ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

TARO ASUKE

Classification of riemannian flows with transverse similarity structures

Annales de la faculté des sciences de Toulouse 6e série, tome 6, nº 2 (1997), p. 203-227

<http://www.numdam.org/item?id=AFST_1997_6_6_2_203_0>

Annales de la Faculté des Sciences de Toulouse Vol. VI, n° 2, 1997

Classification of Riemannian flows with transverse similarity structures(*)

TARO ASUKE(1)

RÉSUMÉ. — Dans cet article, nous classifions les flots riemanniens avec structures transversalement similaires.

ABSTRACT. — In this paper, we give the classification of Riemannian flows with transverse similarity structures.

Introduction

The classification of transversely similar flows is done by Ghys [9] if the codimension of the flow is two and by Nishimori [13] in some particular cases. Thus we restrict ourselves in the case where the codimension of the flow is greater than two. We assume throughout this paper, the foliations and flows are assumed to be oriented and transversely oriented unless otherwise stated.

The classification is based on the following theorem.

THEOREM [1]. — *Let $(M, \mathcal{F})$ be a Riemannian foliation with a transverse similarity structure of a closed manifold M. Then the leaf space $\widehat{M}$ of the lifted foliation of the universal covering $\widetilde{M}$ of M is a simply connected Hausdorff manifold, and the mapping $\Delta : \widehat{M} \longrightarrow \mathbb{R}^q$ induced by the developing map is a covering map onto its image, denoted by X, and the natural*

(*) Reçu le 8 septembre 1995

(1) Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153 (Japan)
Current Address: Department of Mathematics, Faculty of Sciences, Hiroshima University, Kazamiyama 1-3-1, Higachi- Hiroshima 739 (Japan)
email: asuke@math.sci.hiroshima-u.ac.jp

projection p from $\widetilde{M}$ *to* $\widehat{M}$ *is a locally trivial fibration. Moreover, we have either*

1) $(M, \mathcal{F})$ *is an* $\left(\mathrm{Isom}^+(\mathbb{R}^q), \mathbb{R}^q\right)$*-foliation and we have* $X = \mathbb{R}^q$*, or*
2) $(M, \mathcal{F})$ *is a* $\left(\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0}), \mathbb{R}^q \backslash \mathbb{R}^{q_0}\right)$*-foliation and* X *is a connected component of* $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$*, where* $0 \leq q_0 < q$.

The classification of $\left(\mathrm{Isom}^+(\mathbb{R}^q), \mathbb{R}^q\right)$-flows is already done by Carrière [5], thus we treat in this paper $\left(\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0}), \mathbb{R}^q \setminus \mathbb{R}^{q_0}\right)$-flows, which are naturally Riemannian as we shall see later.

Before we state the main result, we make some definitions. First of all, we denote by $\mathbb{R}_+$ the positive real numbers.

DEFINITION .— *We denote by* $\mathrm{Sim}(\mathbb{R}^q)$ *the group of similarity transformations of* $\mathbb{R}^q$*, namely, we have*

$$\mathrm{Sim}(\mathbb{R}^q) = \left\{ g : \mathbb{R}^q \to \mathbb{R}^q \mid g(x) = r_g A_g x + v_g \right\},$$

where $x \in \mathbb{R}^q$*,* $r_g \in \mathbb{R}_+$*,* $A_g \in \mathrm{O}(q)$ *and* $v_g \in \mathbb{R}^q$.

We denote by $\mathrm{Sim}^+(\mathbb{R}^q)$ *the subgroup of* $\mathrm{Sim}(\mathbb{R}^q)$ *which consists of orientation preserving elements. We consider* $\mathrm{Sim}^+(\mathbb{R}^0) = \mathbb{R}_+$*. We put* $|g| = r_g$ *for an element* g *of* $\mathrm{Sim}^+(\mathbb{R}^q)$*. For a subgroup* G *of* $\mathrm{Sim}(\mathbb{R}^q)$*, we denote by* $|G|$ *the subgroup of* $\mathbb{R}_+$ *defined by* $|G| = \left\{ |g| \mid g \in G \right\}$.

Now we define a subgroup $\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0})$ of $\mathrm{Sim}^+(\mathbb{R}^q)$ by putting

$$\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0}) = \left\{ g \in \mathrm{Sim}^+(\mathbb{R}^q) \mid g(\mathbb{R}^{q_0}) = \mathbb{R}^{q_0} \right\}.$$

We denote by $\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0})$ the identity component of $\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0})$, and notice then that we have

$$\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0}) \subset \mathrm{Sim}(\mathbb{R}^{q_0}) \times \mathrm{O}(q_1)$$
$$\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0}) = \mathrm{Sim}^+(\mathbb{R}^{q_0}) \times \mathrm{SO}(q_1)$$

where $q_1 = q - q_0$.

If $q_0 = 0$, we use the following notation, namely, we put

$$\mathrm{CO}^+(q) = \mathrm{Sim}^+(\mathbb{R}^q, \{0\}).$$

Finally by abuse of notation, we often denote by $rAx+v$ an element g of $\mathrm{Sim}^+(\mathbb{R}^q)$ such that $g(x)=rAx+v$.

Let M be an n-dimensional closed manifold and $\mathcal{F}$ a q-codimensional foliation of M. We say $(M,\mathcal{F})$ is transversely similar if $(M,\mathcal{F})$ is a $(\mathrm{Sim}^+(\mathbb{R}^q),\mathbb{R}^q)$-foliation. Then there is a homomorphism from $\pi_1(M)$ to $\mathrm{Sim}^+(\mathbb{R}^q)$, which is called the holonomy homomorphism. We denote by Γ the image of the holonomy homomorphism and by H the closure of Γ in $\mathrm{Sim}^+(\mathbb{R}^q)$. We denote the identity component of H by H_0 and put $\Gamma_0=\Gamma\cap H_0$.

Now we consider $\mathbb{R}^q=\mathbb{R}^{q_0}\times\mathbb{R}^{q_1}$. For a point x in $\mathbb{R}^q$, we write $x=(x_0,x_1)$ according to the decomposition $\mathbb{R}^q=\mathbb{R}^{q_0}\times\mathbb{R}^{q_1}$. We introduce a metric g_H on $X=\mathbb{R}^q\setminus\mathbb{R}^{q_0}$ by the following formula, namely,

$$g_H(x)=\frac{1}{\|x_1\|^2}\,g_0(x)\,,$$

where g_0 denotes the Euclidean metric and $\|\cdot\|$ denotes the Euclidean norm.

Notice that $\mathrm{Sim}(\mathbb{R}^q,\mathbb{R}^{q_0})$ acts on $\mathbb{R}^q\setminus\mathbb{R}^{q_0}$ equipped with the metric g_H as a group of isometries, and hence $(\mathrm{Sim}^+(\mathbb{R}^q,\mathbb{R}^{q_0})\,,\,\mathbb{R}^q\setminus\mathbb{R}^{q_0})$-flows are naturally Riemannian.

Now we state the main result of this paper.

THEOREM A.— *Let $(M,\mathcal{F})$ be a Riemannian flow of a closed manifold M of codimension greater than 2. Suppose that $(M,\mathcal{F})$ is a $(\mathrm{Sim}^+(\mathbb{R}^q,\mathbb{R}^{q_0})\,,\,\mathbb{R}^q\setminus\mathbb{R}^{q_0})$-flow, then M fibres over S^1 and we have the following cases:*

1) *All the orbits of $\mathcal{F}$ are closed and form the fibres of a Seifert fibration. In this case $(M,\mathcal{F})$ is a $(\mathrm{CO}^+(q)\,,\,\mathbb{R}^q\setminus\{0\})$-flow, and $\Gamma_0=\{1\}$.*
2) *We have $|H_0|=\mathbb{R}_+$, then $(M,\mathcal{F})$ is a $(\mathrm{CO}^+(q)\,,\,\mathbb{R}^q\setminus\{0\})$-flow. The flow $\mathcal{F}$ is finitely covered by the flow $(N_g,\mathcal{F}_g)$ defined below with* $\log|g|$ *being irrational. In particular, $\pi_1(M)$ is a finite extension of Γ_0, which is isomorphic to $F\times\mathbb{Z}^2$, where F is a finite Abelian group.*
3) *We have $|H_0|=\{1\}$, then each orbit closure is naturally equipped with a transverse $(\mathrm{Isom}^+(\mathbb{R}^d)\,,\,\mathbb{R}^d)$ structure, where d is the dimension of the orbit closure minus 1. In this case, we have $q_0\le 2$ and there are following cases:*

   3a) $q_0=0$: the fibres of the fibration equipped with the restricted flow is obtained as the suspension of an isometry of a spherical manifold.

3b) $q_0 \neq 0$*: the ambient manifold M is finitely covered by a T^{q_0+1}-bundle over $S^{q-q_0-1} \times S^1$, whose fibre is equipped with a fixed irrational linear flow. In particular, the dimension of orbit closures are the same and at most* 3.

In the cases 1) and 3), the fibration $M \to S^1$ is naturally defined so that each orbit is contained in a fibre.

In the cases 2) and 3b), there is no closed orbit and the holonomy homomorphism ϕ is injective. Moreover $\pi_1(M)$ acts on $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$ freely via ϕ.

Remark.— As we noticed above $(\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0}), \mathbb{R}^q \setminus \mathbb{R}^{q_0})$-flows are naturally Riemannian.

The flows $(N_g, \mathcal{F}_g)$ in Theorem A are constructed as follows.

DEFINITION .— *Let g be an element of $\mathrm{CO}^+(q)$ such that $1 < |q| < e$. Then g induces an automorphism $\gamma(g)$ on $S^1 \times S^{q-1} = (\mathbb{R}^q \setminus \{0\})/\langle e \rangle$. We simply take the suspension of g, namely, we put $\widetilde{N}_g = \mathbb{R} \times S^1 \times S^{q-1}$ and put $\widetilde{\mathcal{F}}_g = \{\mathbb{R} \times \{p\} \mid p \in S^1 \times S^{q-1}\}$, respectively. Now we put $\lambda(g)(t,p) = (t-1\,,\,\gamma(g)(p))$ and define $(N_g, \mathcal{F}_g)$ as the quotient of $(\widetilde{N}_g, \widetilde{\mathcal{F}}_g)$ by $\langle \lambda(g) \rangle$.*

Note that if H_N denotes the closure of the holonomy group of $(N_g, \mathcal{F}_g)$ then $|H_N| = \mathbb{R}_+$ holds if and only if $\log |g|$ is irrational.

As we have already mentioned in [1], we have the following corollary.

COROLLARY B.— *Let $(M, \mathcal{F})$ be a transversely similar flow of codimension q, $q > 2$. Suppose that all the orbits of $\mathcal{F}$ are dense, then $(M, \mathcal{F})$ is an $(\mathrm{Isom}^+(\mathbb{R}^q), \mathbb{R}^q)$-flow and differentiably conjugate to an irrational linear flow on the torus.*

Theorem A is a generalization of the classification of transversely hyperbolic flows by Epstein [6]. For the detail see Theorem 4.2.

Remark.— Noticing that the group of similarity transformations $\mathrm{Sim}^+(\mathbb{R}^q)$ is naturally contained in the group of conformal transformations of the sphere, we can consider transversely flat conformal flows whose holonomy groups are contained in $\mathrm{Sim}^+(\mathbb{R}^q)$. In this case, it turns out that we need to add only $(\mathrm{SO}(q+1)\,,\, S^q)$-flows to the list of above theorems (see [1]). This case is classified by Carrière [5].

This paper is organized as follows. In the first section, we give some definitions and the proof of the first part of Theorem A. In the section 2, we treat the case $q_1 = q - q_0 \neq 2$ and the group H_0 is studied. In particular we will see that we have either $|H_0| = \{1\}$ or $|H_0| = \mathbb{R}_+$, and the latter case is extensively studied. The case $q_1 = 2$ is treated in the section 3. In the section 4, the case $|H_0| = \{1\}$ is again considered and it is shown that there is a restriction concerning dimension of the orbit closures. Then some examples are given in the last section.

1. Prelimenaries

Let M be an n-dimensional closed manifold and $\mathcal{F}$ a q-dimensional foliation of M. We say $(M, \mathcal{F})$ is transversely similar if $(M, \mathcal{F})$ is a $(\mathrm{Sim}^+(\mathbb{R}^q), \mathbb{R}^q)$-foliation. We denote by $\widetilde{M}$ the universal covering of M and by $\widetilde{\mathcal{F}}$ the lifted foliation of $\mathcal{F}$ to $\widetilde{M}$. Then there is a submersion $D : \widetilde{M} \to \mathbb{R}^q$ which is called the developing map and the holonomy homomorphism $\phi : \pi_1(M) \to \mathrm{Sim}^+(\mathbb{R}^q)$. These mappings satisfy the equivariant condition such that

$$D(\gamma x) = \phi(\gamma)\, D(x)\,,$$

where $x \in \widetilde{M}$ and $\gamma \in \pi_1(M)$. The image of $\pi_1(M)$ by ϕ is called the holonomy group and denoted by Γ.

We denote by H the closure of Γ in $\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0})$ and by H_0 the identity component of H. We put $\Gamma_0 = \Gamma \cap H_0$. It is well-known that this group H_0 is Abelian (See [6], [12] or [4]).

If we denote by $\widehat{M}$ the leaf space of the lifted foliation $(\widetilde{M}, \widetilde{\mathcal{F}})$ and by p the natural projection from $\widetilde{M}$ to $\widehat{M}$, then the developing map D factors as $\Delta \circ p$. Since the action of $\pi_1(M)$ on $\widetilde{M}$ preserves $\widetilde{\mathcal{F}}$, we have a natural action of $\pi_1(M)$ on $\widehat{M}$. Then by definition, the mapping Δ satisfies the same equivariant condition as D.

Now we begin the proof of Theorem A. First we show that under the assumption M fibers over S^1.

DEFINITION 1.1.— *We define the radial vector field $\overline{R}$ on $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$ as follows. We consider $\mathbb{R}^q \setminus \mathbb{R}^{q_0} = \mathbb{R}^{q_0} \times (\mathbb{R}^{q_1} \setminus \{0\})$, where $q_1 = q - q_0$, and put*

$$\overline{R}_x = x_1\,, \quad \textit{where } x = (x_0, x_1)\,.$$

Note that $\overline{R}$ is invariant under the action of $\mathrm{Sim}(\mathbb{R}^q, \mathbb{R}^{q_0})$.

PROPOSITION 1.2.— *Let* $(M, \mathcal{F})$ *be a* $(\mathrm{Sim}(\mathbb{R}^q, \mathbb{R}^{q_0}), \mathbb{R}^q \backslash \mathbb{R}^{q_0})$*-foliation Then* M *fibres over* S^1.

Proof.— We define a 1-form $\overline{\omega}$ by the formula

$$\overline{\omega} = g_H(\overline{R}, \,\cdot\,).$$

Then $\overline{\omega}$ is invariant under the action of $\mathrm{Sim}(\mathbb{R}^q, \mathbb{R}^{q_0})$ and hence induces a 1-form ω on M.

An easy computation shows that $\overline{\omega}$ is closed and non-vanishing. Consequently, the form ω is also closed and non-vanishing. Then the proposition follows from the theorem of Tischler [14]. □

We refer this fibration as $\xi : M \to S^1$. Note that the group of the period of ω is equal to

$$|\Gamma| = \{|g| \mid g \in \Gamma\}.$$

Now we identify following spaces in the natural way, namely,

$$\mathbb{R}^q \setminus \mathbb{R}^{q_0} \cong \mathbb{R}^{q_0} \times (\mathbb{R}^{q_1} \setminus \{0\}) \cong \mathbb{R}^{q_0} \times S^{q_1} \times \mathbb{R}_+.$$

Then $\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0}) = \mathrm{Sim}^+(\mathbb{R}^{q_0}) \times \mathrm{SO}(q_1)$ acts on $\mathbb{R}^{q_0} \times S^{q_1} \times \mathbb{R}_+$ by the formula

$$(g_1, g_2)(x, y, t) = (g_1(x),\, g_2(y),\, |g_1|t),$$

where (g_1, g_2) is an element of $\mathrm{Sim}^+(\mathbb{R}^{q_0}) \times \mathrm{SO}(q_1)$.

From now on, by taking a double covering if necessary, we assume that the holonomy group Γ is contained in $\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0})$.

We will make use of the following proposition (see Bröcker and Tom Dieck [2]). We leave the proof to the reader.

PROPOSITION 1.3.— *Let* G *be a Lie subgroup of* $\mathrm{SO}(n)$ *which is a torus. If we consider* $G = \mathbb{R}^d/\mathbb{Z}^d$*, then by taking conjugation, we can find* $a_1, \ldots, a_\ell \in \mathbb{Z}^d$ *such that any element* $x \in G \cong T^d = \mathbb{R}^d/\mathbb{Z}^d$ *is uniquely represented as*

$$x = R(a_1, x) \oplus \cdots \oplus R(a_\ell, x) \oplus 1 \oplus \cdots \oplus 1,$$

where

$$R(a_i, x) = \begin{pmatrix} \cos 2\pi\sqrt{-1}\,\langle a_i, x\rangle & -\sin 2\pi\sqrt{-1}\,\langle a_i, x\rangle \\ \sin 2\pi\sqrt{-1}\,\langle a_i, x\rangle & \cos 2\pi\sqrt{-1}\,\langle a_i, x\rangle \end{pmatrix}$$

and $\langle\,\cdot\,,\,\cdot\,\rangle$ *denotes the Euclidean inner product.*

If we have $q_1 = q - q_0 \neq 2$, then $X = \mathbb{R}^q \setminus \mathbb{R}^{q_0}$ is simply connected and the projection p from $\widetilde{M}$ to $\widehat{M}$ coincides with the developing map D and $\widehat{M}$ is isomorphic to a connected component of $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$. Here if we have $q_0 = q - 1$, then we consider one of the connected components. Then the H_0-orbit of a point x in X corresponds to an orbit closure. However, if we have $q_1 = 2$, then $\widehat{M}$ is the universal covering space of $\mathbb{R}^q \setminus \mathbb{R}^{q-2}$ and Δ is the covering map onto $\mathbb{R}^q \setminus \mathbb{R}^{q-2}$. In particular X is no longer simply connected and we must take care. So we divide the proof into the two cases according to whether q_1 is equal or not equal to 2.

2. The case where q_1 is not equal to 2

First we show the following lemma, which will play a crucial role in the classification.

LEMMA 2.1.— *If we have $q_0 = 0$, then H_0 is contained in $\mathrm{CO}^+(q) = \mathrm{CO}^+(q_1)$. If we have $q_0 > 0$, then as a subgroup of $\mathrm{Sim}^+(\mathbb{R}^{q_0}) \times \mathrm{SO}(q_1)$ we have*

$$H_0 = \mathbb{R}^{q_0} \times H_1 ,$$

where H_1 is a closed, connected subgroup defined by $H_1 = H_0 \cap \mathrm{SO}(q_1)$ and $\mathbb{R}^{q_0}$ denotes the full group of parallel translations of $\mathbb{R}^{q_0}$.

Proof.— It suffices to show the lemma in the case where we have $q_0 > 0$. First note that Γ cannot be contained in $\mathrm{Isom}(\mathbb{R}^q)$ because X/Γ equipped with the quotient topology is compact. In fact, if we suppose the contrary, then any element g of Γ satisfies $|g| = 1$. So the function d on $X = \mathbb{R}^q \setminus \mathbb{R}^{q_0}$ defined by the formula

$$d(x) = ||x_1||$$

is invariant under the action of Γ, and thus defines a continuous unbounded function on M. This is a contradiction. So possibly by changing the model $\mathbb{R}^q$ by a parallel translation, we can find an element γ_0 of Γ of the form $\gamma_0(x) = r_0 A_0 x$, where $r_0 < 1$ and $A_0 \in \mathrm{SO}(q)$.

Let h be an element of H_0 satisfying $h(x) = r\,Ax + v$. We define elements $h_{v,n}$ of H_0 by the formula

$$h_{v,n} = \left[h, \gamma_0^n\right] = h\gamma_0^n h^{-1} \gamma_0^{-n} ,$$

then it is easy to see that we can find an increasing sequence $I = \{i_n\}$ of positive integers such that h_{v,i_n} converges to the element h_v of H_0 given by

$$h_v(x) = x + v\,.$$

Now we define

$$\begin{aligned} V &= \big\{v \in \mathbb{R}^q \mid \text{there is an element } h \in H_0 \text{ of the form } h(x) = r\,Ax + v\big\} \\ &= \big\{v \in \mathbb{R}^q \mid \text{there is an element } h \in H_0 \text{ of the form } h(x) = x + v\big\}\,. \end{aligned}$$

Then it is easy to see that V is a closed, connected $\mathbb{Z}$-module. Thus V is a vector subspace of $\mathbb{R}^q$.

The same argument shows that we have

$$\begin{aligned} &\big\{v \in \mathbb{R}^q \mid \text{there is an element } h \in H \text{ of the form } h(x) = r\,Ax + v\big\} \\ &\quad = \big\{v \in \mathbb{R}^q \mid \text{there is an element } h \in H \text{ of the form } h(x) = x + v\big\}\,. \end{aligned}$$

We denote by V_1 this space, which is by definition H-invariant. Notice that the identity component of V_1 coincides with V.

Now we claim that we have $V = V_1$. Suppose the contrary and fix a point $p = (0, x_0) \in X = \mathbb{R}^{q_0} \times \big(\mathbb{R}^{q_1} \setminus \{0\}\big)$. Then we can find a point $q = (u, x_0) \in V_1 \times \{x_0\} \setminus V \times \{x_0\}$ which is nearest to p with respect to g_H. Notice that by definition of V_1 we can find an element h of H given by the equation $h(x) = x + u$.

We denote by d_H the distance determined by g_H, which is H-invariant. Since $\gamma_0(u) \in \mathbb{R}^{q_0}$ holds we have

$$\begin{aligned} d_H\big(p\,,\, \gamma_0 h \gamma_0^{-1}(p)\big) &= d_H\big(p\,,\, p + r_0\, A_0 u\big) \\ &< d_H\big(p\,,\, p + u\big) = d_H(p, q)\,. \end{aligned}$$

It follows that we have $\gamma_0 h \gamma_0^{-1}(p) = p$ because we have $r_0\, A_0 u \in V_1$. Thus we have $u = 0$, which is a contradiction. Therefore we have $V = V_1$ and V is H-invariant.

Now we show that V coincides with $\mathbb{R}^{q_0}$. Suppose the contrary and decompose $\mathbb{R}^{q_0}$ as $V \oplus W$, where W is the orthogonal complement of V with respect to the Euclidean metric. It is straightforward that the action of H preserves the decomposition

$$\mathbb{R}^q = V \oplus W \oplus \mathbb{R}^{q_1}\,.$$

Then for a point x in $X \subset \mathbb{R}^q$, we write $x = (x_V, x_W, x_1)$ according to this decomposition and we define a function on $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$ by

$$f(x) = \frac{\|x_W\|}{\|x_1\|}.$$

It is easy to see that the function f is invariant under the action of $\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q_0})$ and we have a unbounded continuous function on M. This is a contradiction.

Now the commutativity of H_0 and the fact that we have $\mathbb{R}^{q_0} \neq \{0\}$ show that any element of H_0 is of the form $(x+v, B)$. It follows easily that if we put $H_1 = H_0 \cap \mathrm{SO}(q_1)$, then we have

$$H_0 = \mathbb{R}^{q_0} \times H_1 .$$

Now it is clear that H_1 is connected and closed. □

Note that in the case where $q_0 > 0$ holds, H_0 is contained in $\mathrm{Isom}^+(\mathbb{R}^q)$.

If all the orbits are closed, whe have the following. Recall that we have a fibration $p : \widetilde{M} \to \widehat{M}$, whose fibre is either $\mathbb{R}$ or S^1.

PROPOSITION 2.2.— *If all the orbits are closed, then $(M, \mathcal{F})$ is a $(\mathrm{CO}^+(q), \mathbb{R}^q \setminus \{0\})$-flow whose orbits form the fibres of a Seifert fibration. In this case, each orbit is contained in a fiber of ξ.*

Proof.— It is well-known that if all the orbits are closed, then the orbits of $\mathcal{F}$ form the fibres of a Seifert fibration. The lemma 2.1 shows that we have $\mathbb{R}^{q_0} = \{0\}$ and H is contained in $\mathrm{CO}^+(q)$.

Since hall the orbits are closed, the group

$$|H| = \{|h| \mid h \in H\}$$

is infinite cyclic, say $\langle r_0 \rangle$, where $1 < r_0$.

Now we identify S^1 with $\mathbb{R}_+/|H|$ and denote by Π the natural projection from $\mathbb{R}_+$ to S^1. For any point x in M, we choose a lift $\widetilde{x}$ of x in $\widetilde{M}$ and define

$$\alpha(x) = \Pi(\|D(\widetilde{x})\|) .$$

The discreteness of $|H|$ shows that α is well-defined and coincides with ξ defined above. Thus each orbit is contained in a fibre of ξ. □

Now we look at the group $|H_0|$, then we have either $|H_0| = \mathbb{R}_+$ or $|H_0| = \{1\}$.

First we assume we have $|H_0| = \mathbb{R}_+$, then Lemma 2.1 shows that we have $q_0 = 0$ and there is no closed orbit. It follows easily that Γ acts on $S^{q-1} \times \mathbb{R}_+$ freely, and that the holonomy homomorphism ϕ is injective.

We show that the following theorem.

THEOREM 2.3.— *Let $(M, \mathcal{F})$ be a $\left(\mathrm{CO}^+(q), \mathbb{R}^q \setminus \{0\}\right)$-flow with $|H_0| = \mathbb{R}_+$. Then by changing the similarity structure without changing the flow, we may suppose that the radial vector field induces a non-vanishing global vector field on M which is tangent to the orbit closures and transverse to orbits. Moreover we have $H_0 = \mathbb{R}_+ \times K_0$ and $H = \mathbb{R}_+ \times K$, where K_0 and K denote the kernel of the homomorphism $|\cdot|$, and $\mathbb{R}_+$ acts on $\mathbb{R}^q \setminus \{0\}$ as a group of multiplications.*

Proof.— Let K be as in the statement. Then K is compact and hence admits the Haar measure μ.

Let σ be a section of $|\cdot| : H \to \mathbb{R}_+$. We define a vector field X_0 on $S^{q-1} \times \{1\}$ by the formula

$$X_0(x) = \frac{\mathrm{d}}{\mathrm{d}t}\,\sigma(t)(x)\bigg|_{t=1}$$

Then X_0 is tangent to the H_0-orbits.

Now we define a vector field X on S^{q-1} by the formula

$$X(x) = \int_{k \in K} k^{-1}{}_* X_0(kx)\,\mathrm{d}\mu\,.$$

Then X is invariant under the action of K.

We extend X to $S^{q-1} \times \mathbb{R}_+$ by the formula

$$X(x,t) = g_* X\left(g^{-1}(x,t)\right),$$

where g is an element of H with $|g| = t$.

The invariance of X under the action of K shows that X is well-defined. Moreover we see that for an element h of H we have

$$\begin{aligned} h_*X(x,t) &= h_* g_* X\big(g^{-1}(x,t)\big) \\ &= (hg)_* X\big((hg)^{-1}h(x,t)\big) = X\big(h(x,t)\big)\,. \end{aligned}$$

That is, X is an H-invariant vector field.

Let φ_t be the 1-parameter transformation group associated with X, where $t \in \mathbb{R}_+$ a multiplicative group. We define a diffeomorphism $\rho : \mathbb{R}^q \setminus \{0\} \to \mathbb{R}^q \setminus \{0\}$ by the formula

$$\rho(x,t) = \varphi_t(x,1)\,,$$

where we consider $\mathbb{R}^q \setminus \{0\} = S^{q-1} \times \mathbb{R}_+$ as usual.

Then clearly we have $\rho_*^{-1} X = \overline{R}$, the radial vector field, and that $H(x,t) = Kx \times \mathbb{R}_+$ for $(x,t) \in S^{q-1} \times \mathbb{R}_+$. By the invariance of X we have for an element h of H that

$$\begin{aligned} \rho^{-1} \circ h \circ \rho(x,t) &= \rho^{-1} \circ h\big(\varphi_t(x,1)\big) = \rho^{-1}\,\big(\varphi_t\big(h(x,1)\big)\big) \\ &= \Big(\varphi_{|h|^{-1}t^{-1}} \circ \varphi_t\big(h(x,1)\big)\,,\ |h|t\Big) = \Big(\varphi_{|h|^{-1}}\big(h(x,1)\big)\,,\ |h|t\Big)\,. \end{aligned}$$

Since $X(x,1)$ can be regarded as an element of $\mathfrak{co}(q)$, we see that φ_t is an element of $\mathrm{CO}^+(q)$ for each t. Thus $\rho^{-1}H\rho$ acts on $\mathbb{R}^q \setminus \{0\}$ as a subgroup of $\mathrm{CO}^+(q)$.

We consider φ as a section of $|\cdot|$, then we see that from the above calculation that $\varphi(\mathbb{R}_+)$ acts on $\mathbb{R}^q \setminus \{0\}$ as a group of multiplications. Notice also that if we have $|h| = 1$ then we have $\rho^{-1}h\rho = h$ as elements of $\mathrm{CO}^+(q)$.

Finally, we choose a lift $\widetilde{X}$ of X to $\widetilde{M}$ so that $\widetilde{X}$ induces a vector field Y on M. Since X is tangent to H_0-orbits, we see that Y is tangent to orbit closures. It is clear that Y is transverse to orbits of $\mathcal{F}$. This completes the proof. □

We go back to the proof of Theorem A. Since Γ_0 is dense in $\mathbb{R}_+$, we can find two elements g_1 and g_2 such that $\big\langle |g_1|\,,\,|g_2| \big\rangle$ is dense in $\mathbb{R}_+$. Here we assume that $1 < |g_1| < |g_2|$ holds.

Since H_0 is commutative and Γ acts freely, we see from Proposition 1.3 that we have

$$H_0 \subset \mathbb{R}_+ \times \big(\mathrm{SO}(2) \oplus \cdots \oplus \mathrm{SO}(2)\big)\,.$$

Thus we can write

$$g_2 = |g_2|\big(R(t_1) \oplus \cdots \oplus R(t_\ell)\big)\,,$$

where

$$R(t_i) = \begin{pmatrix} \cos 2\pi\sqrt{-1}\,t_i & -\sin 2\pi\sqrt{-1}\,t_i \\ \sin 2\pi\sqrt{-1}\,t_i & \cos 2\pi\sqrt{-1}\,t_i \end{pmatrix}$$

We define an automorphism ψ of $\mathbb{R}^q \setminus \{0\}$ by the formula

$$\psi(x,r) = \exp\left(\frac{\log r}{\log|g_2|}\right)\left(R\Big(-\frac{\log r}{\log|g_2|}t_1\Big) \oplus \cdots \oplus R\Big(-\frac{\log r}{\log|g_2|}t_\ell\Big)\right)x\,.$$

Let $h = |h|\big(R(s_1) \oplus \cdots \oplus R(s_\ell)\big)$ be an element of H_0, then routine computations show that we have

$$\begin{aligned}&\psi \circ h \circ \psi^{-1}(x,r) = \\ &= r\exp\left(\frac{\log|h|}{\log|g_2|}\right)\left(R\Big(s_1 - \frac{\log|h|}{\log|g_2|}t_1\Big) \oplus \cdots \oplus R\Big(s_\ell - \frac{\log|h|}{\log|g_2|}t_\ell\Big)\right)x\,.\end{aligned}$$

Thus we have $\psi H_0 \psi^{-1} \subset CO^+(q)$. In particular, by putting $h = g_2$ we have

$$\psi \circ g_2 \circ \psi^{-1}(x,r) = erx\,.$$

Now let γ_1 and γ_2 be elements of $\pi_1(M)$ such that we have $\phi(\gamma_1) = g_1$ and $\phi(\gamma_2) = g_2$, respectively. Then we consider a fibration

$$\psi \circ D : \widetilde{M} \to \mathbb{R}^q \setminus \{0\}\,.$$

By taking quotient corresponding to $\langle\gamma_2\rangle$ we have a fibration

$$\widetilde{M}/\langle\gamma_2\rangle \longrightarrow (\mathbb{R}^q \setminus \{0\})/\langle e\rangle = S^1 \times S^{q-1}\,.$$

It is easy to see that $N = \widetilde{M}/\langle\gamma_1,\gamma_2\rangle$ is compact, we see from the theorem of Fried [8] that the flow on N is obtained as a suspension. Now it is clear that the monodromy is given by g_1, thus we showed Theorem A in the case $|H_0| = \mathbb{R}_+$ holds.

In this case we can describe Γ_0 more explicitly. We choose an orbit L of $\mathcal{F}$, which is not closed. Then $\overline{L}$ is a smoothly embedded torus equipped with an irrational linear flow from $\mathcal{F}$. We denote by $\widehat{L}$ one of the lifts of $\overline{L}$ to $\widetilde{M}$. Then we can choose an embedded torus T_L in $\overline{L}$ which is transverse to

$\mathcal{F}\restriction_{\overline{L}}$, so that we have $\widehat{L} \cong \mathbb{R} \times \widehat{T_L}$, where $\mathbb{R}$ corresponds to the orbits and $\widehat{T_L}$ denotes one of the lifts of T_L contained in $\widehat{L}$. It follows that D restricted on $\widehat{T_L}$ is a diffeomorphism onto its image, denoted by $\widehat{C_L}$ and $\widehat{T_L}$ is a covering space of T_L. Since we have $H_0 = \mathbb{R}_+ \times K_0$ and $\widehat{C_L}$ is determined from an orbit closure, we have $\widehat{C_L} = \widehat{U_L} \times \mathbb{R}_+$, where $\widehat{U_L}$ is an embedded torus in S^{q-1} defined by $\widehat{U_L} = \widehat{C_L} \cap (S^{q-1} \times \{1\})$.

In particular there are elements $g_1, \ldots, g_k$ and h of Γ_0 such that $g_1, \ldots, g_k$ generate the covering transformations of $\widehat{T_L} \to T_L$, and h induces the holonomy of $(\overline{L}, \mathcal{F}\restriction_{\overline{L}})$. Here we may assume the group $\langle g_1, \ldots, g_{k-1}\rangle$ is contained in $\mathrm{SO}(q)$ and finite, and that $|g_k| > 1$ holds.

Now we claim that we have $\Gamma_0 = \langle g_1, \ldots, g_k, h\rangle$. In fact, lef f be an element of Γ_0, then we have

$$f\restriction_{\widehat{C_L}} = g_1^{a_1} g_2^{a_2} \cdots g_k^{a_k} h^b \restriction_{\widehat{C_L}},$$

where $a_1, \ldots, a_k$ and b are integers. Since Γ_0 acts freely, we have $f = g_1^{a_1} \cdots g_k^{a_k} h^b$.

Noticing that there is an injection from Γ/Γ_0 to $H/H_0 = K/K_0$, which is finite, we see that Γ and hence $\pi_1(M)$ is a finite extension of $F \times \mathbb{Z}^2$, where F is a finite Abelian group.

Before we end the case where $|H_0| = \mathbb{R}_+$, we make one more remark. The existence of the flow $(N_q, \mathcal{F}_g)$ shows that there is no dimensional restriction like the case where we have $|H_0| = \{1\}$ and $q_0 > 0$. However, if we assume the singular foliation $\overline{\mathcal{F}}$ defined by the orbit closures is regular, we have the following.

THEOREM 2.4.— *Suppose that $|H_0| = \mathbb{R}_+$ and $\overline{\mathcal{F}}$ is regular, then we have either $K_0 = \{1\}$ or $K_0 = S^1$, where K_0 is defined as in Theorem 2.3. In particular the dimension of orbit closures is either 2 or 3, and each orbit closure inherits a natural transverse similarity structure from $(M, \mathcal{F})$.*

Proof.— Consider the foliation $\mathcal{K}$ defined by the action of K_0 on S^{q-1}. Since $\overline{\mathcal{F}}$ is regular, $\mathcal{K}$ is also regular. On the other hand it is known that orbit closures are tori. Thus the leaves of $\mathcal{K}$ consist of either points or tori. It follows now from the theorem of Lu [11] that we have either $K_0 = \{1\}$ or $K_0 = S^1$. The last statement follows from Theorem 2.3 □

Now we treat the case where we have $|H_0| = \{1\}$. Noticing that in this case we have $H_0 \subset \mathrm{Isom}^+(\mathbb{R}^d)$ for some positive integer d, we see that the group H_0 has the following structure.

LEMMA 2.5.— *If* $|H_0| = \{1\}$ *holds, then we have*

$$H_0 = \mathbb{R}^{q_0} \times H_0^{\pitchfork},$$

where $H_0^{\pitchfork} = H_0 \cap \mathrm{SO}(q_1)$ *and* $\mathbb{R}^{q_0}$ *acts on* $\mathbb{R}^{q_0}$ *as a group of parallel translations.*

Notice that we have also $H_0^{\pitchfork} = \mathrm{pr}_2(H_0)$, where pr_2 is the projection pr_2 from $\mathrm{Isom}^+(\mathbb{R}^{q_0}) \times \mathrm{SO}(q - q_0)$ to $\mathrm{SO}(q - q_0)$.

Now we show the following.

PROPOSITION 2.6.— *If we have* $|H_0| = \{1\}$, *then* M *fibres over* S^1 *so that each orbit of* $\mathcal{F}$ *is contained in a fibre. The universal covering of the fibre is* $\mathbb{R} \times \mathbb{R}^{q_0} \times S^{q-q_0-1}$ *and naturally inherits a transverse* $(\mathbb{R}^{q_0} \times \mathrm{SO}(q-q_0), \mathbb{R}^{q_0} \times S^{q-q_0-1})$ *structure. Each connected component of the lift of each orbit closure projects down to* $\mathbb{R}^{q_0} \times T^d$, *where* T^d *is a torus contained in* S^{q-q_0-1}. *The collection of such tori forms a singular foliation* $\mathcal{F}^{\pitchfork}$ *of* S^{q-q_0-1} *defined by the action of the torus* $H_0^{\pitchfork}$.

Remark.— Later we will show that q_0 and d are under a strong restriction.

Proof.— In this case H_0 and Γ_0 are contained in $\mathrm{Isom}^+(\mathbb{R}^q)$ and the group

$$|H| = \{|h| \mid h \in H\}$$

is infinite cyclic, say $\langle r_0 \rangle$, where $r_0 > 1$.

Now we define a mapping $\alpha : M \to S^1$ as follows. First we identify S^1 with $\mathbb{R}_+/|H|$, and denote by Π the natural projection from $\mathbb{R}_+$ to S^1. Let x be a point in M and choose a lift $\widetilde{x}$ of x in $\widetilde{M}$. We write $D(\widetilde{x}) = (z_0, z_1)$, and put

$$\alpha(x) = \Pi(\|z_1\|).$$

It is easy to see by using the same argument as in the proof of Proposition 2.2 that the mapping α is a well-defined and coincides with ξ, whose fibre is equal to $\pi(\mathbb{R} \times (\mathbb{R}^{q_0} \times S^{q-q_0-1}))$ and union of orbits of $\mathcal{F}$.

Now we choose an orbit L of $\mathcal{F}$ and let T be its closure. We denote by $\widetilde{T}$ one of the connected components of the lift of T to the universal covering of M. The projected image $\widehat{T}$ of $\widetilde{T}$ in X by the developing map $D = p$ is contained in $\mathbb{R}^q \times S^{q-q_0-1}$. Noticing that $\mathbb{R} \times \widehat{T}$ covers a torus, Lemma 2.1 shows that $\widehat{T}$ is homeomorphic to $\mathbb{R}^q \times T^q$, where T^d is a torus in S^{q-q_0-1}. The correspondence between T and $\widehat{T}$ shows that the collection $\mathcal{F}^{\pitchfork}$ of such tori as $\widehat{T}$ forms a singular Riemannian foliation of S^{q-q_0-1}.

It is clear from the definition that $H_0^{\pitchfork}$ is a torus, which acts on S^{q-q_0-1} faithfully. Then $(S^{q-q_0-1}, \mathcal{F}^{\pitchfork})$ is obtained as the orbits of the action of $H_0^{\pitchfork}$. □

In the case where we have $|H_0| = \{1\}$ and $q_0 > 0$, there are no closed orbits and we can repeat the same argument as in the case where we have $|H_0| = \mathbb{R}_+$ to deduce that Γ acts freely on $\mathbb{R}^q \setminus \mathbb{R}^{q_0}$ and the holonomy homomorphism ϕ is injective.

3. The case where q_1 is equal to 2

In this section we assume that we have $q_1 = 2$, namely, $\mathbb{R}^{q_0} = \mathbb{R}^{q-2}$ holds. In this case, we take the universal covering of $X = \mathbb{R}^q \setminus \mathbb{R}^{q-2}$ and then follow the argument in the case where q_1 is not equal to 2.

We denote by R_θ the rotation by angle θ around $\mathbb{R}^{q-2}$. Notice that R_θ commutes with any element of $\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q-2})$. Now we put

$$\mathbb{H}^{q-1} = \left\{x = (x_1, \dots, x_{q-1}, x_q) \in \mathbb{R}^q \mid x_{q-1} > 0\,,\ x_q = 0\right\},$$

and define a diffeomorphism $h : \mathbb{H}^{q-1} \times S^1 \to \mathbb{R}^q \setminus \mathbb{R}^{q-2}$ by the formula

$$h(x,t) = R_{2\pi t}x\,.$$

Then we identify the universal covering space of $\mathbb{R}^q \setminus \mathbb{R}^{q-2}$ with $\mathbb{H}^{q-1} \times \mathbb{R}$ and the lift of $\mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q_0})$ with $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$.

The action of $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$ on $\mathbb{H}^{q-1} \times \mathbb{R} = \mathbb{R}^{q-2} \times \mathbb{R}_+ \times \mathbb{R}$ is given by the formula

$$(h,\theta)(x,r,t) = \big(h(x),\ |h|r,\ t+\theta\big)\,,$$

where (h,θ) is an element of $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$.

Now it is clear that the pair of mappings

$$(D,\phi) : (\widetilde{M}, \pi_1(M)) \longrightarrow (\mathbb{R}^q \setminus \mathbb{R}^{q-2}\,,\, \mathrm{Sim}_0^+(\mathbb{R}^q, \mathbb{R}^{q-2}))$$

lifts up to a pair

$$(\widetilde{D},\widetilde{\phi}) : (\widetilde{M}, \pi_1(M)) \longrightarrow (\mathbb{H}^{q-1} \times \mathbb{R}\,,\, \mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R})\,.$$

We denote by $\widetilde{\Gamma}$ the image of the holonomy homomorphism $\widetilde{\phi}$, by $\widetilde{H}$ its closure in $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$ and by $\widetilde{H}_0$ the identity component of $\widetilde{H}$. Notice that the metric g_H on $\mathbb{R}^q \setminus \mathbb{R}^{q-2}$ lifts up to $\widetilde{g}_H$ given by

$$\widetilde{g}_H(x) = \left(\frac{1}{r^2}\, g_E(v)\right) \oplus g_E(t)\,,$$

where $x = (v, r, t) \in \mathbb{R}^{q-2} \times \mathbb{R}_+ \times \mathbb{R} = \mathbb{H}^{q-1} \times \mathbb{R}$. Then $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$ acts as a group of isometries. For an element $\gamma = (g_1, g_2)$ of $\mathrm{Sim}^+(\mathbb{R}^{q-2}) \times \mathbb{R}$, we put $|\gamma| = |g_1|$.

We have the following lemma, an analogue to Lemma 2.1.

LEMMA 3.1.— *We can write $\widetilde{H}_0$ as follows, namely,*

$$\widetilde{H}_0 = \mathbb{R}^{q_0} \times \widetilde{H}_1\,,$$

where $\widetilde{H}_1$ is a closed, connected subgroup defined by $\widetilde{H}^1 = \widetilde{H}_0 \cap \mathbb{R}$ and $\mathbb{R}^{q_0}$ denotes the full group of parallel translations of $\mathbb{R}^{q_0}$.

Notice that $\widetilde{H}_0$ is contained in $\mathrm{Isom}^+(\mathbb{R}^{q_0}) \times \mathbb{R}$, *and that $\widetilde{H}_1$ is equal to either $\{0\}$ or $\mathbb{R}$.*

Proof.— As in the proof of Lemma 2.1, we can find an element γ_0 of Γ being of the form $\gamma_0 = r_0\, A_0$, where $r_0 < 1$ and $A_0 \in \mathrm{SO}(q)$. This element γ_0 lifts to an element η_0 of $\widetilde{\Gamma}$ which satisfies

$$\eta_0(x,t) = (r_0 A x,\, t + \theta)\,.$$

Let h be an element of $\widetilde{H}_0$ satisfying

$$h(x,t) = (s\, Bx + v,\, t + \psi)\,.$$

Then the similar argument as in the proof of Lemma 2.1, we see that the element h_v given by the formula

$$h_v(x,t) = (x + v\ ,\ t)$$

belongs to $\widetilde{H}_0$. Moreover we see that $\widetilde{H}_0$ contains the full group of parallel transformations along $\mathbb{R}^{q_0} = \mathbb{R}^{q-2}$, i.e.,

$$\widetilde{H}_0 \supset \{(x,t) \mapsto (x+t,t) \mid v \in \mathbb{R}^{q_0}\}\,.$$

The rest of the proof is parallel to that of Lemma 2.1 and left to the reader. □

We end this section with the following theorem.

THEOREM 3.2.— *There is no closed orbit and each orbit is contained in a fibre of $\xi : M \to S^1$. The universal covering of the fibre is $\mathbb{R}^{q_0+2}$ and naturally equipped with a structure of a Lie-$\mathbb{R}^{q_0+1}$ flow. Each connected component of the lift of the closure of each orbit projects down to $\mathbb{R}^{q_0} \times$ {one point} or $\mathbb{R}^{q_0} \times \mathbb{R}$.*

Proof.— The first assertion is a direct consequence of Lemma 3.1. Then note that we have

$$\widetilde{M} \cong \mathbb{R} \times (\mathbb{H}^{q-1} \times \mathbb{R}) \cong \mathbb{R} \times (\mathbb{H}^{q-2} \times \mathbb{R}_+) \times \mathbb{R}\,.$$

Now notice that the additive group

$$|\widetilde{H}| = \{|h| \mid h \in \widetilde{H}\}$$

is infinite cyclic, say $\langle r_0 \rangle$, where $r_0 > 1$.

We define a mapping $\widetilde{\alpha} : \widetilde{M} \cong \mathbb{R} \times \mathbb{R}^{q-2} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by the formula

$$\widetilde{\alpha}(h,v,r,t) = r\,.$$

Then as in the proof of Proposition 2.6, we see that α is well-defined and coincides with ξ, which is constant along each leaf closure.

Finally we recall that we have $\widetilde{H}_0 = \mathbb{R}^{q_0} \times \widetilde{H}_1$, where we have $\widetilde{H}_1 = \mathbb{R}$ or $\widetilde{H}_1 = \{0\}$. This completes the proof. □

4. The dimension restriction

Here we give the proof of the remainder part of the theorem.

PROPOSITION 4.1.— *Suppose that we have* $|H_0| = \{1\}$*, then* $q_0 \leq 2$ *holds. Moreover we have the following cases:*

1) We have $q_0 = 1$ *and then* d *is an even number. In particular, the dimension of orbit closures in even.*

2) We have $q_0 = 2$ *and then* $d = 0$ *holds. In particular, the dimension of orbit closures is* 3*. Hence* M *admits a Riemannian foliation whose leaves are* 3*-dimensional tori.*

Proof.— We may assume we have $q_0 \neq 0$. We fix an orbit L of $\mathcal{F}$, then in the case where $q_1 \neq 2$, we have $\overline{L} = \pi(\mathbb{R} \times \mathbb{R}^{q_0} \times T^d \times \{t_0\})$ for some $t_0 \in \mathbb{R}_+$. More precisely, if we fix a point x of $\mathbb{R}^{q_0} \times T^d$, then we have $H_0 x = \mathbb{R}^{q_0} \times T^d$. Here we consider $\mathbb{R}^{q_0} \times T^d \times \{t_0\} = \mathbb{R}^{q_0} \times T^d$.

Since we have $|H| \cong \mathbb{Z}$, we can choose an element $\gamma_0 = r_0 A_0$ of Γ such that $|H| = \langle r_0 \rangle$, where $r_0 < 1$. Let G be the group obtained by dividing H_0 by the isotropy group at a point of $\mathbb{R}^{q_0} \times T^d$. Then G is naturally identified with $\mathbb{R}^{q_0} \times T^d$. Now from the construction of the fibration α, if we denote by μ the monodromy of this fibration, then we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R}^{q_0} \times T^d & \xrightarrow{\varphi} & \mathbb{R}^{q_0} \times T^d = G \\ \mu \downarrow & & \downarrow \rho \\ \mathbb{R} \times \mathbb{R}^{q_0} \times T^d & \xrightarrow{\varphi} & \mathbb{R}^{q_0} \times T^d = G \end{array}$$

where φ is the restriction of the developing map D to $\mathbb{R} \times \mathbb{R}^{q_0} \times T^d$, and ρ is given by conjugation by γ_0.

Now we go up to the universal coverings, then the above diagram lifts up as follows.

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R}^{q_0} \times \mathbb{R}^d & \xrightarrow{\widetilde{\varphi}} & \mathbb{R}^{q_0} \times \mathbb{R}^d \\ \widetilde{\mu} \downarrow & & \downarrow \widetilde{\rho} \\ \mathbb{R} \times \mathbb{R}^{q_0} \times \mathbb{R}^d & \xrightarrow{\widetilde{\varphi}} & \mathbb{R}^{q_0} \times \mathbb{R}^d \end{array}$$

Here we identify $\mathbb{R}^{q_0} \times \mathbb{R}^d$ with $\widetilde{G}$.

Note that we may assume $\widetilde{\varphi}$ is a linear surjective mapping, and that by Lemma 4.7 of [6] we may assume that $\widetilde{\mu}$ is of the form $x \mapsto Ax + b$, where $A \in \mathrm{SL}(q_0 + d + 1\,,\, \mathbb{Z})$ and $b \in \mathbb{R}^{q_0+d+1}$.

In the case where we have $q_1 = 2$, we have $\overline{L} = \pi\big(\mathbb{R} \times \mathbb{R}^{q_0} \times \{s_0\} \times \{t_0\}\big)$ or $\overline{L} = \pi\big(\mathbb{R} \times \mathbb{R}^{q_0} \times \mathbb{R} \times \{t_0\}\big)$ for some $s_0 \in \mathbb{R}$ and some $t_0 \in \mathbb{R}_+$. Thus we can proceed as in the case where $q_1 \neq 2$ holds.

Now let $\widetilde{\gamma}_0$ be the lift of the action of γ_0 on $\mathbb{R}^{q_0} \times T^d$, then the mapping $\widetilde{\gamma}_0 : \mathbb{R}^{q_0} \times \mathbb{R}^d \times \{t_0\} \to \mathbb{R}^{q_0} \times \mathbb{R}^d \times \{rt_0\}$ is given by the formula

$$\widetilde{\gamma}_0(x_1, x_2) = (r_0\, B_0 x_1\,,\, B_1 x_2 + \theta_0)\,,$$

where $B_0 \in \mathrm{SO}(q_0)$, $B_1 \in \mathrm{SO}(d)$ and $\theta_0 \in \mathbb{R}^d$. It follows that we have

$$\widetilde{\rho}(x_1, x_2) = (r_0\, B_0 x_1\,,\, B_1 x_2)\,.$$

Notice that, since we have $\widetilde{\varphi}(b) = \widetilde{\varphi} \circ \widetilde{\mu}(0) = \widetilde{\rho}(0) = 0$, the lifted monodromy $\widetilde{\mu}$ can be replaced by the mapping

$$A : x \longmapsto Ax\,.$$

We will observe the eigenvalues of A. First note that the kernel of $\widetilde{\varphi}$ is one of the eigenspace of A and we let $\{\lambda_0, \ldots, \lambda_\ell\}$ be the collection of the eigenvalues of A. Let $\{v_i^j\}$ be the basis of the generalized eigenspace of λ_j, where $i = 1, \ldots, n_j$ and $j = 0, \ldots, \ell$. Namely, we have

$$\big(A - \lambda_j\big)^k v_i^j = 0$$

for some positive integer k.

We write $\widetilde{\varphi}(z) = \big(\widetilde{\varphi}_1(z), \widetilde{\varphi}_2(z)\big)$, where $z \in \mathbb{R} \times \mathbb{R}^{q_0} \times \mathbb{R}^d$, then we have

$$\varphi\big((A - \lambda)^k\big) = \big((r_0 B_0 - \lambda)^k \widetilde{\varphi}_1\,,\, (B_1 - \lambda)^k \widetilde{\varphi}_2\big)\,.$$

It follows that we have

$$(r_0 B_0 - \lambda_j)^k \widetilde{\varphi}_1(v_i^j) = 0 \quad \text{and} \quad (B_1 - \lambda_j)^k \widetilde{\varphi}_2(v_i^j) = 0\,.$$

On the other hand, since the collection $\{v_i^j \mid i = 1, \ldots, n_j\,,\, j = 0, \ldots, \ell\}$ forms a basis of $\mathbb{R}^{q+d_0+1}$, we have $\widetilde{\varphi}(v_i^j) = 0$ only for $i = 1$ and $j = 0$.

Now we assume that we have $j \neq 0$, then there are two cases:

1) $\widetilde{\varphi}_1(v_i^j) \neq 0$ holds. We have $|\lambda_j| = r_0$ and $\widetilde{\varphi}_2(v_i^j) = 0$

2) $\widetilde{\varphi}_2(v_i^j) \neq 0$ holds. We have $|\lambda_j| = 1$ and $\widetilde{\varphi}_1(v_i^j) = 0$.

Since $\widetilde{\varphi}$ is surjective, we have by changing the order of λ_j that

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_t| = r_0\,,$$
$$|\lambda_{t+1}| = |\lambda_{t+2}| = \ldots = |\lambda_{t+s}| = 1 \quad \text{and} \quad |\lambda_0| = r_0^{-q_0}\,,$$

where $1 \leq t \leq q_0$, $1 \leq s \leq d$ and $t + s = \ell$.

We show that the characteristic polynomial g_A of A is irreducible. Let f be the minimal polynomial of λ_0 over $\mathbb{Q}$, then we have $f(A)v_0 = 0$. Noticing that v_0 corresponds to the direction of the induced flow on $\overline{L}$, $f(A)$ is a constant mapping on $\mathbb{R}^{q_0+d+1}$. Since $f(A)v_0 = 0$ holds, we have $f(A) = 0$ and hence the minimal polynomial of A is irreducible over $\mathbb{Q}$.

It is clear that g_A divides the polynomial f^{q_0+d+1}. Since $\mathbb{Q}[x]$ is an UFD and f is irreducible, we see that g_A is a power of f by a positive integer. Since we have $r_0^{-q_0} > 1$, λ_0 is a simple root. It follows that $f = g_A$ holds and hence g_A is irreducible over $\mathbb{Q}$. Thus any root of f is a simple root.

Now we suppose we have $q_0 > 2$, then we may assume that we have $\lambda_1 \neq \overline{\lambda_1}$, where denotes the complex conjugate. Of course $\overline{\lambda_1}$ is an eigenvalue and we may assume $\lambda_2 \neq \overline{\lambda_1}$.

Let F be the minimum splitting field of f. We consider $F \subset \mathbb{C}$. Let θ be an automorphism of F over $\mathbb{Q}$ such that $\theta(\lambda_1) = \lambda_0$ holds, then θ induces a permutation among the eigenvalues. Since θ is an automorphism of a field, we have

$$\theta(\lambda_1) \cdot \theta(\overline{\lambda_1}) = \theta(\lambda_2) \cdot \theta(\overline{\lambda_2})\,.$$

This implies we have

$$r_0^{-q_0} = r_0^m\,, \quad m = -1,\, 0,\, 1 \text{ or } 2\,.$$

But this is impossible since we assumed that $q_0 > 2$ and we have $r_0 < 1$.

If we have $q_0 = 1$ and d is an odd number, the the number s defined above is odd. It follows that A has 1 or -1 as its eigenvalue. This contradicts the irreducibility of g_A.

If we have $q_0 = 2$ and $d \neq 0$, then we can find an automorphism θ of F such that $\theta(\lambda_3) = \lambda_0$. Since we have $|\lambda_3| = 1$, λ_3 is not a real number and hence $\overline{\lambda_3}$ is another eigenvalue of g_A satisfying $|\overline{\lambda_3}| = 1$. Then we have

$$\theta(\lambda_3) \cdot \theta(\overline{\lambda_3}) = 1 .$$

This shows that we have $r_0 = 1$, which is a contradiction. □

As we mentioned in the introduction, we have another proof of the following theorem of Epstein [6]. This is the case where we have $q_1 = 1$.

THEOREM 4.2.— *Let $(M, \mathcal{F})$ be an $(\mathrm{Isom}^+(\mathbb{H}^q), \mathbb{H}^q)$-flow. If we assume that $(M, \mathcal{F})$ is both oriented and transversely oriented, then we have the following cases:*

1) *All the orbits are closed and form the fibres of a Seifert fibration.*
2) *If there is a non-closed orbit, then $(M, \mathcal{F})$ is differentiably conjugate to the suspension of $(T, \mathcal{F}', f)$ over S^1, where T is a d-dimensional torus and $\mathcal{F}'$ is an irrational linear flow of T, where d is either 2 or 3, and f is a diffeomorphism of T preserving $\mathcal{F}'$.*

Proof.— The case 1) follows from the general theory of Riemannian foliations. So we show the case 2).

As in [6], we see that in this case the flow is a $(\mathrm{Sim}^+(\mathbb{R}^q, \mathbb{R}^{q-1}), \mathbb{H}^q)$-flow and we have $q - q_0 = 1$. Hence $|H_0| = \{1\}$ holds and we have a fibration $\alpha : M \to S^1$, whose fibre has a natural transverse $(\mathrm{Isom}^+(\mathbb{R}^{q-1}), \mathbb{R}^{q-1})$ structure.

The proposition 4.1 shows that we have $q_0 \leq 2$ and the dimension of the leaf closure is equal to q. It is easy to see that the closure of each orbit is precisely the fibre of the fibration α. Then by the general theory of the Riemannian foliation, we see that the induced foliation of each fibre is locally constant. □

COROLLARY 4.3.— *If we have $q_0 \neq 0$ and $q_1 = 2$, then M fibres over T^2 and the fibre is $(q_0 + 1)$-dimensional torus equipped with an irrational linear flow.*

Proof.— If we have $q_0 = 2$, then Proposition 4.1 shows that the $\widetilde{H}_0$-orbit of a point $x = (x_h, x_s, p)$ of $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+$ is $\mathbb{R}^2 \times \{x_s\} \times \{p\}$. It follows that the additive group

$$\begin{aligned}[\widetilde{H}] &= \{\theta \mid \text{there is an element } h \in \widetilde{H} \text{ of the form } h(x,t) = (Ax + v\,,\, t + \theta)\} \\ &= \{\theta \mid \text{there is an element } h \in \widetilde{H} \text{ of the form } h(x,t) = (Ax\,,\, t + \theta)\}\end{aligned}$$

is discrete, say $\mathbb{Z}\theta_0$.

We write $\widetilde{D}(\widetilde{x}) = \big(D_1(\widetilde{x}),\, D_2(\widetilde{x}),\, D_3(\widetilde{x})\big)$ and define a mapping $\widetilde{\beta} : \widetilde{M} \to S^1 \times S^1 = \mathbb{R}/\mathbb{Z}\theta_0 \times \mathbb{R}_+/\langle r_0 \rangle$ by the formula

$$\widetilde{\beta}(\widetilde{x}) = \big(D_2(\widetilde{x}) \bmod \theta_0\,,\, D_3(\widetilde{x}) \bmod r_0\big)\,.$$

If we have $\widetilde{y} = \gamma\widetilde{x}$, then we have $D(\widetilde{y}) = \phi(\gamma)(\widetilde{x})$. The above argument shows that we can write $\phi(x_h, x_s, p) = (r_0^k A x_h + v,\, x_s + \ell\theta_0,\, r_0^k)$. Thus $\widetilde{\beta}$ defines a mapping β from $M \to T^2$. It is easy to see that β is a fibration. The above construction shows that the fibre of β is a 3-dimensional torus equipped with an irrational linear flow.

If we have $q_0 = 1$, then again Proposition 4.1 shows that the $\widetilde{H}_0$-orbit of a point $x = (x_h, x_s, p)$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ is $\mathbb{R} \times \{x_s\} \times \{p\}$. Then the same argument applies and we obtain the result. □

From now on, we assume that we have $q_1 \geq 3$ and that $|H_0| = \{1\}$ holds. Recall that in this case, we have the natural fibration $\alpha : M \to S^1$, whose fibres are foliated by restricting $\mathcal{F}$.

THEOREM 4.4.— *If we have $q_0 = 0$ and $|H_0| = \{1\}$, then the fibre of α equipped with the flow is obtained as the suspension of an isometry of a spherical manifold.*

Proof.— Let F be a fibre of α. Recall that in this case $\big(F, \mathcal{F}\restriction_F\big)$ is an $\big(\mathrm{SO}(q)\,,\, S^{q-1}\big)$-flow. Then the theorem of Carrière [5] applies and the result follows. □

THEOREM 4.5.— *If we have $q_0 \neq 0$ and $|H_0| = \{1\}$, then M is finitely covered by a T^{q_0+1}-bundle over $S^{q-q_0-1} \times S^1$. The flow restricted to each fibre is a fixed irrational linear flow.*

Proof.— First we assume $q_0 = 2$ holds. Let L be an orbit of $\mathcal{F}$, then Proposition 4.1 shows that the developing image of a connected component of $\pi^{-1}(\overline{L})$ is $\mathbb{R}^2 \times \{p\}$, where $p \in S^{q_1-1}$. Thus H_0 and Γ_0 act on $S^{q_1-1} \times \mathbb{R}_+$ trivially.

We put $\check{M} = \widetilde{M}/\phi^{-1}(\Gamma_0)_{\vee}$ Since Γ_0 acts on $S^{q_1-1} \times \mathbb{R}_+$ trivially, a simple argument shows that $\check{M}$ is diffeomorphic to $T^{q_0+1} \times S^{q_1-1} \times \mathbb{R}_+$, where T^{q_0+1} is equipped with a fixed irrational linear flow induced by the action of $\phi^{-1}(\Gamma_0)$ on $\pi^{-1}(\overline{L})$.

Now recall that in this case we can find an element γ_0 of Γ_0 such that $|\gamma_0|$ generates $|H|$. We put $g = \phi^{-1}(\gamma_0)$ and write $g = (g_1, g_2) \in \mathrm{Sim}^+(q_0) \times \mathrm{SO}(q_1)$, then we can write

$$g(x,y,t) = \left(\overline{g_1}(x,y,t),\, g_2 y,\, |\gamma_0| t\right),$$

where $(x,y,t) \in T^{q_0+1} \times S^{q_1-1} \times \mathbb{R}_+$ and $\overline{g_1}$ is the mapping induced from g_1.

From this we deduce that if we put $M_0 = \check{M}/\langle g \rangle$ then M_0 is a T^{q_0+1}-bundle over $S^{q_1-1} \times S^1$.

Secondly we assume that $q_0 = 1$ holds. In this case we have the associated singular Riemannian foliation $(S^{q-q_0-1}, \mathcal{F}^{\text{th}})$ defined in Proposition 2.6. The proposition 4.1 shows that the leaves of this foliation is of even dimension. It follows from Proposition 1.3 that $(S^{q-q_0-1}, \mathcal{F}^{\text{th}})$ is trivial. Now we can repeat the same argument as in the first case and the theorem is proved. □

5. Examples

First we show examples related to the flows $(N_g, \mathcal{F}_g)$ defined in the introduction.

Example 5.1

Consider the Hopf fibration $S^3 \to S^2$ and let X be a vector field tangent to the fibres. We choose X appropriately so that if we consider its 1-parameter group φ_t, then $\{\varphi_n(x)\}_{n \in \mathbb{Z}}$ is dense in the fibre passing through x.

Now we put $X = S^3 \times \mathbb{R}_+$ and define transformations φ, ψ of X by the formula $\varphi(x,t) = (\varphi_1(x), rt)$ and $\varphi(x,t) = (x, 2t)$, respectively. Notice that if we consider S^3 as the unit sphere in $\mathbb{C}^2$ then we have

$$\varphi_1(z_1, z_2) = \left(e^{2\pi\sqrt{-1}\,\theta} z_1 ,\, e^{2\pi\sqrt{-1}\,\theta} z_2\right),$$

where θ is an irrational number. Therefore φ is an orthogonal transformation.

We put $N = X/\langle\psi\rangle$ and consider the suspension $(M, \mathcal{F})$ of (N, f) over S^1, where f is the automorphism induced by φ. It is clear that $(M, \mathcal{F})$ is a $(\mathrm{CO}^+(4), \mathbb{R}^4 \setminus \{0\})$-flow and that for a suitable choice of r, we can construct both examples where the dimension of each orbit closure is equal to 2, or where the dimension of each orbit closure is equal to 1.

Example 5.2

Let us consider $T^d = \mathbb{R}^d/\mathbb{Z}^d$ and let φ be an automorphism of T^d induced from the parallel translation of $\mathbb{R}^d$ by a vector $(\theta_1, \ldots, \theta_d)$. We put

$$g = e^{\theta_d}\big(R(\theta_1) \oplus \cdots \oplus R(\theta_{d-1})\big) .$$

If we choose φ so that all the orbits are dense in T^d, we can deduce from Fried [7] that $(N_g, \mathcal{F}_g)$ contains an orbit L such that the closure $\overline{L}$ of L in N_g equipped with the restricted flow does not inherit any transverse similarity structure from $\mathcal{F}_g$.

Now we give an example where $|H_0| = \{1\}$ holds but Γ_0 is non-trivial.

Example 5.3

Let $(M, \mathcal{F})$ be a $(\mathrm{Sim}^+(\mathbb{R}^3, \mathbb{R}^2), \mathbb{H}^3)$-flow such that the closure of each orbit is of dimension 3 and forms a fibre of the fibration $\alpha : M \to S^1$, whose monodromy map μ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

This is just an example of Epstein [6]. We denote by ϕ the holonomy homomorphism and by D the developing map defining $(M, \mathcal{F})$.

Now we put $(M', \mathcal{F}') = (M \times S^d, \mathcal{F} \times S^d)$, where $\mathcal{F} \times S^d = \{L \times \{p\} \mid L \in \mathcal{F} \text{ and } p \in S^d\}$, $d > 1$.

We define mappings

$$\phi' : \pi_1(M') = \pi_1(M) \longrightarrow \mathrm{Sim}^+(\mathbb{R}^{d+3}, \mathbb{R}^2)$$

and

$$D' : \widetilde{M}' = \widetilde{M} \times S^d \longrightarrow \mathbb{R}^{d+3} \setminus \mathbb{R}^2 = \mathbb{H}^3 \times S^d$$

as follows, namely, we put

$$\phi'(\gamma') = \phi(\gamma) \times \mathrm{id}_{\mathbb{R}^d} \quad \text{and} \quad D'(x, p) = \big(D(x), p\big) .$$

As is easily seen, the pair (ϕ', D') defines a transversely similar flow of M' obtained as the suspension of the pair $(T^3 \times S^d \,, \mu \times \mathrm{id})$ over S^1.

This example shows that the codimension itself admits no restriction while the closure of each orbit is of dimension at most three.

Acknowledgments

Finally the author would like to express his gratitude to Professors T. Tsuboi and S. Matsumoto for their helpful suggestions and encouragement.

References

[1] ASUKE (T.) .— *On transversely flat conformal foliations with good measures*, Trans. A.M.S. **348** (1996), pp. 1939-1958.

[2] BRÖCKER (T.) and TOM DIECK (T.) .— *Representations of Compact Lie Groups*, Graduate Texts in Mathematics **98**, Springer-Verlag, 1985.

[3] BRUNELLA (M.) .— *On transversely holomorphic flows* I, Invent Math. **126** (1996), pp. 265-279.

[4] CARRIÈRE (Y.) .— *Flots riemanniens – Structures transverses des feuilletages*, Astérique **116** (1984), pp. 31-52.

[5] CARRIÈRE (Y.) .— *Variations on Riemannian Flows, Riemannian Foliations*, Progress in Mathematics **73** (1988), pp. 217-234.

[6] EPSTEIN (D. B. A.) .— *Tranversely hyperbolic* 1*-dimensional foliations, Structures transverses des feuilletages*, Astérique **116** (1984), pp. 53-69.

[7] FRIED (D.) .— *Closed similarity manifolds*, Comment. Math. Helvetici **55** (1980), pp. 576-582.

[8] FRIED (D.) .— *The geometry of cross sections to flows*, Topology **21** (1982), pp. 353-371.

[9] GHYS (E.) .— *Flots transversalement affines et tissus feuilletés*, Mem. Soc. Math. France (N.S.) **46** (1991), pp. 123-150.

[10] HERMANN (R.) .— *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc. **11** (1960), pp. 236-242.

[11] LU (D.) .— *Homogeneous Riemannian foliations of spheres*, Trans. A.M.S. **340** (1993), pp. 95-102.

[12] MOLINO (P.) .— *Riemannian Foliations*, Progress in Mathematics **73**, Birkhäuser (1988).

[13] NISHIMORI (T.) .— *A note on the classification of nonsingular flows with transverse similarity structures*, Hokkaido Math. J. **21** (1992), pp. 381-393.

[14] TISCHLER (D.) .— *On fibering certain foliated manifolds over* S^1, Topology **9** (1970), pp. 153-154.