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On the Hasse Principle for Homogeneous Spaces with Finite Stabilizers^(*)

MIKHAIL BOROVOI⁽¹⁾ and BORIS KUNYAVSKII⁽²⁾

RÉSUMÉ. — Nous construisons un espace homogène X à stabilisateur fini sous le groupe $SL_{m,k}$ qui est défini sur un corps de nombres k et qui possède les propriétés suivantes :

- (1) $X(k_v) \neq \emptyset$ pour tous les complétés k_v de k ;
- (2) la “première obstruction de Brauer–Manin” au principe de Hasse pour X s’annule;
- (3) le groupe de Brauer non ramifié de $X_{\bar{k}}$ est nul;
- (4) mais $X(k) = \emptyset$.

Cela signifie que pour les espaces homogènes à stabilisateurs finis, la “première obstruction de Brauer–Manin” au principe de Hasse n’est pas la seule.

ABSTRACT. — We construct a homogeneous space X of the group $SL_{m,k}$ over a number field k , with finite stabilizer, such that X has the following properties:

- (1) $X(k_v) \neq \emptyset$ for any completion k_v of k ;
- (2) the “first Brauer–Manin obstruction” to the Hasse principle for X is zero;
- (3) the unramified Brauer group of $X_{\bar{k}}$ is zero;
- (4) but $X(k) = \emptyset$.

This means that for homogeneous spaces with finite stabilizers, the “first Brauer–Manin obstruction” to the Hasse principle is not the only one.

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0. Introduction

In this paper we construct a homogeneous space X with a finite stabilizer for which the Hasse principle fails and this fact cannot be explained in the same way as for homogeneous spaces with connected stabilizers.

In more detail, let X be an algebraic variety over a number field k . The variety X is called a counter-example to the Hasse principle, if X has a k_v -point for any completion k_v of k , but has no k -points. Manin ([Man1], [Man2]) proposed a general method of explaining obstructions to the Hasse principle with the use of the Brauer group of X . For a k -variety X such that $X(k_v)$ is nonempty for any place v of k , Manin’s method gives an obstruction for X to have a k -point.

We assume X to be nonsingular. Consider the cohomological Brauer group $\mathrm{Br} X = H_{\acute{e}t}^2(X, \mathbf{G}_m)$. We have canonical maps $X_{\bar{k}} \rightarrow X \rightarrow \mathrm{Spec} k$ where \bar{k} is an algebraic closure of k . Set

$$\begin{aligned} \mathrm{Br}_0 X &= \mathrm{im}[\mathrm{Br} k \rightarrow \mathrm{Br} X] \\ \mathrm{Br}_1 X &= \ker[\mathrm{Br} X \rightarrow \mathrm{Br} X_{\bar{k}}] \\ \mathrm{Br}_a X &= \mathrm{Br}_1 X / \mathrm{Br}_0 X, \end{aligned}$$

the “algebraic” Brauer group of X . Set

$$\mathfrak{B}(X) = \ker \left[\mathrm{Br}_a X \longrightarrow \prod_v \mathrm{Br}_a X_{k_v} \right],$$

v running over all the places of k .

We refer to [Man2] and [CTS1] for a definition of the Brauer–Manin obstruction to the Hasse principle for X . For a few classes of smooth projective rational varieties it was proved that it is the only obstruction to the Hasse principle. We refer to [CT1], [CT2] and [MatTs] for a discussion and relevant bibliography.

Sansuc [San] introduced the Brauer–Manin obstruction related to $\mathfrak{B}(X)$ (“the first obstruction”, in the terminology of [CTS1]). This obstruction $m(X)$ lives in the group $\mathfrak{B}(X)^\sim$ dual to $\mathfrak{B}(X)$. For a class \mathcal{C} of k -varieties a natural question arises whether the obstruction $m(X)$ related to $\mathfrak{B}(X)$ is the only one, i.e., is it true that if $X \in \mathcal{C}$, $X(k_v) \neq \emptyset$ for any place v of k , and $m(X) = 0$, then X must have a k -point.

Let G be k -group. A (right) homogeneous space X of G is a k -variety X together with an action $X \times G \rightarrow X$ of G on X defined over k , such that $G(\bar{k})$ acts on $X(\bar{k})$ transitively, where \bar{k} is an algebraic closure of k . Concerning homogeneous spaces, the obstruction $m(X)$ was proved to be the only obstruction to the Hasse principle for principal homogeneous spaces of tori (Vokresenskiĭ [Vos1], [Vos2]) and more general for principal homogeneous spaces of connected (affine) k -groups (Sansuc [San]). Moreover, the similar assertion was proved for homogeneous spaces of connected k -groups with connected stabilizers [Bor].

We are interested here in homogeneous spaces of simply connected groups. Let X be a homogeneous space of a semisimple simply connected group G defined over a number field k . Let \bar{x} be a \bar{k} -point of X , and $\bar{H} = \text{Stab}(\bar{x})$ its stabilizer in $G_{\bar{k}}$. Assume that $X(k_v) \neq \emptyset$ for every place v of k and that $m(X) = 0$. If \bar{H} is connected, then X must have a k -point. Let us now consider the case when \bar{H} is nonconnected, say, finite. If \bar{H} is abelian, then again X must have a k -point (cf. [Bor]). However if \bar{H} is nonabelian, it turns out that vanishing of $m(X)$ does not imply the existence of a k -point in X . Moreover, it may happen that the unramified Brauer group $\text{Br}_{\text{nr}} X_{\bar{k}}$ also vanishes. Therefore we have a phenomenon different from Harari's counter-example [Har]. In his examples $\text{Br}_{\mathbf{a}} X$ vanishes, hence $m(X) = 0$, but there exists a "geometric" obstruction related to the unramified Brauer group $\text{Br}_{\text{nr}} X_{\bar{k}}$.

Our main result is the following theorem.

THEOREM 0.1. — *There exists a homogeneous spaces X of SL_m for some m , with finite stabilizer F , with the following properties:*

- (1) $X(k_v) \neq \emptyset$ for every place v of k ;
- (2) $m(X) = 0$;
- (3) $\text{Br}_{\text{nr}} X_{\bar{k}} = 0$;
- (4) but $X(k) = \emptyset$.

Theorem 0.1 means that for homogeneous spaces with finite stabilizers the Brauer–Manin obstruction related to $\mathfrak{B}(X)$ is not the only obstruction to the Hasse principle, and this fact cannot be always explained with the help of a "geometric" obstruction related to $\text{Br}_{\text{nr}} X_{\bar{k}}$.

It would be interesting to say something about a smooth compactification X^c of X ; namely, we do not know whether X^c has a k -point. Of course, the

major problem is to compute the unramified Brauer group $\text{Br}_{\text{nr}} X = \text{Br } X^c$ and to decide whether the Brauer–Manin obstruction on X^c vanishes.

Let us briefly describe the idea of proof of Theorem 0.1. We construct F as a nilpotent group of class 2, as in Bogomolov’s version [Bog] of Saltman’s counter-example to Noether’s problem [Sal]. Let L/k be a Galois extension of degree n , and let $\mathfrak{g} = \text{Gal}(L/k)$ be the Galois group. To construct a desired example we need several ingredients:

- (1) a finite k -group F trivialized by L , nilpotent of class 2, given as an extension

$$1 \longrightarrow Z \longrightarrow F \longrightarrow F/Z \longrightarrow 1$$

with abelian k -groups Z and F/Z ;

- (2) a nonzero cohomology class $\eta \in H^2(\mathfrak{g}, Z)$, locally trivial and not coming from $H^1(\mathfrak{g}, F/Z)$;
- (3) an embedding of Z into an abelian k -group A , killing η .

Having these data at our disposal, we define $N = (F \times A)/Z$ and embed N into SL_m for some m . The group A/Z acts on $F \setminus \text{SL}_m$. Let $c \in Z^1(\mathfrak{g}, A/Z)$ be a cocycle whose image in $H^2(\mathfrak{g}, Z)$ is η . We define X as a twisted form ${}_c(F \setminus \text{SL}_m)$. Since η does not come from $H^1(\mathfrak{g}, F/Z)$, we have $X(k) = \emptyset$ (Prop. 3.1). Since η is locally trivial, we have $X(k_v) \neq \emptyset$ for all v (Prop. 3.2). In our example $\mathfrak{B}(X) = 0$ and $\text{Br}_{\text{nr}} X_{\bar{k}} = 0$.

Notation and conventions

k is a field of characteristic 0. We denote by L/k a finite Galois extension with Galois group \mathfrak{g} . When k is a number field, we write k_v for the completion of k at a place v and denote by $\mathfrak{g}_v = \text{Gal}(L_w/k_v)$ a decomposition group, where w is a place of L lying over v .

By a k -group we mean an affine algebraic group defined over k , usually finite. For a k -group C we write $\mathbf{X}(C)$ for the character group of C , i.e., $\mathbf{X}(C) = \text{Hom}(C, \mathbf{G}_{m, \bar{k}})$. Usually we assume that both C and $\mathbf{X}(C)$ are trivialized by L , i.e., all the elements and characters of C are defined over L . By $R_{L/k}C$ we denote the k -group obtained from C_L by Weil’s restriction of the ground field. When C is abelian, we set,

$$\text{III}^i(\mathfrak{g}, C) = \ker \left[H^i(\mathfrak{g}, C) \longrightarrow \prod_v H^i(\mathfrak{g}_v, C) \right],$$

where v runs over all the places of k , and

$$\text{III}_\omega^i(\mathfrak{g}, C) = \ker \left[H^i(\mathfrak{g}, C) \longrightarrow \prod_{\mathfrak{h} \subset \mathfrak{g}} H^i(\mathfrak{h}, C) \right],$$

where \mathfrak{h} runs over all the cyclic subgroups of \mathfrak{g} .

For any finite abelian group A , let A^\sim denote the dual group, $A^\sim = \text{Hom}(A, \mathbf{Q}/\mathbf{Z})$.

For any group F denote by $Z(F)$ its center and by $[F, F]$ the derived group.

Denote by μ_n the group of n -th roots of unity, and set $\mathbf{Z}/n = \mathbf{Z}/n\mathbf{Z}$.

1. Cocycle

Let k be a field of characteristic 0, and let

$$1 \longrightarrow Z \longrightarrow F \longrightarrow F/Z \longrightarrow 1 \tag{1.1}$$

be an exact sequence of finite k -groups. Suppose $Z = Z(F)$. Denote by L the field of definition of all the points of F . Set $\mathfrak{g} = \text{Gal}(L/k)$. Let $\eta \in H^2(\mathfrak{g}, Z)$ be a cohomology class represented by a cocycle $d = (d_{\sigma, \tau})$, $\sigma, \tau \in \mathfrak{g}$.

Our aim is to associate to these data a 1-cocycle which can be used to construct a desired homogeneous space.

Choose an embedding $j : Z \hookrightarrow A$ where A is finite abelian k -group trivialized by L , such that $j_*\eta = 0$ (e.g. one may take $A = R_{L/k}Z$). Let $i : Z \hookrightarrow F$ denote the inclusion. We have an embedding $Z \hookrightarrow F \times A$ defined by $z \mapsto (i(z), j(z)^{-1})$. Set $N = (F \times A)/Z$. We have embeddings $F \hookrightarrow N$, $A \hookrightarrow N$, so we may and will regard A and F as subgroups of N . Note that A is central in N , and F is normal in N .

Since $j_*\eta = 0$ in $H^2(\mathfrak{g}, A)$, we have

$$d_{\sigma, \tau}^{-1} c_{\sigma\tau} = c_\sigma \sigma c_\tau$$

for some cochain $c : \mathfrak{g} \rightarrow A$. Set $\bar{c}_\sigma = c_\sigma \pmod{Z}$, $\bar{c}_\sigma \in A/Z$. Then $\bar{c}_{\sigma\tau} = \bar{c}_\sigma \cdot \sigma \bar{c}_\tau$, so \bar{c} is a cocycle, $\bar{c} \in Z^1(\mathfrak{g}, A/Z)$. Note that the cohomology class of \bar{c} can be obtained from η using the exact sequence

$$H^1(\mathfrak{g}, A/Z) \longrightarrow H^2(\mathfrak{g}, Z) \longrightarrow H^2(\mathfrak{g}, A).$$

2. Homogeneous space

Now we use the cocycle \bar{c} obtained in Section 1 to construct a homogeneous space with stabilizer F .

We start by choosing an embedding $N \hookrightarrow \mathrm{SL}_{m,k}$ for some m . The group A acts on the left on $F \backslash \mathrm{SL}_m$ by $a \cdot Fg = Fag$ where $a \in A$, $g \in \mathrm{SL}_m$. This action commutes with the natural right action of SL_m and is defined over k ; indeed

$$\sigma_a \cdot F \cdot \sigma g = F \cdot \sigma_a \cdot \sigma g = \sigma(Fag).$$

Moreover $Z = F \cap A$, and therefore Z acts trivially, so we obtain an action of A/Z on $F \backslash \mathrm{SL}_m$.

We use the cocycle $\bar{c} \in Z^1(\mathfrak{g}, A/Z)$ to define a twisted form

$$X = \bar{c}(F \backslash \mathrm{SL}_m)$$

of $F \backslash \mathrm{SL}_m$ (see [Ser2, Chap. I, 5.3], for the definition of a twisted form).

3. k -points

Let k be a field of characteristic 0. Let $\Delta : H^1(\mathfrak{g}, F/Z) \rightarrow H^2(\mathfrak{g}, Z)$ be the connecting map associated to (1.1), and let X be as in Section 2.

PROPOSITION 3.1. — *X has a k -point if and only if $\eta \in \mathrm{im} \Delta$.*

Proof. — Assume that X has a k -point $x_0 = Fg$. Then for any $\sigma \in \mathfrak{g}$ we have ${}^\sigma x_0 = x_0$, i.e.,

$$F c_\sigma {}^\sigma g = Fg$$

(it follows from the definition of a twisted form). Hence $f_\sigma c_\sigma {}^\sigma g = g$ for some $f_\sigma \in F$, or

$$f_\sigma c_\sigma = g \cdot {}^\sigma g^{-1}.$$

We have

$$(g \cdot {}^\sigma g^{-1}) \cdot {}^\sigma (g \cdot {}^\tau g^{-1}) = g \cdot {}^{\sigma\tau} g^{-1},$$

whence

$$(f_\sigma c_\sigma) \cdot {}^\sigma (f_\tau c_\tau) = f_{\sigma\tau} c_{\sigma\tau},$$

or

$$f_\sigma c_\sigma {}^\sigma f_\tau {}^\sigma c_\tau = f_{\sigma\tau} c_{\sigma\tau}.$$

Since $c_\sigma \in A$ and A is central in N , we see that

$$f_\sigma {}^\sigma f_\tau f_{\sigma\tau}^{-1} \cdot c_\sigma {}^\sigma c_\tau c_{\sigma\tau}^{-1} = 1.$$

Since

$$c_\sigma {}^\sigma c_\tau c_{\sigma\tau}^{-1} = d_{\sigma,\tau}^{-1},$$

we obtain

$$f_\sigma {}^\sigma f_\tau f_{\sigma\tau}^{-1} = d_{\sigma,\tau}.$$

Let \bar{f}_σ denote the image of f_σ in F/Z . Then $\bar{f}_\sigma \cdot {}^\sigma \bar{f}_\tau = \bar{f}_{\sigma\tau}$, i.e., $\bar{f} \in Z^1(\mathfrak{g}, F/Z)$. Let ξ denote the class of \bar{f} in $H^1(\mathfrak{g}, F/Z)$. Then $\Delta(\xi)$ is the cohomology class of d , hence $\Delta(\xi) = \eta$.

Conversely, assume that $\eta = \Delta(\xi)$ for some $\xi \in H^1(\mathfrak{g}, F/Z)$. Then

$$d_{\sigma,\tau} f_{\sigma\tau} = f_\sigma \cdot {}^\sigma f_\tau$$

for some cochain $f : \mathfrak{g} \rightarrow F$. Set $c'_\sigma = f_\sigma c_\sigma$. Then

$$c'_{\sigma\tau} = f_{\sigma\tau} c_{\sigma\tau} = d_{\sigma,\tau}^{-1} f_\sigma {}^\sigma f_\tau d_{\sigma,\tau} c_\sigma {}^\sigma c_\tau = f_\sigma c_\sigma {}^\sigma f_\tau {}^\sigma c_\tau = c'_\sigma {}^\sigma c'_\tau,$$

because $c_\sigma \in A$, A is central in N , Z is central in F . We see that (c'_σ) is a cocycle, $c' \in Z^1(\mathfrak{g}, N)$.

Since $H^1(\mathfrak{g}, \mathrm{SL}_m(L)) = 1$ (cf. [Ser1, Chap. X, § 1]), there exists $g \in \mathrm{SL}_m(L)$ such that

$$c'_\sigma = g \cdot {}^\sigma g^{-1}$$

for every $\sigma \in \mathfrak{g}$. We set $x_0 = Fg \in X(L)$. Then

$${}^\sigma x_0 = F c_\sigma {}^\sigma g = F f_\sigma c_\sigma {}^\sigma g = F c'_\sigma {}^\sigma g = Fg = x_0.$$

Thus ${}^\sigma x_0 = x_0$, hence $x_0 \in X(k)$. \square

Now let k be a number field. Let v be a place of k , and $\mathfrak{g}_v \subseteq \mathfrak{g}$ a decomposition group. Let $\mathrm{loc}_v : H^i(\mathfrak{g}, \cdot) \rightarrow H^i(\mathfrak{g}_v, \cdot)$ denote the restriction map. Again let X be as in Section 2.

PROPOSITION 3.2. — *For a place v of k , if $\mathrm{loc}_v \eta = 0$, then $X(k_v) \neq \emptyset$.*

Proof. — Consider X_{k_v} . Since $\text{loc}_v \eta = 0$, the class $\text{loc}_v \eta$ is in the image of $\Delta_v : H^1(\mathfrak{g}_v, F/Z) \rightarrow H^2(\mathfrak{g}_v, Z)$. By Proposition 3.1, X_{k_v} has a k_v -point. \square

4. Bilinear map

We construct an extension

$$1 \longrightarrow Z \longrightarrow F \longrightarrow F/Z \longrightarrow 1 \tag{4.1}$$

and a cohomology class $\eta \in H^2(\mathfrak{g}; Z)$ with the properties

- (1) $Z(F) = Z$ and $[F, F] = Z$;
- (2) η is locally trivial, i.e., $\text{loc}_v \eta = 0$ in $H^2(\mathfrak{g}_v, Z)$ for all v ;
- (3) η is not in the image of $\Delta : H^1(\mathfrak{g}, F/Z) \rightarrow H^2(\mathfrak{g}, Z)$.

Having such a triple (F, Z, η) we can apply the construction of Sections 1 and 2 to obtain X satisfying (1) and (4) of Theorem 0.1, according to Section 3.

The idea is the following: we construct an extension (4.1) satisfying (1) with $H^1(\mathfrak{g}, F/Z) = 0$ and $\text{III}^2(\mathfrak{g}, Z) \neq 0$. Then a nonzero $\eta \in \text{III}^2(\mathfrak{g}, Z)$ will satisfy (2) and (3).

In this section we present a general method for constructing such extensions. We state some sufficient conditions under which (1) holds. An explicit construction of (4.1) is postponed to Section 5.

Further on we write the group composition in Z and F/Z as addition.

Let us start with a linear map (homomorphism) $\varphi : M \otimes M \rightarrow Z$, where M and Z are abelian groups. It corresponds to a bilinear (bi-additive) map $\varphi_* : M \times M \rightarrow Z$. Consider the map

$$\Phi : (M \oplus M) \times (M \oplus M) \longrightarrow Z$$

defined by

$$\Phi((x, y), (x', y')) = \varphi_*(x, y').$$

Then $\Phi(a, b) - \Phi(a, b + c) + \Phi(a + b, c) - \Phi(b, c) = 0$, and one may regard Φ as a 2-cocycle for the trivial action of $M \oplus M$ on Z . We obtain a central extension

$$1 \longrightarrow Z \longrightarrow F \longrightarrow M \oplus M \longrightarrow 1$$

defined by Φ . It has a canonical section $s : M \oplus M \rightarrow F$, and we write an element of F as $zs(a)$, where $z \in Z$, $a = (x, y) \in M \oplus M$. Then the multiplication law in F is given by

$$zs(a) \cdot z's(a') = zz'\Phi(a, a')s(a + a'),$$

where $z, z' \in Z$, $a = (x, y)$, $a' = (x', y') \in M$. It follows that

$$s(a)s(a') = \Phi(a, a')s(a + a').$$

Hence for the commutator of $s(a)$ and $s(a')$ we have

$$[s(a), s(a')] = \Phi(a, a') - \Phi(a', a) = \varphi_*(x, y') - \varphi_*(x', y).$$

LEMMA 4.3. — *If φ is surjective, then $Z = [F, F]$.*

Proof. — Indeed, $[s(x, 0), s(0, y)] = \varphi_*(x, y)$. Hence $Z \subseteq [F, F]$. But F/Z is abelian, thus $[F, F] = Z$. \square

Let

$$\text{lker } \varphi_* = \{x \in M \mid \varphi_*(x, y) = 0, \forall y \in M\}$$

$$\text{rker } \varphi_* = \{y \in M \mid \varphi_*(x, y) = 0, \forall x \in M\}$$

denote the left and the right kernels of φ_* , respectively.

LEMMA 4.4. — *If $\text{lker } \varphi_* = \text{rker } \varphi_* = 0$, then $Z = Z(F)$.*

Proof. — Clearly $Z \subseteq Z(F)$. Let $zs(a) \in Z(F)$, $z \in Z$, $a = (x, y) \in M \oplus M$. We have $[zs(a), s(a')] = 0$ for any $a' = (x', y')$. But $[zs(a), s(a')] = \varphi_*(x, y') - \varphi_*(x', y)$. Set $x' = 0$. We get $\varphi_*(x, y') = 0$ for any $y' \in M$, so $x = 0$. Quite similarly we get $y = 0$. Thus $a = 0$ and $zs(a) \in Z$. \square

5. Group extension

In this section we construct a desired extension (4.1) explicitly. Imposing some conditions on k and L and applying results of Section 4 we check (1)-(3) of Section 4.

From now on we assume that our ground field k contains the n -th roots of unity, where $n = [L : k]$ and L denotes the field trivializing F . Denote $\mathfrak{g} = \text{Gal}(L/k)$. We fix a primitive n -th root of unity and use it to identify \mathbf{Z}/n and μ_n .

Notations 5.1.— Let us define M and Z . Take $M = R_{L/k}\mu_n = R_{L/k}(\mathbf{Z}/n)$. We have canonical embeddings $\mathbf{Z}/n \hookrightarrow M$ and

$$j : \mathbf{Z}/n \xrightarrow{\sim} \mathbf{Z}/n \otimes \mathbf{Z}/n \hookrightarrow M \otimes M.$$

Set $Z = M \otimes M / j(\mathbf{Z}/n)$ and denote by $\varphi : M \otimes M \rightarrow Z$ the projection. We have an exact sequence of (\mathbf{Z}/n) -modules

$$0 \longrightarrow \mathbf{Z}/n \longrightarrow M \otimes M \longrightarrow Z \longrightarrow 0 \tag{5.1}$$

Note that since j is \mathfrak{g} -equivariant, one can view (5.1) as a sequence of $(\mathbf{Z}/n)[\mathfrak{g}]$ -modules.

The canonical linear map φ gives rise to a bilinear map $\varphi_* : M \times M \rightarrow Z$ which in turn gives rise to a cocycle $\Phi : (M \oplus M) \times (M \oplus M) \rightarrow Z$, (see Sect. 4). Denote by F the extension of Z by $M \oplus M$ with the help of Φ .

LEMMA 5.2. — *The $(\mathbf{Z}/n)[\mathfrak{g}]$ -module $M \otimes M$ is free.*

Proof.— Consider the module M_g generated over \mathbf{Z}/n by $\{s \otimes sg\}_{s \in \mathfrak{g}}$. Then M_g is $(\mathbf{Z}/n)[\mathfrak{g}]$ -free of rank 1, and $M \otimes M = \bigoplus_{g \in \mathfrak{g}} M_g$ is $(\mathbf{Z}/n)[\mathfrak{g}]$ -free of rank n , where $n = [L : k]$ is the order of \mathfrak{g} . \square

Now we want to apply Lemmas 4.3 and 4.4.

LEMMA 5.3. — *The map φ is surjective with the property*

$$\text{lker } \varphi_* = \text{rker } \varphi_* = 0$$

Proof.— The map φ is surjective by construction. Let us regard all the terms of (5.1) as \mathbf{Z}/n -modules. Denote by $\{e_g\}_{g \in \mathfrak{g}}$ the canonical free basis of M , then we can choose $\{e_g \otimes e_h\}$ as a free basis of $M \otimes M$. The embedding of \mathbf{Z}/n into $M \otimes M$ is written as

$$j(m) = \sum_{g, h \in \mathfrak{g}} m e_g \otimes e_h. \tag{5.2}$$

Suppose $x \in \text{lker } \varphi_*$, i.e., $\varphi(x \otimes y) = 0$ for any $y \in M$. Then $x \otimes y \in \text{im } j$ for any y . Write $x = \sum_{g \in \mathfrak{g}} a_g e_g$. An arbitrary element of $M \otimes M$ of the form

$$\sum_{g, h \in \mathfrak{g}} m_{g,h} e_g \otimes e_h. \quad (5.3)$$

lies in the image of j if and only if all $m_{g,h}$ are equal, see (5.2). Take $y = e_h$. Then $x \otimes y = \sum_{g \in \mathfrak{g}} a_g e_g \otimes e_h$, so in (5.3) only n coefficients a_g may happen to be nonzero and the other $(n^2 - n)$ coefficients are zero. So $a_g = 0$ for any g , and $x = 0$. Thus $\text{lker } \varphi_* = 0$. Quite similarly $\text{rker } \varphi_* = 0$. \square

To satisfy (2) and (3) (Sect. 4), some additional assumptions are needed.

LEMMA 5.4. — *Assume that $\mathfrak{g} = \mathbf{Z}/p \times \mathbf{Z}/p$ but all the decomposition groups \mathfrak{g}_v are cyclic. Then $\text{III}^2(\mathfrak{g}, Z) \neq 0$.*

Proof. — Under the hypotheses of the lemma we have $\text{III}^2(\mathfrak{g}, Z) = \text{III}_\omega^2(\mathfrak{g}, Z)$. Let us show that $\text{III}_\omega^2(\mathfrak{g}, Z) \neq 0$. Let $n = p^2$. From (5.1) we get $\text{III}_\omega^2(\mathfrak{g}, Z) \simeq \text{III}_\omega^3(\mathfrak{g}, \mathbf{Z}/p^2)$ because $M \otimes M$ is a free module. Multiplication by p^2 gives an exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}/p^2 \longrightarrow 0$$

and an exact sequence of cohomology groups

$$H^3(\mathfrak{g}, \mathbf{Z}) \longrightarrow H^3(\mathfrak{g}, \mathbf{Z}) \longrightarrow H^3(\mathfrak{g}, \mathbf{Z}/p^2).$$

Since the first arrow is induced by multiplication by p^2 and all the terms are groups of exponent p , we conclude that this arrow is zero. Thus we obtain an inclusion

$$H^3(\mathfrak{g}, \mathbf{Z}) \hookrightarrow \text{III}^3(\mathfrak{g}, \mathbf{Z}/p^2).$$

whence an inclusion

$$\text{III}_\omega^3(\mathfrak{g}, \mathbf{Z}) \hookrightarrow \text{III}_\omega^3(\mathfrak{g}, \mathbf{Z}/p^2).$$

But $\text{III}^3(\mathfrak{g}, \mathbf{Z}) = \mathbf{Z}/p$ (cf. [ANT, § VII.11.4] for $p = 2$, [Vos2, § 6.46] in general), so $\text{III}_\omega^3(\mathfrak{g}, \mathbf{Z}/p^2) \neq 0$. This proves the lemma. \square

This lemma allows us to find a needed cocycle η and thus to complete the construction of a desired triple (F, Z, η) .

PROPOSITION 5.5. — *Let M, Z, F, φ be as above. Assume that $\mathfrak{g} = \mathbf{Z}/p \times \mathbf{Z}/p$ but all the decomposition groups are cyclic. Then there exists $\eta \in H^2(\mathfrak{g}, Z)$ such that $\eta \in \text{III}^2(\mathfrak{g}, Z)$ and η is not in the image of $\Delta : H^1(\mathfrak{g}, F/Z) \rightarrow H^2(\mathfrak{g}, Z)$.*

Proof. — By Lemma 5.4, $\text{III}^2(\mathfrak{g}, Z) \neq 0$. Take $\eta \in H^2(\mathfrak{g}, Z)$, $\eta \neq 0$. We have $H^1(\mathfrak{g}, M) = 0$ because M is $(\mathbf{Z}/n)[\mathfrak{g}]$ -free, hence $H^1(\mathfrak{g}, F/Z) = 0$ and $\eta \notin \text{im } \Delta$. \square

We conclude that the triple (F, Z, η) satisfies all the conditions (1)-(3) (Sect. 4).

6. The group $\mathfrak{B}(X)$

Let $1 \rightarrow Z \rightarrow F \rightarrow M \oplus M \rightarrow 1$ be the extension constructed in paragraph 5.1. Let $\eta \in H^2(\mathfrak{g}, Z)$ be a nonzero class constructed in Proposition 5.5. Denote by $\bar{c} \in Z^1(\mathfrak{g}, A/Z)$ the cocycle constructed in Section 1, and let $X = {}_{\bar{c}}(F \setminus \text{SL}_m)$ be the twisted form constructed in Section 2.

As η does not lie in the image of Δ , by Proposition 3.1 X has no k -points. As η is locally trivial, by Proposition 3.2 X has k_v -points for all v . Since the obstruction $m(X)$ lies in $\mathfrak{B}(X)^\sim$ (see Sect. 0), to prove (2) (Sect. 0), it suffices to prove the following result.

PROPOSITION 6.1. — $\mathfrak{B}(X) = 0$.

We need the following lemma which we prove imitating a construction of [Mum, Chap. I, Sect. 2].

LEMMA 6.2. — *Let Y be a variety over an algebraically closed field k . Denote $U(Y) = k[Y]^\times / k^\times$. Let Γ be a finite group acting freely on Y . Set $X = \Gamma \setminus Y$. If $\text{Pic } Y = U(Y) = 0$, then $\text{Pic } X = \text{Hom}(\Gamma, k^\times)$.*

Proof. — Let π be the canonical map $Y \rightarrow X$. Let $\mathcal{L} \in \text{Pic } X$ be a line bundle. Then $\pi^*\mathcal{L}$ is a line bundle on Y . Since $\text{Pic } Y = 0$, $\pi^*\mathcal{L}$ is a trivial line bundle.

Fix an isomorphism $\kappa : \pi^*\mathcal{L} \xrightarrow{\sim} \mathbf{G}_a \times Y$ where \mathbf{G}_a is the one-dimensional vector space over k . The group Γ acts on $\pi^*\mathcal{L}$, hence on $\mathbf{G}_a \times Y$. Write

$$\gamma(\alpha, y) = (\varphi_\gamma(y) \cdot \alpha, \gamma y)$$

where $\gamma \in \Gamma$, $\alpha \in k = \mathbf{G}_a(k)$, $y \in Y$, and $\varphi_\gamma : Y \rightarrow k^\times$ is a function without zeros. Since $U(Y) = 0$ we see that $\varphi_\gamma \in k^\times$, hence $\varphi_\gamma(y)$ does not depend on y . Thus $\gamma(\alpha, y) = (\varphi_\gamma \cdot \alpha, \gamma y)$, where $\varphi_\gamma \in k^\times$. We write

$$\begin{aligned}\gamma(\gamma' \cdot (\alpha, y)) &= (\varphi_\gamma \varphi_{\gamma'} \alpha, \gamma \gamma' y) \\ (\gamma \gamma') \cdot (\alpha, y) &= (\varphi_{\gamma \gamma'} \alpha, \gamma \gamma' y),\end{aligned}$$

whence $\varphi_{\gamma \gamma'} = \varphi_\gamma \varphi_{\gamma'}$. We conclude that $\varphi : \Gamma \rightarrow k^\times$ is a homomorphism.

If $\kappa' : \pi^* \mathcal{L} \xrightarrow{\sim} \mathbf{G}_a \times Y$ is another trivialization, then $\kappa' = \chi \cdot \kappa$ where $\chi \in k^\times$ is a constant, and we obtain the same homomorphism φ . Isomorphic line bundles define the same homomorphism. Thus we obtain a map $\text{Pic } X \rightarrow \text{Hom}(\Gamma, k^\times)$. One checks immediately that this map is a homomorphism.

In the other direction, let $\varphi : \Gamma \rightarrow k^\times$ be a homomorphism. Then φ defines an action of Γ on $\mathbf{G}_a \times Y$ by

$$\gamma \cdot (\alpha, y) = (\varphi_\gamma \cdot \alpha, \gamma y).$$

We define $\mathcal{L}_\varphi = \Gamma \backslash (\mathbf{G}_a \times Y)$. Thus we obtain a homomorphism $\text{Hom}(\Gamma, k^\times) \rightarrow \text{Pic } X$ which is inverse to the constructed map $\text{Pic } X \rightarrow \text{Hom}(\Gamma, k^\times)$. Thus the $\text{Pic } X \rightarrow \text{Hom}(\Gamma, k^\times)$ is an isomorphism. \square

Remark. — Let k be a nonclosed field. One can check that the isomorphism $\text{Pic } X_{\bar{k}} \simeq \text{Hom}(\Gamma, \bar{k}^\times)$ is $\text{Gal}(\bar{k}/k)$ -equivariant.

The preceding lemma allows us to compute $\text{Br}_a X$. Denote by $\mathbf{X}(F) = \text{Hom}(F, \mathbf{G}_m)$ the group of characters of F . We regard $\mathbf{X}(F)$ as a $\text{Gal}(\bar{k}/k)$ -module.

LEMMA 6.3. — $\text{Br}_a X = H^1(k, \mathbf{X}(F))$.

Proof. — By [San, § 6.3(iv)], we have an exact sequence

$$H^2(k, U(X_{\bar{k}})) \longrightarrow \text{Br}_a X \longrightarrow H^1(k, \text{Pic } X_{\bar{k}}) \longrightarrow H^3(k, U(X_{\bar{k}})),$$

where $U(X_{\bar{k}}) = \bar{k}[X]^\times / \bar{k}^\times$. By Rosenlicht's theorem [Ros], $U(\text{SL}_{m, \bar{k}}) = 0$, hence $U(X_{\bar{k}}) = 0$. Thus $\text{Br}_a X = H^1(k, \text{Pic } X_{\bar{k}})$.

We have $\text{Pic } \text{SL}_{m, \bar{k}} = 0$ (cf. [San, (6.9)]) and by Lemma 6.2

$$\text{Pic}(F_{\bar{k}} \backslash \text{SL}_{m, \bar{k}}) = \mathbf{X}(F).$$

Now A is central in N , hence the left action of A on $F \setminus \mathrm{SL}_m$, defined in Section 2, induces a trivial action of F and on $\mathrm{Pic}(F \setminus \mathrm{SL}_m) = \mathbf{X}(F)$. We conclude that twisting by the cocycle \bar{c} with values in A/Z does not change the Picard group of $F \setminus \mathrm{SL}_m$. It follows that $\mathrm{Pic} X_{\bar{k}} = \mathbf{X}(F)$. Thus $\mathrm{Br}_a X = H^1(k, \mathbf{X}(F))$. \square

Proof of Proposition 6.1. — By Lemma 6.3,

$$\begin{aligned} \mathfrak{B}(X) &= \ker \left(H^1(k, \mathbf{X}(F)) \longrightarrow \prod_v H^1(k_v, \mathbf{X}(F)) \right) \\ &= \mathrm{III}^1(k, \mathbf{X}(F)) = \mathrm{III}^1(k, \mathbf{X}(F/[F, F])) . \end{aligned}$$

By (5.2) we have $[F, F] = Z$, so

$$\mathfrak{B}(X) = \mathrm{III}^1(k, \mathbf{X}(F/Z)) = \mathrm{III}^1(k, \mathbf{X}(M \oplus M)) = \mathrm{III}^1(k, (\mathbf{Z}/n)[\mathfrak{g}])^2 .$$

By Shapiro's lemma, $\mathrm{III}^1(k, (\mathbf{Z}/n)[\mathfrak{g}]) = \mathrm{III}^1(L, \mathbf{Z}/n)$. But Chebotarev's density theorem gives $\mathrm{III}^1(L, \mathbf{Z}/n) = 0$ (cf. [San, § 2a]), so $\mathfrak{B}(X) = 0$. \square

7. The group $\mathrm{Br}_{\mathrm{nr}} X_{\bar{k}}$

Let X be as in Section 6. We compute the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}} X_{\bar{k}}$. According to [Bog] and [CTS3, 3.6], this group does not depend on the embedding $F \hookrightarrow \mathrm{SL}_m$ but only on F . Denote it by $B_0(F)$. To compute it, proceed as in [Bog] and [CTS2].

Suppose $f : \wedge^2 \Gamma \rightarrow Z$ is a linear map, where Γ and Z are abelian groups, and \wedge^2 denotes the second exterior power. One can view f as a skew-symmetric bilinear map $\Gamma \times \Gamma \rightarrow Z$, or as a 2-cocycle. Consider an extension

$$1 \longrightarrow Z \longrightarrow F \longrightarrow \Gamma \longrightarrow 1 ,$$

defined by this cocycle. If $Z = \wedge^2 \Gamma/S$, then $B_0(F) = (S \setminus S_{\mathrm{bi}})^\sim$ (cf. [CTS2, § 4d]), where S_{bi} is the group generated by the elements of the form $\gamma_i \wedge \gamma_j$ lying in S with $\gamma_i, \gamma_j \in \Gamma$.

With the notation of 5.1, set $\Gamma = M \oplus M$, then

$$\bigwedge^2 \Gamma = \left(\bigwedge^2 (M \oplus 0) \right) \oplus \left(\bigwedge^2 (0 \oplus M) \right) \oplus (M \otimes M).$$

Set

$$S = \left(\bigwedge^2 (M \oplus 0) \right) \oplus \left(\bigwedge^2 (0 \oplus M) \right) \oplus (\mathbf{Z}/n)$$

and embed S into $\bigwedge^2 \Gamma$ by the map $j_* = (\text{id}, \text{id}, j)$, where id denotes the identity map and j is as in Notations 5.1. We get an exact sequence

$$0 \longrightarrow S \xrightarrow{j_*} \left(\bigwedge^2 (M \oplus 0) \right) \oplus \left(\bigwedge^2 (0 \oplus M) \right) \oplus (M \otimes M) \xrightarrow{f} Z \longrightarrow 0,$$

where $f = (\text{id}, \text{id}, \varphi)$ with the notation of 5.1. Thus the extension

$$1 \longrightarrow Z \longrightarrow F \longrightarrow M \oplus M \longrightarrow 1,$$

defined by the cocycle f , coincides with one constructed in Section 5.

PROPOSITION 7.1. — $B_0(F) = 0$.

Proof. — By (5.2), for any $m \in \mathbf{Z}/n$ we have

$$j(m) = \sum_{g, h \in \mathfrak{g}} m e_g \otimes e_h.$$

Take $\gamma = \sum_{g \in \mathfrak{g}} e_g \in M$, set $\gamma_1 = (\gamma, 0)$, $\gamma_2 = (0, \gamma) \in M \oplus M$, then

$$\gamma_1 \wedge \gamma_2 = \sum_{g, h \in \mathfrak{g}} e_g \otimes e_h$$

is an element of S_{bi} generating $j(\mathbf{Z}/n)$. The elements of the form $(a, 0) \wedge (b, 0)$ and $(0, a) \wedge (0, b)$, where $a, b \in M$, also belong to S_{bi} and generate $\bigwedge^2 (M \oplus 0)$ and $\bigwedge^2 (0 \oplus M)$, respectively. Thus $S/S_{\text{bi}} = 0$ and $B_0(F) = 0$. \square

Proposition 7.1 concludes the proof of Theorem 0.1, modulo the assumptions of Proposition 5.5 concerning the choice of k and L .

8. Field extension

It remains to note that according to [Lan, Chap. 5, § 4], one can construct infinitely many unramified extensions L/k with a given Galois group \mathfrak{g} . Thus all the decomposition groups \mathfrak{g}_v will be cyclic, and the hypotheses of Proposition 5.5 will be satisfied if $\mathfrak{g} = \mathbf{Z}/p \times \mathfrak{g}/p$ and k contains the p^2 -th roots of unity.

Let us give an explicit numerical example. Take $n = 4$, $k = \mathbf{Q}(\sqrt{-1})$, $L = k(\sqrt{13}, \sqrt{17})$. Then $\mathfrak{g} = \text{Gal}(L/k) = \mathbf{Z}/2 \times \mathbf{Z}/2$, but all the decomposition groups \mathfrak{g}_v are cyclic. Indeed, we have to check it for v lying over $p = 2, 13, 17$. But for $p = 2$ it is clear, and for $p = 13, 17$ it follows from the fact that the corresponding decomposition groups of $\mathbf{Q}(\sqrt{13}, \sqrt{17})/\mathbf{Q}$ are cyclic (cf. [ANT, Ex. 5]).

Note that the order of the resulting group F equals $n^{n^2+2n-1} = 2^{46}$.

Thus the assumptions of Proposition 5.5 are satisfied. Theorem 0.1 is completely proved. \square

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