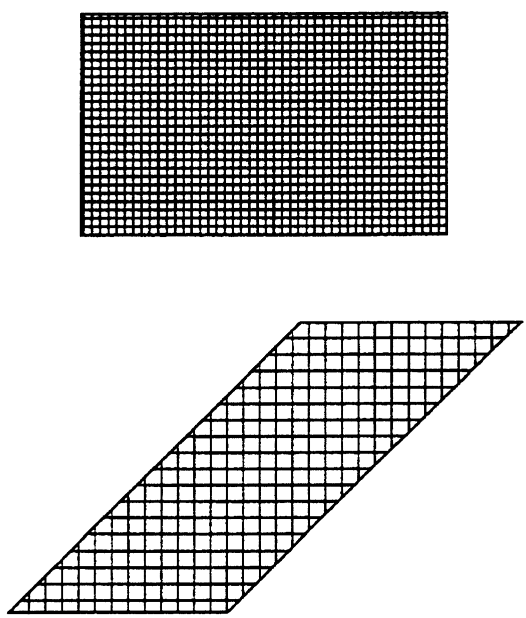
Thus, the universal covering $\widetilde{\mathbf{S}_1^2}$ of the de Sitter space $\mathbf{S}_1^2$ is conformally flat. In contrast, it is not conformal to E_1^2 since each boundary point is approached by two null geodesics (compare with diagram 2). Of course, $\widetilde{\mathbf{S}_1^2}$ is not geodesically connected since $\mathbf{S}_1^2$ is not.

Together, the two null foliations define a grid of null geodesics on S called the *null grid*. Locally there is a system of (local) coordinates x, y on S for which $g = f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ for some non-vanishing function f.

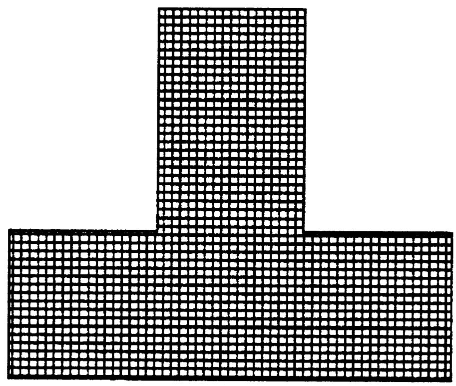
Thus, locally, the null grid can be made as the net of level lines for the above system of coordinates.

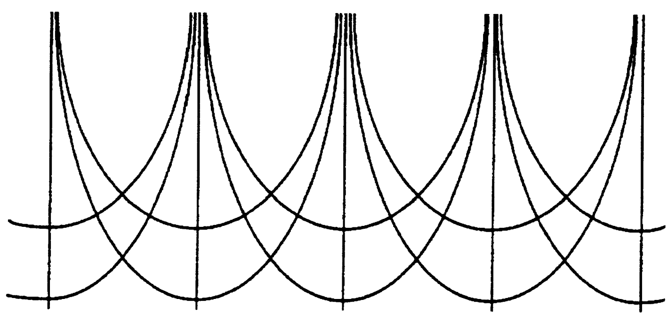
A basic fact of Lorentz geometry is that a diffeomorphism is conformal if and only if the induced map on the tangent bundle preserves the null-cones. This means that, for Lorentz surfaces, a diffeomorphism is conformal iff it preserves the null grids.

Diagrams 1 to 4 show the null grids for the metric $\mathrm{d}x\,\mathrm{d}y$ on four open subsets of the plane $\mathbb{R}_1^2$. The strip shown in diagram 1 is conformal to E_1^2, but none of those in diagrams 2 to 4 is conformal to E_1^2. This fact is related to the behaviours of $\mathcal{F}_1$ and $\mathcal{F}_2$.



Diagrams 1 and 2





Diagrams 3 and 4

2.3 The conformal boundary

Let l, m be two leaves of the two null foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, and let us attach ideal boundary points l_+, l_- and m_+, m_- in the positive and the negative directions respectively.

The conformal boundary of (S, g) denoted by $\partial_c S$ (and which is due to Kulkarni [3]) will be the set of equivalence classes relatively to a certain equivalence relation on the set of all ideal boundary points. Thus, it is invariant under conformal changes.

A topology is then defined on $\overline{S} = S \cup \partial_c S$ which induces a topology on $\partial_c S$. In general, $\partial_c S$ may be complex and need not be Hausdorff. (See [3] for more details).

Now, it is not difficult to see that the four regions in diagrams 1 to 4 represent different conformal classes. Of course, not every simply connected Lorentzian surface is conformally flat. In fact, for the region shown in diagram 4, some boundary points are approached by infinitely many null geodesics. Note also that this region may be seen as the universal covering of a punctured disc with the metric $\mathrm{d}x\,\mathrm{d}y$.

A remarkable conformal invariant

Let $i(p)$ be the number of all possible null geodesics in S ending at the boundary point p. It is a conformal invariant of (S, g).

In the examples of the diagrams above, $\partial_c S$ can be identified with the subset of all topological boundary points which are end points of null geodesics living in S. Then, $i(p) \neq 0$ for all points of $\partial_c S$.

Thus, in diagram 1, $i(p) = 1$ for all points of $\partial_c S$. In diagram 2, $i(p) = 2$ since each boundary point is approached by exactly two null geodesics, one of each of the two null foliations. In diagram 3, the six boundary points at outer corners are approached along by no null geodesic, while those at the two inner corners are approached by exactly two null geodesics. This means that only the latter (i.e., the two inner corners) are in $\partial_c S$. Thus, $i(p) = 1$ except at the two inner corners on $\partial_c S$ where $i(p) = 2$. In diagram 4, $i(p) = \infty$ for every boundary point on $\partial_c S$ corresponding to the origin of the punctured disk. This is an example of a simply connected Lorentzian surface which cannot be conformally flat.

The following result will be easily proved, by a close reading of [3].

PROPOSITION 2.1.— *Let (S, g) be a connected, simply connected Lorentz surface with smoothable $\partial_c S$. Then,*

1) (S, g) is (globally) conformally flat iff $i(p) \leq 3$ for every p in $\partial_c S$;

2) (S, g) is conformal to E_1^2 iff $i(p) = 1$ for every p in $\partial_c S$;

3) (S, g) is conformal to $\widetilde{S_1^2}$ iff $i(p) = 2$ for every p in $\partial_c S$.

The boundary points $p \in \partial_c S$ such that $i(p) \neq 2$ were called, by Kulkarni, *characteristic* points.

Remark.— Another conformal invariant is the number of corner points on $\partial_c S$. Thus, by varying the number of horizontal and vertical line segments making up the polygonal boundary of a certain region, we can construct an infinite number of conformally distinct simply connected Lorentz surfaces. This contrasts completely with the three well known conformal classes of simply connected Riemannian surfaces.

The next result points out an equivalence between the fact to be conformal to a subset of E_1^2 and certain assumptions on the behaviours of $\mathcal{F}_1$ and $\mathcal{F}_2$. But, before stating it, we shall need a definition.

DEFINITION .— *We say that the pair* $(\mathcal{F}_1, \mathcal{F}_2)$ *make up a product (which is always true locally) if there exist global coordinates* (x, y) *such that the metric is given by* $g = f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ *for a certain non-vanishing function* f *on* S.

PROPOSITION 2.2.— *Under the assumptions of proposition 2.1, we have:*

1) (S, g) *is conformally flat iff* $(\mathcal{F}_1, \mathcal{F}_2)$ *is a product;*

2) (S, g) *is globally hyperbolic iff each leaf of* $\mathcal{F}_1$ *intersects each leaf of* $\mathcal{F}_2$ *and conversely.*

3. Geodesic connectedness

In this Section we shall discuss the main purpose of this paper.

3.1 General case

In the simply connected case, we shall prove that geodesic connectedness is obtained when (S, g) is globally hyperbolic.

The idea is to prove that every inextendible geodesic starting at a fixed point p of S may be unbounded, that is, it does not remain in any compact subset of S. Therefore, we can apply the implicit function theorem to conclude that every point of S can be joined to the fixed point p.

We are going to state our main result, but first we need the next lemma.

LEMMA 3.1.— *Let* $\gamma : [\,0\,, b\,[\; \mapsto S$ *be a given null geodesic of a simply connected Lorentzian surface* (S, g). *Then, each other geodesic intersects* γ *at most once.*

Proof.— Suppose that σ is another geodesic in S which intersects γ twice (possibly at the same point). Then, by the transversality of the null foliations, we can construct a non-vanishing vector field which will be transversal to the boundary of the disk made up of γ and σ, but this is obviously absurd. Hence, the geodesic σ must intersect γ at most once. □

THEOREM 3.2.— *A simply connected Lorentzian surface (S, g) is geodesically connected if it is globally hyperbolic.*

Proof.— Let p, q be two distinct points in S and let $\mathcal{R}$ be the rectangle made up of the null geodesics through p and q as in figure 1. Since (S, g) is globally hyperbolic it follows that $\mathcal{R}$ is compact.

CLAIM 1.— *Every geodesic starting at p and going in $\mathcal{R}$ must go out.*

Proof.— Let $\gamma : [0, b[\mapsto S$, $b < \infty$, be an inextendible geodesic. We are going to show that γ does not remain in $\mathcal{R}$ when $t \to b$.

Since $\mathcal{R}$ is compact, if γ were entirely contained in $\mathcal{R}$ then there would be a sequence $s_n \to b$ such that $\gamma(s_n) \to r$, with $r \in \mathcal{R}$.

Thus, γ intersects one of the two null geodesics through r infinitely many times. Otherwise, γ would be extendible past b, contradicting the assumption that γ is inextendible beyond b. But, according to lemma 3.1, γ cannot intersect a null geodesic twice. □

Now, we parametrize the geodesics starting at p and going in $\mathcal{R}$ by the angle between such geodesics and the fixed null geodesic joining p and b as shown in figure 1.

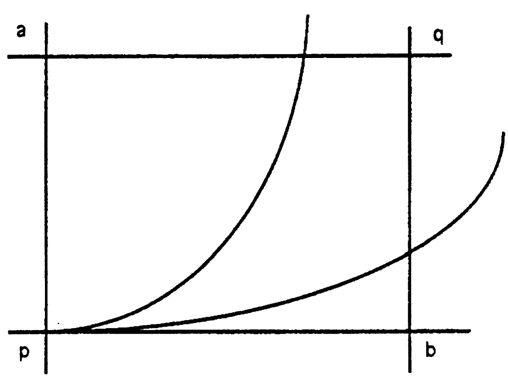


Fig. 1

If $A(a,q,b)$ denotes the reunion of the segments $]a\,,\,q]$ and $[q\,,\,b[$ where a, b are determined by the intersections of the null geodesics through p and q, then we get a map $\phi :]\alpha\,,\,\beta[\;\rightarrow A(a,q,b)$ which associates, for every geodesic starting at p and going in $\mathcal{R}$ its intersection point with $A(a,q,b)$.

Now, it is obvious by Claim 1 that ϕ is well defined.

CLAIM 2.— *The map ϕ is continuous.*

Proof.— Since (S,g) is globally hyperbolic then, by Proposition 2.2, there are global coordinates (x,y) such that null geodesics of $\mathcal{F}_1$ (for example) are given by $x = constant$.

Assume that the null geodesic joining b and q is given by $x = c$, and let $(t,u) \mapsto \big(x(t,u)\,,\,y(t,u)\big)$ be the two parameters family of geodesics starting at p and going in $\mathcal{R}$. This provides that the map ϕ is given by :

$$(t,u) \mapsto \big(c\,,\,y(t,u)\big)\,.$$

On the other hand, every geodesic starting at p and going in $\mathcal{R}$ is transversal to $A(a,q,b)$ and consequently, at any (t,u) such that $x(t,u) = c$, $\partial\phi(t,u)/\partial t$ is never vertical. But this implies that, for every (t,u) such that $x(t,u) = c$, $\partial x(t,u)/\partial t \neq 0$.

Now, for every (t,u) such that $x(t,u) = c$, we can apply the implicit function theorem to guarantee the existence of a neighborhood of (t,u) for which t can be expressed as a continuous function of u. In other words, $y\big(t(u),u\big)$ is continuous in u and consequently ϕ is continuous too. □

We return now to the proof of Theorem 3.2. Since $]\alpha\,,\,\beta[$ is connected and since a, b are attained by geodesics starting at p and going in $\mathcal{R}$, the map ϕ (which is continuous) will be onto and consequently q may be joined to p by a geodesic, as we wished. □

3.2 Compact case

Suppose now that S is a compact surface and recall that S could be Lorentzian if and only if its Euler–Poincaré characteristic vanishes. So, the 2-torus $\mathbb{T}^2$ is the only orientable surface susceptible to be Lorentzian. Furthermore, we remark that:

a) the unit disk $(D, \mathrm{d}x\,\mathrm{d}y)$ is not geodesically complete;

b) if (M, g) is a simply connected Lorentzian surface, then there is a smooth conformal factor $\Omega : M \to \,]0\,,\infty[$ such that $(M, \Omega g)$ is geodesically complete (cf. [1]).

Because of these two facts, it might be natural to state the following conjecture : *a simply connected complete Lorentz surface is geodesically connected iff it is globally hyperbolic.*

So, in this context we shall now prove the following.

THEOREM 3.3.— *Let $(\mathbb{T}^2, g)$ be a Lorentzian 2-torus. Then, its universal covering is geodesically connected iff it is globally hyperbolic.*

Proof.— By theorem 3.2, we have only to prove the converse.

Suppose that the universel covering is not globally hyperbolic. Then there exist causally related points $p < q$ such that $J^+(p) \cap J^-(q)$ is noncompact. This means, by proposition 2.2, that there exists at least a leaf of $\mathcal{F}_1$ which does not meet certain leaves of $\mathcal{F}_2$.

Because the surface is the universal covering of a torus, it must contain at least one Reeb's component.

Now, let x, y be two points as shown in figure 2.

CLAIM 3.— *There is no geodesic joining x and y.*

Proof.— We first assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are oriented as shown in figure 2. It is clear that no null geodesic cannot join x and y.

Also, a timelike geodesic starting at x could not even cross the line l of figure 2, otherwise, it would intersect the null geodesic l_x or l_y twice, which contradicts Lemma 3.1.

Equally, a spacelike geodesic starting at x cannot neither reach the point y, for almost the same reason. Indeed, according to lemma 3.1, if such a geodesic meets the line l at a certain point z then it would be entirely imprisoned in the right half-cone made up of the two null geodesics segments $l_z = l$ and m_z through the point z (fig. 2). Thus such a geodesic could not reach the point y, which concludes the proof of Theorem 3.3. □

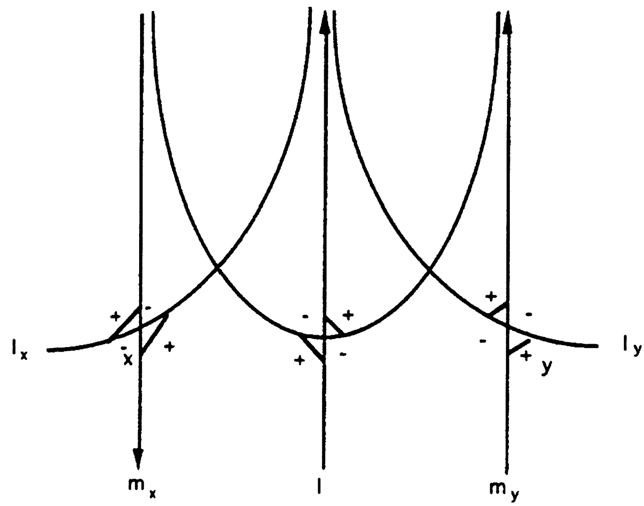


Fig. 2

3.3 Final remarks

1) It is obvious that global hyperbolicity is invariant (at least for surfaces) under conformal changes. However, there is no reason for that geodesic connectedness should be invariant by conformal changes. We can think about the universel covering of the de Sitter space. We know that this simply connected surface is conformal to a subset of E_1^2 although it is not geodesically connected.

2) In [1] and [4], we can find a result due to Seifert which says that for a globally hyperbolic Lorentz manifold any two causally related points may be joined by a nonspacelike geodesic.

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References

[1] BEEM (J. K.) and EHRLICH (P. E.) .— *Global Lorentzian Geometry*, Dekker, New York, 1981.

[2] CARRIÈRE (Y.) and ROZOY (L.) .— *Complétude des métriques lorentziennes de $\mathbb{T}^2$ et difféomorphismes du cercle*, Bul. Soc. Bras. Math. **25**, n° 2 (1994), pp. 223-235.

[3] KULKARNI (R.S.) .— *An analogue of the Riemann mapping theorem for Lorentz metrics*, Proc. R. Soc. London **A401** (1985), pp. 117-130.

[4] O'NEILL (B.) .— *Semi-Riemannian geometry*, Academic Press, 1983.