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Positive quadratic differential forms : linearization, finite determinacy and versal unfolding

Positive quadratic differential forms: linearization, finite determinacy and versal unfolding[(*)]

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Résumé. — Des problèmes locaux autour des singularités tels que linéarisation, détermination finie, et déploiements versels sont considérés pour un type de forme différentielle quadratique C^∞ sur des surfaces et pour leurs paires de feuilletages (avec singularités) associées.

Abstract. — Local problems around singularities such as linearization, finite determinacy, and versal unfolding are considered for a kind of smooth quadratic differential forms on surfaces and for their associated pair of foliations (with singularities).

1. Introduction

A C^r-quadratic differential form on an oriented, connected, smooth two-manifold M is an element of the form $\omega = \sum_{k=1}^n \phi_k \psi_k$ where ϕ_k and ψ_k are 1-forms on M of class C^r. Therefore, for each point p in M, $\omega(p)$ is a quadratic form on the tangent space $T_p M$. We say that ω is *positive* if for every point p in M the subset $\omega(p)^{-1}(0)$ of $T_p M$ is either the union of two transversal lines (in this case p is called a *regular point* of ω) or all $T_p M$ (in this case p is called a *singular point* of ω). If p is a regular point of such ω, we call $L_1(\omega)(p)$ (resp. $L_2(\omega)(p)$) the line of $\omega(p)^{-1}(0)$ characterized as follows. Let C be a positively oriented circle around the origin of $T_p M$.

Then, $q \in C \cap L_1(\omega)(p)$ (resp. $q \in C \cap L_2(\omega)(p)$) if there exists a small open arc (q_1, q_2) on C containning q such that $\omega(p)$ is positive (resp. negative) on (q_1, q) and negative (resp. positive) on (q, q_2). In this form, associated to each positive C^r-quadratic differential form ω on M we have a triple $C(\omega) = \{f_1(\omega), f_2(\omega), \mathrm{Sing}(\omega)\}$ called the *configuration associated to* ω, where $\mathrm{Sing}(\omega)$ is the set of singular points of ω and $f_1(\omega)$ and $f_2(\omega)$ are the two transversal C^r *one-dimensional foliations* defined on $M \setminus \mathrm{Sing}(\omega)$ whose tangent lines at each regular point p are respectively $L_1(\omega)(p)$ and $L_2(\omega)(p)$.

Here, we study local problems around generic singular points of smooth positive quadratic differential forms; more precisely, we deal with linearization, finite determinacy and versal unfolding. This paper and the corresponding methods have been inspired in the work developed by F. Dumortier, R. Roussari, J. Sotomayor and F. Takens. See for instance [Du], [Ta], [DRS] and [So].

Throughout this paper PQD will refer to the expression: "positive quadratic differential" and M will denote an oriented, connected, smooth two-manifold.

In general PQD forms appear related with lines of principal curvature and asymptotic lines (cf. [Gu3], [GS1], [GS2]). The configurations that can be achieved by lines of principal curvature and asymptotic lines seem to be very limited (see [Kl], [SX]); however, it follows from [Gu1, Theorem A] that every configuration of two transverse C^r-one-dimensional foliations with common singularities is realized as the configuration of a positive C^r-quadratic differential form.

Two PQD forms ω_1 and ω_2 are said to be *equivalent* if there exists a homeomorphism h of M such that $h(C(\omega_1) = C(\omega_2))$. That is, h maps $\mathrm{Sing}(\omega_1)$ onto $\mathrm{Sing}(\omega_2)$ and maps leaves of $f_1(\omega_1)$ and $f_2(\omega_1)$ onto leaves of $f_1(\omega_2)$ and $f_2(\omega_2)$, respectively. A PQD form ω is said to be *structurally stable* if any PQD form $\widetilde{\omega}$ sufficiently C^1-close to ω, is equivalent to ω. It was obtained in [Gu1] and [Gu3], a complete characterization of structurally stable positive C^∞-quadratic differential forms, with the C^2-topology.

In this article we study simple singular points of PQD forms. In local (x, y)-coordinates, the origin $(0, 0)$ is a simple singular point of

$$\omega = a(x, y)\, dy^2 + 2b(x, y)\, dx\, dy + c(x, y)\, dx^2$$

if (and only if) by a linear transformation of coordinates, the function (a, b, c) can be put in the form

$$a(x, y) = y + M_1(x, y)$$
$$b(x, y) = b_1 x + b_2 y + M_2(x, y)$$
$$c(x, y) = -y + M_3(x, y)$$

where $b_1 \neq 0$ and $M_1(x, y)$, $M_2(x, y)$, $M_3(x, y) = O\big((x^2 + y^2)^{1/2}\big)$. In Section 2, we list some of their properties; for instance, they are generic and persistent (see [Gu1]). Among other things, in Section 3 we show that, under generic conditions on b_1 and via an analytic change of coordinates, the function (a, b, c) can take the form:

$$a(x, y) = y + p_2(x, y) + \cdots + p_n(x, y) + P_n(x, y)$$
$$b(x, y) = b_1 x + b_2 y + Q_n(x, y)$$
$$c(x, y) = -y + R_n(x, y)$$

where p_m are homogeneous polynomials of degree m, for all $2 \leq m \leq n$, and P_n, Q_n, $R_n = O\big(|(x, y)|^n\big)$. Moreover if we allow b_1 and b_2 to be arbitrary, with $b_1 \cdot b_2 \neq 0$, and permit a smooth change of coordinates, (a, b, c) becomes:

$$a(x, y) = y$$
$$b(x, y) = (b_1 x + b_2 y)\, Q_2(x, y)$$
$$c(x, y) = -y + M_3(x, y)$$

with $M_3(x, y) = O\big((x^2 + y^2)^{1/2}\big)$ and $Q_2(0, 0) = 1$. It will be clear that these sorts of simplifications are optimal. In Section 4 we prove that a smooth PQD form around a simple singular point is 1-determined (i.e. it is locally topologically equivalent to its linear part). Finally, in Section 5, we determine the versal unfolding of an arbitrary nonlocally stable simple singular point: It is either of codimension 1 (§ 5.2) or codimension 2 (§ 5.3).

2. Simple singular points

In this section we introduce the definition of a simple singular point and list some of its properties.

Let $Q(M)$ be the manifold made up of the pairs (p, α) such that $p \in M$ and $\alpha = \sum_{i=1}^{n} \phi_i \psi_i$, with ϕ_i and ψ_i in the cotangent space $(T_p M)^*$. Then every C^∞ quadratic differential form ω on M can be considered as a C^∞ section $\omega : M \to Q(M)$. With this representation, the usual derivative of ω at each p in M, $D\omega_p$, is a (linear) quadratic differential form on $T_p M$ [Gu1, p. 479].

Let $\mathcal{F}(M)$ be the set of PQD forms defined on M.

DEFINITION 2.1. — *Let $\omega \in \mathcal{F}(M)$. A singular point p of ω is said to be simple if $D\omega_p$ is a PQD form on $T_p M$.*

These singular points are persistent. More precisely, if p_0 is a simple singular point of ω_0 there exist neighborhoods $\mathcal{N} \subset \mathcal{F}(M)$ (with the C^1 topology) of ω_0 and $V \subset M$ of p_0 and a C^∞ map $p : (\mathcal{N}, \omega_0) \to (V, p_0)$ called the C^∞ *continuation* of the singular point p_0 in V, such that, for every $\omega \in \mathcal{N}$,

$$\mathrm{Sing}(\omega) \cap V = \{p(\omega)\}$$

and $p(\omega)$ is a simple singular point [Gu1, Prop. 5.2]. Also, these types of singular points are C^2-generic [Gu1, Theorem B(iii)].

Associated to a singular point p of a PQD form $\omega \in \mathcal{F}(M)$ and to coordinates (u, v) in the tangent space $T_p M$ we have a homogeneous polynomial of degree 3: $D\omega_p(u, v)(u, v)$ called the *separatrix polynomial*. This denomination comes from the fact that, for a simple singular point, the roots of this polynomial correspond to all possible directions of asymptotic convergence to the point p by the leaves of both foliations $f_1(\omega)$ and $f_2(\omega)$. In this respect, we have the following result.

DEFINITION 2.2. — *Let p be a simple singular point of $\omega \in \mathcal{F}(M)$ and (u, v) coordinates in $T_p M$. Then the point p is called:*

  a) *A hyperbolic singular point if the separatrix polynomial $D\omega_p(u, v)(u, v)$ has only simple roots.*

  b) *A D_{12} singular point if $D\omega_p(u, v)(u, v)$ has one simple and one double root.*

  c) *A $\widetilde{D}_1$ singular point if $D\omega_p(u, v)(u, v)$ has a triple root.*

Remark 2.3. — Every simple singular point p of a PQD form $\omega \in \mathcal{F}(M)$ may be expressed in an appropiate local chart $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0))$ in the form:

$$(x, y)^*(\omega) = \big(y + M_1(x, y)\big)\, \mathrm{d}y^2 + 2\big(b_1 x + b_2 y + M_2(x, y)\big)\, \mathrm{d}x\, \mathrm{d}y +$$
$$+ \big(-y + M_3(x, y)\big)\, \mathrm{d}x^2$$

with $M_i(x, y) = \mathrm{O}\big((x^2 + y^2)^{1/2}\big)$, $i = 1,\, 2,\, 3$ and $b_1 \neq 0$ [Gu1, Prop. 5.1]). In these coordinates:

1) The singular point is hyperbolic if and only if

$$b_2^2 - 2b_1 + 1 \neq 0 \quad \text{and} \quad b_1 \neq \frac{1}{2}\,.$$

2) The singular point is D_{12} if and only if

$$b_2 \neq 0 \quad \text{and either} \quad b_2^2 - 2b_1 + 1 = 0 \quad \text{or} \quad b_1 = \frac{1}{2}\,.$$

3) The singular point is $\widetilde{D}_1$ if and only if

$$b_1 = \frac{1}{2} \quad \text{and} \quad b_2 = 0\,.$$

The hyperbolic singular points are C^1 locally stable [Gu1, Prop. 6.2]. Moreover, there are three different topological types of them. In the above coordinates they are characterized by the following inequalities:

$$D_1 \quad : \quad b_2^2 - 2b_1 + 1 < 0$$
$$D_2 \quad : \quad b_2^2 - 2b_1 + 1 > 0 \quad \text{and} \quad b_1 \neq \frac{1}{2},\, b_1 > 0$$
$$D_3 \quad : \quad b_1 < 0\,.$$

The corresponding local configurations are the ones given in figure 1.
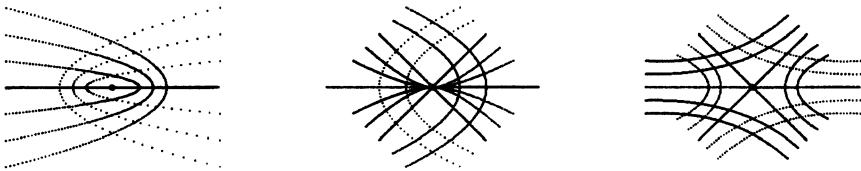

Fig. 1

3. Normal Forms

Denote by

$$H^m = H^m(\mathbb{R}^2)$$

the space of real valued homogeneous polynomial functions, of degree m, on two real variables u, v. Let H_k^m be the polynomial vector fields on $\mathbb{R}^k$ whose coordinate functions belong to H^m.

Let A, B, $C : \mathbb{R}^2 \to \mathbb{R}$ be linear maps that satisfy the condition

$$B^2 - AC \geq 0 \quad \text{everywhere in } \mathbb{R}^2 .$$

Associated to (A, B, C), let us consider the linear map

$$\mathrm{ad}_m : H_2^m \to H_3^m$$

defined by

$$\mathrm{ad}_m(\alpha, \beta) = (A, B, C)(\alpha, \beta) + A\left(2\frac{\partial\beta}{\partial v}, \frac{\partial\beta}{\partial u}, 0\right) +$$

$$+ B\left(2\frac{\partial\alpha}{\partial v}, \frac{\partial\alpha}{\partial u} + \frac{\partial\beta}{\partial v}, 2\frac{\partial\beta}{\partial u}\right) + C\left(0, \frac{\partial\alpha}{\partial v}, 2\frac{\partial\alpha}{\partial u}\right) .$$

PROPOSITION 3.1.— *Let*

$$\omega = a(x, y)\,\mathrm{d}y^2 + 2b(x, y)\,\mathrm{d}x\,\mathrm{d}y + c(x, y)\,\mathrm{d}x^2$$

be a PQD *form such that the origin* $(0, 0)$ *is a singular point. Let* $(A, B, C) = D(a, b, c)(0, 0)$. *Given* $k \geq 2$ *fixed, let us consider, for each* $2 \leq m \leq k$, *the linear map* $\mathrm{ad}_m : H_2^m \to H_3^m$ *associated to* (A, B, C) *and choose a complement* G^m *for* $\mathrm{ad}_m(H_2^m)$ *in* H_3^m, *i.e.:*

$$H_3^m = \mathrm{ad}_m(H_2^m) \oplus G^m .$$

Then there exists an analytic change of coordinates in a neighborhood of the origin $(x, y) = h(u, v)$ *such that if*

$$(u, v)^*(\omega) = \widetilde{a}(u, v)\,\mathrm{d}v^2 + 2\widetilde{b}(u, v)\,\mathrm{d}u\,\mathrm{d}v + \widetilde{c}(u, v)\,\mathrm{d}u^2$$

then

$$(\widetilde{a}, \widetilde{b}, \widetilde{c})(u, v) = (A, B, C)(u, v) + (p_2, q_2, r_2)(u, v) + \cdots +$$
$$+ (p_k, q_k, r_k)(u, v) + (P_k, Q_k, R_k)(u, v)$$

with $(p_m, q_m, r_m) \in G^m$, *for all* $2 \le m \le k$, *and* $P_k, Q_k, R_k = O\big(|(u, v)|^k\big)$.

Proof. — First, let us observe that if $(x, y) = G(u, v)$ is a change of coordinates, then ω in the new coordinates, takes the form

$$(u, v)^*(\omega) = (\mathrm{d}u\ \mathrm{d}v) \begin{pmatrix} \widetilde{c}(u, v) & \widetilde{b}(u, v) \\ \widetilde{b}(u, v) & \widetilde{a}(u, v) \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

with

$$\begin{pmatrix} \widetilde{c}(u, v) & \widetilde{b}(u, v) \\ \widetilde{b}(u, v) & \widetilde{a}(u, v) \end{pmatrix} = \mathfrak{D}G(u, v)^T M(G(u, v))\mathfrak{D}G(u, v)$$

where $\mathfrak{D}G(u, v)$ is the jacobian matrix of G in (u, v), $\mathfrak{D}G(u, v)^T$ is its transposed matrix and $M(u, v)$ is the matrix

$$M(u, v) = \begin{pmatrix} c(u, v) & b(u, v) \\ b(u, v) & a(u, v) \end{pmatrix}.$$

Moreover if $G(u, v) = (u, v) + F(u, v)$ with $\mathfrak{D}F(0) = 0$, we have that

$$\mathfrak{D}G(u, v)^T M(G(u, v))\mathfrak{D}G(u, v) =$$
$$= M((u, v) + F(u, v)) + \mathfrak{D}F(u, v)^T M(u, v) + M(u, v)\mathfrak{D}F(u, v)$$
$$+ \mathfrak{D}F(u, v)^T M(u, v)\mathfrak{D}F(u, v).$$

To proceed inductively with the proof, let us assume — by a change of coordinates — that the functions a, b, c of our positive quadratic differential form $\omega = a(x, y)\,\mathrm{d}y^2 + 2b(x, y)\,\mathrm{d}x\,\mathrm{d}y + c(x, y)\,\mathrm{d}x^2$ can be written as:

$$(a, b, c)(u, v) = (A, B, C)(u, v) + (p_2, q_2, r_2)(u, v) + \cdots +$$
$$+ (p_{s-1}, q_{s-1}, r_{s-1})(u, v) + (P_{s-1}, Q_{s-1}, R_{s-1})(u, v)$$

with $(p_m, q_m, r_m) \in G^m$ for all $2 \le m < s - 1$, and $P_{s-1}, Q_{s-1}, R_{s-1} = O\big(|(u, v)|^{s-1}\big)$.

Then let us consider a coordinate transformation of the form $(x, y) = (u, v) + \big(\alpha(u, v), \beta(u, v)\big)$ where α and β are homogeneous polynomials of degree s whose coefficients will be determined later. In this way, if

$$(u, v)^*(\omega) = \widetilde{a}(u, v)\, \mathrm{d}v^2 + \widetilde{b}(u, v)\, \mathrm{d}u\, \mathrm{d}v + \widetilde{c}(u, v)\, \mathrm{d}u^2$$

then a small computation gives us that

$$(\widetilde{a}, \widetilde{b}, \widetilde{c})(u, v) = (A, B, C)(u, v) + (p_2, q_2, r_2)(u, v) + \cdots +$$
$$+ (p_s, q_s, r_s)(u, v) + \mathrm{ad}_s(\alpha, \beta)(u, v) + (P_s, Q_s, R_s)(u, v)$$

with $P_s, Q_s, R_s = \mathrm{O}\big(|(u, v)|^s\big)$.

Therefore, the terms of degree less than s are preserved and the resulting terms of degree s are

$$(p_s, q_s, r_s)(u, v) + \mathrm{ad}_s(\alpha, \beta)(u, v)\,.$$

It is clear that α and β can be taken so that

$$(p_s, q_s, r_s)(u, v) + \mathrm{ad}_s(\alpha, \beta)(u, v) \in G^s\,,$$

as it was required. $\square$

Given the linear maps

$$A\,,\ B\,,\ C : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

as above, let M_m be the matrix of $\mathrm{ad}_m\ :\ H_2^m\ \rightarrow\ H_3^m$ with respect to the canonical bases of H_2^m and H_3^m. Then M_m is a matrix with $3(m + 1)$ rows and $2(m + 1)$ columns and therefore the space $\mathrm{ad}_m(H_2^m)$ has complementary spaces in H_2^m of dimension at least $m + 1$ For this reason, besides $\mathrm{ad}_m = (\mathrm{ad}_m^1, \mathrm{ad}_m^2, \mathrm{ad}_m^3)$ we shall also consider the linear maps

$$\mathrm{ad}_m^{i,j} : H_2^m \longrightarrow H_2^m$$

defined by $\mathrm{ad}_m^{i,j} = (\mathrm{ad}_m^i, \mathrm{ad}_m^j)$ for $i,\, j = 1,\, 2,\, 3$ with $i \neq j$.

3.1 Analytic Change of Coordinates

Using the notation of the previous proposition, we say that ω is C^r linearizable if there exists a C^r change of coordinates h such that the resulting functions $\widetilde{a},\, \widetilde{b}$ and $\widetilde{c}$ are linear. As, for all $m \geq 1$, the dimension of H_2^m is less than the dimension of H_3^m; in most cases we cannot linearize ω. Next proposition states what we get in this direction.

Let $\omega = a(x,y)\,dy^2 + 2b(x,y)\,dx\,dy + c(x,y)\,dx^2$ be a PQD form such that the origin is a singularity and

$$D(a,b,c)(0,0) = \begin{pmatrix} 0 & 1 \\ b_1 & b_2 \\ 0 & -1 \end{pmatrix}$$

PROPOSITION 3.2. — *Given $m \geq 2$, the determinant $d_m^{3,2}$ of the linear map $\mathrm{ad}_m^{3,2}$ is given by*

$$d_m^{3,2} = (-1)^{m-1}(-m + b_1)(-1 + 2mb_1) \times$$

$$\times \prod_{k=1}^{m} \left(k - 1 + (m+k-2)b_1 + 2(m-k+2)(m-k)b_1^2 \right).$$

Moreover, if $n \geq 2$ is a natural number and $d_m^{3,2} \neq 0$ for all $2 \leq m \leq n$ (which is a generic condition on b_1), there exists an analytic coordinate transformation $(x,y) = G(u,v)$ such that if

$$(u,v)^*(\omega) = \widetilde{a}(u,v)\,dv^2 + 2\widetilde{b}(u,v)\,du\,dv + \widetilde{c}(u,v)\,du^2$$

then

$$\widetilde{a}(u,v) = v + p_2(u,v) + \cdots + p_n(u,v) + P_n(u,v)$$
$$\widetilde{b}(u,v) = b_1 u + b_2 v + Q_n(u,v)$$
$$\widetilde{c}(u,v) = -v + R_n(u,v)$$

with $p_m \in H^m$ for all $2 \leq m \leq n$ and $P_n, Q_n, R_n = O\big(|(v,u)|^n\big)$.

Proof. — Let $n \geq 2$ be a natural number and suppose that $d_m^{3,2} \neq 0$ for all $2 \leq m \leq n$. Under these conditions we have that if $G^m = H^m \times \{0\} \times \{0\} \subset H^m \times H^m \times H^m = H_3^m$ then

$$H_3^m = \mathrm{ad}_m(H_2^m) \oplus G^m.$$

From the previous proposition follows that, to finish the proof of this proposition, it remains to check the formula for $d_m^{3,2}$.

Observe that for $j = 0, \ldots, m$

$$\mathrm{ad}_m^{3,2}(u^{m-j}v^j, 0) =$$
$$= \left(\left[-2(m-j)u^{m-j-1}v^{j+1} \right], \left[(-j + (m-j+1)b_1)u^{m-j}v^j + \right. \right.$$
$$\left. \left. + (m-j)b_2 u^{m-j-1}v^{j+1} \right] \right)$$

and

$$\mathrm{ad}_m^{3,2}(0, u^{m-j}v^j) =$$

$$= \Big(\big[(-1 + 2(m-j)b_1)u^{m-j}v^j + 2(m-j)b_2 u^{m-j-1}v^{j+1} \big] ,$$

$$\big[jb_1 u^{m+1-j}v^{j-1} + (j+1)b_2 u^{m-j}v^j + (m-j)u^{m-j-1}v^{j+1} \big] \Big)$$

Hence, the matrix $U(m) = (u_{ij})$, $i, j = 1, \ldots, 2(m+1)$ of $\mathrm{ad}_m^{3,2}$ with respect to the basis

$$(u^m, 0), \ (v^{m-1}v, 0), \ \ldots, \ (v_m, 0), \ (0, u^m), \ (0, u^{m-1}v), \ \ldots, \ (0, v^m)$$

is of the form

$$U(m) = \begin{pmatrix} A(m) & C(m) \\ B(m) & D(m) \end{pmatrix}$$

with $A(m) = (a_{ij})$, $B(m) = (b_{ij})$, $C(m) = (c_{ij})$, $D(m) = (d_{ij})$, $i, j = 1, \ldots, m+1$ and

$$\begin{cases} a_{i+1,i} = -2(m+1-i) & i = 1, \ldots, m \\ a_{i,j} = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} b_{i,i} = 1 - i + (m+2-i)b_1 & i = 1, \ldots, m+1 \\ b_{i+1,i} = (m+1-i)b_2 & i = 1, \ldots, m \\ b_{i,j} = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} c_{i,i} = -1 + 2(m+1-i)b_1 & i = 1, \ldots, m+1 \\ c_{i+1,i} = 2(m+1-i)b_2 & i = 1, \ldots, m \\ c_{ij} = 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} d_{i,i+1} = ib_1 & i = 1, \ldots, m \\ d_{i,i} = ib_2 & i = 1, \ldots, m+1 \\ d_{i+1,i} = (m+1-i) & i = 1, \ldots, m \\ d_{ij} = 0 & \text{otherwise.} \end{cases}$$

For instance, the matrix $U(3)$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & -1+6b_1 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & 6b_2 & -1+4b_1 & 0 & 0 \\
0 & -4 & 0 & 0 & 0 & 4b_2 & -1+2b_1 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 2b_2 & -1 \\
4b_1 & 0 & 0 & 0 & b_2 & b_1 & 0 & 0 \\
3b_2 & -1+3b_1 & 0 & 0 & 3 & 2b_2 & 2b_1 & 0 \\
0 & 2b_2 & -2+2b_1 & 0 & 0 & 2 & 3b_2 & 3b_1 \\
0 & 0 & b_2 & -3+b_1 & 0 & 0 & 1 & 4b_2
\end{pmatrix}$$

First, we compute the determinant of $U(m)$ by cofactor expansion along its first row and then we compute, the resulting determinant, by cofactor expansion along its $m+1$ column. In this way

$$\det U(m) = -(-1 + 2mb_1)(-m + b_1)\det V(m)$$

where $V(m) = (v_{ij})$, $i, j = 1, \ldots, 2m$, is given by

$$v_{i,j} = \begin{cases} u_{i+1,j} & i = 1, \ldots, 2m, j = 1, \ldots, m \\ u_{i+1,j+2} & i = 1, \ldots, 2m, j = m+1, \ldots, 2m. \end{cases}$$

For instance, when $m = 3$, we have that

$$V(3) = \begin{pmatrix} -6 & 0 & 0 & -1 + 4b_1 & 0 & 0 \\ 0 & -4 & 0 & 4b_2 & -1 + 2b_1 & 0 \\ 0 & 0 & -2 & 0 & 2b_2 & -1 \\ 4b_1 & 0 & 0 & b_1 & 0 & 0 \\ 3b_2 & -1 + 3b_1 & 0 & 2b_2 & 2b_1 & 0 \\ 0 & 2b_2 & -2 + 2b_1 & 2 & 3b_2 & 3b_1 \end{pmatrix}.$$

Then we define the matrix $W(m) = (w_{ij})$, $j = 1, \ldots, 2m$, by

$$\begin{cases} w_{i,j} = v_{i,j} & j = 1, \ldots, m \\ w_{i,m+k} = 2(m - k + 1)v_{i,m+k} + \\ \quad + (-1 + 2(m - k)b_1)v_{i,k} & j = m+1, \ldots, 2m \end{cases}$$

for $i = 1, \ldots, 2m$. Hence,

$$\det V(m) = \frac{1}{\prod_{k=1}^{m} 2(m - k + 1)} \det W(m).$$

For instance, the matrix $W(3)$ is

$$\begin{pmatrix} -6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 24\,b_2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 8b_2 & 0 \\ 4b_1 & 0 & 0 & 2b_1(1 + 8b_1) & 0 & 0 \\ 3b_2 & -1 + 3b_1 & 0 & * & 1 + 3b_1 + 6b_1^2 & 0 \\ 0 & 2b_2 & -2 + 2b_1 & * & * & 2(1 + 2b_1) \end{pmatrix}.$$

Observe that an entry $w_{i,j}$ of $W(m)$, with $i < j$, is zero unless $i = 2, \ldots, m$ and $j = m - 1 + i$ in which case

$$w_{i,m-1+i} = 4(m - i + 2)(m - i + 1)b_2.$$

We define the matrix $Z(m) = (z_{i,j})$, i, $j = 1, 2, \ldots, 2m$, by

$$
\begin{cases}
z_{i,j} = w_{i,j} & i = 1, \ldots, 2m, \, j = 1, \ldots, m \\
z_{i,m+k} = w_{i,m+k} + 2(m+1-k)w_{i,k+1} & i = 1, \ldots, 2m, \, k = 1, \ldots, m.
\end{cases}
$$

As $Z(m)$ can be obtained from $W(m)$ by iteration of the process of adding a multiple of one column to another column, we have that the determinant of $Z(m)$ is the same as that of $W(m)$. Since $Z(m)$ is a lower triangular matrix and $z_{i,i} = w_{i,i}$, for all $i = 1, \ldots, 2m$, we have that

$$
\det W(m) = \prod_{k=1}^{2m} w_{k,k} \, .
$$

Finally, for all $k = 1, \ldots, m$, we have that $w_{k,k} = -2(m+1-k)$ and

$$
\begin{aligned}
w_{m+k,m+k} &= \\
&= 2(m+1-k)v_{m+k,m+k} + \big(-1 + 2(m-k)b_1\big)v_{m+k,k} \\
&= 2(m+1-k)u_{m+k+1,m+k+2} + \big(-1 + 2(m-k)b_1\big)u_{m+k+1,k} \\
&= 2(m+1-k)d_{k,k+1} + \big(-1 + 2(m-k)b_1\big)b_{k,k} \\
&= 2(m+1-k)kb_1 + \big(-1 + 2(m-k)b_1\big)\big(1 - k + (m+2-k)b_1\big).
\end{aligned}
$$

Thus

$$
\det W(m) = (-1)^m \prod_{k=1}^{m} \big(2(m+1-k)\big) \times
$$

$$
\times \prod_{k=1}^{m} \big(k - 1 + (m-2+k)b_1 + 2(m+2-k)(m-k)b_1^2\big) \, .
$$

Therefore

$$
\det U(m) = (-1)^{m-1}(-m+b_1)(-1+2mb_1) \times
$$

$$
\times \prod_{k=1}^{m} \big(k - 1 + (m+k-2)b_1 + 2(m-k+2)(m-k)b_1^2\big) \, .
$$

This finishes the proof. □

3.2 Smooth Change of Coordinates

Let $\omega = a(x, y) \, dy^2 + 2b(x, y) \, dx \, dy + c(x, y) \, dx^2$ be a PQD form such that

$$a(x, y) = y + M_1(x, y)$$
$$b(x, y) = b_1 x + b_2 y + M_2(x, y)$$
$$c(x, y) = -y + M_3(x, y)$$

with $M_i(x, y) = O\big((x^2 + y^2)^{1/2}\big)$, $i = 1, 2, 3$, and $b_1 \neq 0$.

Then if $(x, y) = G(u, v)$ is a change of coordinates, then ω in the new coordinates, takes the form

$$(u, v)^*(\omega) = (du \, dv) \begin{pmatrix} \widetilde{c}(u, v) & \widetilde{b}(u, v) \\ \widetilde{b}(u, v) & \widetilde{a}(u, v) \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

with

$$\begin{pmatrix} \widetilde{c}(u, v) & \widetilde{b}(u, v) \\ \widetilde{b}(u, v) & \widetilde{a}(u, v) \end{pmatrix} = \mathfrak{D}G(u, v)^T M\big(G(u, v)\big) \, \mathfrak{D}G(u, v)$$

where $\mathfrak{D}G(u, v)$ is the jacobian matrix of G in (u, v), $\mathfrak{D}G(u, v)^T$ is its transposed and $M(u, v)$ is the matrix

$$M(u, v) = \begin{pmatrix} c(u, v) & b(u, v) \\ b(u, v) & a(u, v) \end{pmatrix}.$$

In this way, if $G(u, v) = \big(g(u, v), h(u, v)\big)$ we obtain:

$$\widetilde{a}(u, v) = a\big(G(u, v)\big) \left(\frac{\partial h}{\partial v}(u, v) \right)^2 +$$
$$+ 2b\big(G(u, v)\big) \frac{\partial g}{\partial v}(u, v) \frac{\partial h}{\partial v}(u, v) + c\big(G(u, v)\big) \left(\frac{\partial g}{\partial v}(u, v) \right)^2 \quad (1)$$

$$\widetilde{b}(u, v) = a\big(G(u, v)\big) \frac{\partial h}{\partial u}(u, v) \frac{\partial h}{\partial v}(u, v) +$$
$$+ b\big(G(u, v)\big) \left(\frac{\partial g}{\partial u}(u, v) \frac{\partial h}{\partial v}(u, v) + \frac{\partial h}{\partial u}(u, v) \frac{\partial g}{\partial v}(u, v) \right) +$$
$$+ c\big(G(u, v)\big) \frac{\partial g}{\partial u}(u, v) \frac{\partial g}{\partial v}(u, v) \quad (2)$$

$$\widetilde{c}(u, v) = a\big(G(u, v)\big) \left(\frac{\partial h}{\partial u}(u, v) \right)^2 +$$
$$+ 2b\big(G(u, v)\big) \frac{\partial g}{\partial u}(u, v) \frac{\partial h}{\partial u}(u, v) + c\big(G(u, v)\big) \left(\frac{\partial g}{\partial u}(u, v) \right)^2. \quad (3)$$

PROPOSITION 3.3.— *There exists a smooth change of coordinates* $(x, y) = G(u, v)$ *with* $G(0, 0) = (0, 0)$, *such that if*

$$(u, v)^*(\omega) = \tilde{a}(u, v)\, dv^2 + 2\tilde{b}(u, v)\, du\, dv + \tilde{c}(u, v)\, du^2$$

then

$$\tilde{a}(u, v) = v$$

$$\tilde{b}(u, v) = b_1 u + b_2 v + \widetilde{M_2}(u, v)$$

$$\tilde{c}(u, v) = -v + \widetilde{M_3}(u, v)$$

and $\widetilde{M_i}(u, v) = O\big((u^2 + v^2)^{1/2}\big)$, $i = 2, 3$.

Proof.— As $(\partial a / \partial y)(0, 0) = 1$, it follows from the C^∞-Weierstrass Preparation Theorem [Ma] that there exists a small neighborhood of the origin where $a(x, y)$ can be written in the form:

$$a(x, y) = \big(y + f(x)\big) Q(x, y)$$

with f, Q of class C^∞, $Q(0, 0) = 1$ and $f(0) = 0$.

Regarding the expresion (1) above, we see that the change of coordinates $G(u, v) = \big(u\,,\, v - f(u)\big)$ gives

$$\tilde{a}(u, v) = v\big(1 + F(u, v)\big)$$

with $F(0, 0) = 0$.

The factor $1 + F(u, v)$ can be made identically equal to 1 by a smooth change of coordinates of the form $(u, v) = H(s, t) = \big(s\,,\, t(1 + R(s, t))\big)$, with $R(0, 0) = 0$, provided that the C^∞ map $R(s, t)$ is a solution of the equation

$$\left(1 + R(s, t) + t\frac{\partial R}{\partial t}(s, t)\right)^2 \big(1 + R(s, t)\big) \Big(1 + F\big(s\,,\, t(R(s, t))\big)\Big) = 1. \quad (4)$$

To solve this equation we consider the vector field $X(r, s, t)$ with associated differential equation

$$\dot{r} = 1 - (1 + r)\sqrt{1 + r + (1 + r)F\big(s\,,\, t(1 + r)\big)}$$

$$\dot{s} = 0$$

$$\dot{t} = t\sqrt{1 + r + (1 + r)F\big(s\,,\, t(1 + r)\big)}\,.$$

Since

$$\mathfrak{D}X(0,0,0) = \begin{pmatrix} -2 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the local central unstable manifold, around the origin, associated to X, is the graphic of a smooth map $r = R(s, t)$ defined at a neighborhood of $(0, 0)$. Such map R satisfies the equation

$$\frac{\partial R}{\partial t}(s, t) =$$

$$= \frac{1 - (1 + R(s, t))\sqrt{1 + R(s, t) + (1 + R(s, t))\, F\left(s\,,\, t(1 + R(s, t))\right)}}{t\sqrt{1 + R(s, t) + (1 + R(s, t))\, F\left(s\,,\, t(1 + R(s, t))\right)}}$$

which implies that $R(s, t)$ solves our equation (4). We observe that the usual proof of the existence and differentiability of the local central unstable manifold provides in the particular case considered here, the claimed smoothness [HPS].

As $\mathfrak{D}G(0, 0)$ and $\mathfrak{D}H(0, 0)$ are the identity function, the linear part of the functions $(\tilde{b}, \tilde{c})$ and (b, c) at the origin are the same. The proof is finished. $\square$

PROPOSITION 3.4.— *If $b_2 \neq 0$ there exists a smooth change of coordinates $(x, y) = G(u, v)$ such that if*

$$(u, v)^*(\omega) = \tilde{a}(u, v)\, dv^2 + 2\tilde{b}(u, v)\, du\, dv + \tilde{c}(u, v)\, du^2$$

then

$$\tilde{a}(u, v) = v$$
$$\tilde{b}(u, v) = (b_1 u + b_2 v)Q_2(u, v)$$
$$\tilde{c}(u, v) = -v + \widetilde{M_3}(u, v)$$

with $\widetilde{M_3}(u, v) = \mathrm{O}\big((u^2 + v^2)^{1/2}\big)$ and $Q_2(0, 0) = 1$.

Proof.— By using the previous proposition we may assume that

$$a(x, y) = y$$
$$b(x, y) = b_1 x + b_2 y + M_2(x, y)$$
$$c(x, y) = -y + M_3(x, y)$$

with $M_i(x, y) = \mathrm{O}\big((x^2 + y^2)^{1/2}\big)$, $i = 2, 3$.

Again as $(\partial b/\partial y)(0,0) = b_2 \neq 0$, we have that there exists a neighborhood of the origin and maps k and Q_1, of class C^∞, such that

$$b(x, y) = b_2 (y + \lambda k(x)) Q_1(x, y)$$

with $\lambda = b_1/b_2$, $k(0) = 0$, $k'(0) = 1$ and $Q_1(0,0) = 1$.

By taking an arbitrary change of coordinates of the form $G(u, v) = (g(u), v)$ it can be obtained

$$\widetilde{a}(u, v) = v$$

$$\widetilde{b}(u, v) = b_2 \Big(v + \lambda k(g(u))\Big) Q_1(g(u), v) g'(u)$$

$$\widetilde{c}(u, v) = (g(u), v) g'(u)^2.$$

Therefore, the proof of this proposition follows by putting $g(u) = k^{-1}(u)$. $\square$

PROPOSITION 3.5.— *If $b_2 = 0$, $b_1 < 1/2$ and $b_1 \neq -1$, there exists change of coordinates $(x, y) = G(v, u)$ such that if*

$$(u, v)^*(\omega) = \widetilde{a}(u, v) \, dv^2 + 2\widetilde{b}(u, v) \, du \, dv + \widetilde{c}(u, v) \, du^2$$

then

$$\widetilde{a}(u, v) = v$$

$$\widetilde{b}(u, v) = (\widetilde{b}_1 u + \widetilde{b}_2 v) Q_2(u, v)$$

$$\widetilde{c}(u, v) = -\widetilde{c}_2 v + \widetilde{M}_3(u, v)$$

with $\widetilde{M}_3(u, v) = O((u^2 + v^2)^{1/2})$ and $Q_2(0,0) = 1$.

Proof. — We shall prove that in this case there exists a linear change of coordinates $G(u, v) = (\alpha u + \beta v, \gamma u + \delta v)$ such that if

$$(u, v)^*(\omega) = \widetilde{a}(u, v) \, dv^2 + 2\widetilde{b}(u, v) \, du \, dv + \widetilde{c}(u, v) \, du^2$$

then the Jacobian matrix of $g(u, v) = (\widetilde{a}, 2\widetilde{b}, \widetilde{c})(u, v)$ is of the form

$$\mathfrak{D}g(0,0) = \begin{pmatrix} 0 & 1 \\ 2\widetilde{b}_1 & 2\widetilde{b}_2 \\ 0 & -1 \end{pmatrix}$$

with $\widetilde{b}_2 \neq 0$. From this point, the result will follow from the former proposition.

By taking $\gamma = \sqrt{1 - 2b_1}$, $\alpha = \delta = 1$ and $\beta = 0$ it is obtained

$$\mathfrak{D}g(0,0) = \begin{pmatrix} \sqrt{1 - 2b_1} & 1 \\ 2(1 - b_1) & 2(b_1 + \sqrt{1 - 2b_1}) \\ 0 & -2b_1 \end{pmatrix}.$$

Observe that the condition $(\partial \widetilde{c}/\partial u)(0,0) = 0$ is preserved by a change of coordinates of the form $G(u,v) = (\alpha u + \beta v, \gamma u + \delta v)$ with $\gamma = \sqrt{1 - 2b_1}$.

So, let us put $G(u,v) = (u + \beta v, \sqrt{1 - 2b_1}u + v)$ with β as one of the solutions of the equation

$$\frac{\partial \widetilde{a}}{\partial u}(0,0) = \sqrt{1 - 2b_1} + 2b_1\beta - \sqrt{1 - 2b_1}\beta^2 = 0.$$

Then

$$\frac{\partial \widetilde{b}}{\partial v}(0,0) = \sqrt{1 - 2b_1} + (b_1 - 1)\beta + b_1\sqrt{1 - 2b_1}\beta^2.$$

As both of the equations have the same roots only when $b_1 = -1$, we may choose β in such a way that $(\partial \widetilde{b}/\partial v)(0,0)$ is not zero.

It is easy to see that to the resulting PQD form we can apply (if necessary) another linear change of coordinates in such a way that new PQD form is as desired. $\square$

4. Finite determinacy

PROPOSITION 4.1. — *Let p be a simple singular point of a PQD form ω. Then there exist neighborhoods $V \subset M$ of p and $W \subset T_pM$ of the origin such that ω/V is topologically equivalent to $\mathrm{D}\omega_p/W$.*

Proof. — Let $(x,y) : (U,p) \to (\mathbb{R}^2, (0,0))$ be a local chart such that

$$(x,y)^*(\omega) = a(x,y)\,dy^2 + 2b(x,y)\,dx\,dy + c(x,y)\,dx^2$$

with

$$a(x,y) = y + M_1(x,y)$$
$$b(x,y) = b_1x + b_2y + M_2(x,y)$$
$$c(x,y) = -y + M_3(x,y),$$

$M_i(x,y) = \mathrm{O}((x^2 + y^2)^{1/2})$, $i = 1, 2, 3$, and $b_1 \neq 0$.

We shall consider the cases:

1) $b_2^2 - 2b_1 + 1 \neq 0$ and $b_1 \neq 1/2$;

2) $b_2 \neq 0$ and $(b_2^2 - 2b_1 + 1 = 0$ or $b_1 = 1/2)$;

3) $(b_1, b_2) = (1/2 , 0)$.

In the first case, the point p is a hyperbolic singularity of ω and the result follows from [Gu1, Prop. 6.2].

In the second and third case, by taking a linear change of coordinates if necessary, we may suppose $b_1 = 1/2$.

As in [Gu2, Sect. 4] let us consider the blowing-up

$$x = u, \quad y = uv$$

and the PQD forms $\omega_0 = (u, v)^*(\omega)$ and ω_1 defined by

$$\begin{aligned}
\omega_0 &= u^2 A_0(u, v)\, dv^2 + 2u B_0(u, v)\, du\, dv + C_0(u, v)\, du^2 \\
&= u\, \omega_1 \\
&= u\big(u^2 A_1(u, v)\, dv^2 + 2u B_1(u, v)\, du\, dv + C_1(u, v)\, du^2\big)
\end{aligned}$$

where

$$\begin{aligned}
A_0(u, v) &= a(u, uv) = u\big(v + u N_1(u, v)\big) \\
B_0(u, v) &= b(u, uv) + va(u, uv) \\
&= u\left(\frac{1}{2} + b_2 v + v^2 + u\big(N_2(u, v) + v N_1(u, v)\big)\right) \\
C_0(u, v) &= c(u, uv) + 2vb(u, uv) + v^2 a(u, uv) \\
&= u\Big(2b_2 v^2 + v^3 + u\big(N_3(u, v) + 2v N_2(u, v) + v^2 N_1(u, v)\big)\Big)
\end{aligned}$$

with $M_i(u, uv) = u^2 N_i(u, v)$, $i = 1, 2, 3$.

Observe that away from $\{(u, v) : u = 0\}$, $f_i(\omega_1) = f_i(\omega_0)$ for $i = 1, 2$. To study the phase portrait of the foliations $f_i(\omega_1)$, $i = 1, 2$, in a neighborhood of $\{(u, v) : u = 0\}$, we consider the vector fields

$$X_i(\omega_1) = \left(u^2 A_1, -u B_1 + (-1)^i \big(u^2(B_1^2 - A_1 C_1)\big)^{1/2}\right)$$

$$Y_i(\omega_1) = \left(u A_1, -B_1 + (-1)^i (B_1^2 - A_1 C_1)^{1/2}\right)$$

$$Z_i(\omega_1) = \left(u\big(B_1 + (-1)^i (B_1^2 - A_1 C_1)^{1/2}\big), -C_1\right).$$

For $i = 1$, 2, the following is satisfied:

a1) $X_i(\omega_1)$ is tangent to $f_i(\omega_1)$;

a2) $Y_i(\omega_1)$ and $Z_i(\omega_1)$ (evaluated at the same point) are linearly dependent;

a3) restricted to $\{(u, v) : u > 0\}$ (resp. to $\{(u, v) : u < 0\}$), $Y_i(\omega_1)$ and $X_i(\omega_1)$ (resp. $X_{3-i}(\omega_1)$) are linearly dependent (see [Gu2, Sect. 4]).

Moreover,

a4)

$$Y_1(\omega_1) =$$
$$= \left(\left[u\big(v + uN_1(u, v)\big) \right], \left[-\frac{1}{2} - b_2 v - v^2 - u(N_2 + vN_1)(u, v) + \right. \right.$$
$$\left. \left. - \left(\left(\frac{1}{2} + b_2 v + uN_2(u, v) \right)^2 - \big(v + uN_1(u, v)\big)\big(-v + uN_3(u, v)\big) \right)^{1/2} \right] \right)$$

has no singularities along $u = 0$;

a5)

$$Z_2(\omega_1) =$$
$$= \left(\left[u\left(\left(\frac{1}{2} + b_2 v + v^2 + u(N_2 + vN_1)(u, v) \right) + \right. \right. \right.$$
$$\left. \left. + \left(\left(\frac{1}{2} + b_2 v + uN_2(u, v) \right)^2 - \big(v + uN_1(u, v)\big)\big(-v + uN_3(u, v)\big) \right)^{1/2} \right) \right],$$
$$\left. \left[- \big(v^2(2b_2 + v) + u(N_3 + 2vN_2 + v^2 N_1)(u, v)\big) \right] \right)$$

has singularities at the points $(0, 0)$ and $(0, -2b_2)$

a6)
$$\mathfrak{D}Z_2(\omega_1)(0, 0) = \begin{pmatrix} 1 & 0 \\ * & 0 \end{pmatrix}$$

and $Z_2(\omega_1)(0, v) = (0 \, , \, -2b_2 v^2 - v^3)$;

a7) if $b_2 \neq 0$ the point $(0, -2b_2)$ is a hyperbolic saddle for $Z_2(\omega_1)$.

Using properties a1)-a7) the following can be obtained: in case 2), i.e. $b_2 \neq 0$ (resp. in case (3), i.e. $b_2 = 0$) the phase portraits of $X_1(\omega_1)$ and $X_2(\omega_1)$ in a neighborhood of $\{(u, v) : u = 0\}$ are homeomorphic to the ones shown in figure 2(a) (resp. in fig. 2(b)).

These are the configurations of the foliations $f_1(\omega_1)$ and $f_2(\omega_1)$ in a neighborhood of $u = 0$. Hence the configuration of ω in a neighborhood of the origin, in case 2) (resp. in case 3)) is homeomorphic to the one presented in figure 3(a) (resp. in fig. 3(b)). This finishes the proof. $\square$
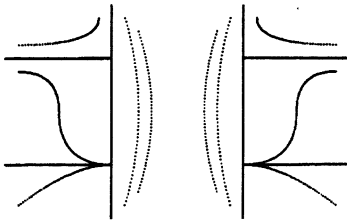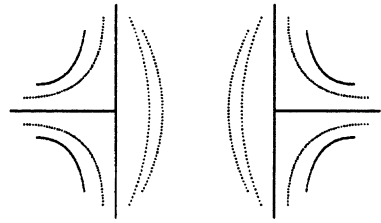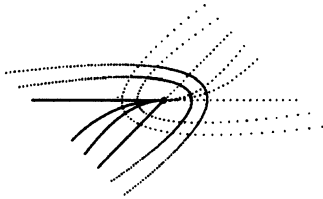

Fig. 2(a)   **Fig. 2(b)**
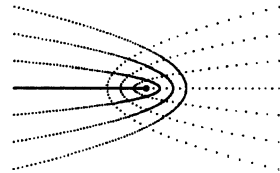

Fig. 3(a)   **Fig. 3(b)**

5. Versal unfolding

In this section, we shall obtain the versal unfolding of the D_{12} and $\widetilde{D}_1$ singularities.

DEFINITION 5.1.— *Two smooth families (ω_μ) and $(\widetilde{\omega}_\mu)$ of PQD forms with (the same) parameter $\mu \in \mathbb{R}^k$ are called C^0-equivalent (over the identity) if there exist homeomorphisms $h_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ such that for each $\mu \in \mathbb{R}^k$, h_μ is a C^0-equivalence between the forms ω_μ and $\widetilde{\omega}_\mu$.*

Remark 5.2.— For local families around the origin of $\mathbb{R}^2 \times \mathbb{R}^k$ one imposes the conditions that $h_0(0,0) = (0,0)$ and $h_\mu(x,y)$ must only be defined for $((x,y),\mu)$ belonging to a neighborhood $V \times W$ of $((0,0),0)$ in $\mathbb{R}^2 \times \mathbb{R}^k$, with $\{(h_\mu(x,y),\mu) \mid ((x,y),\mu) \in V \times W\}$ a neighborhood of $((0,0),0)$.

DEFINITION 5.3.— *If $\phi : (\mathbb{R}^\ell, 0) \rightarrow (\mathbb{R}^k, 0)$ (or ϕ defined only in a neighborhood of the origin) is a smooth mapping and (ω_μ) is a smooth family of PQD forms with parameter $\mu \in \mathbb{R}^k$ (defined in a neighborhood of the origin in $\mathbb{R}^k$) we call family C^∞-induced by ϕ the family $(v_\alpha) = (\omega_{\phi(\alpha)})$ with parameter $\alpha \in \mathbb{R}^\ell$.*

By an *unfolding of a smooth* PQD *form ω* is meant any smooth family ω_μ of PQD forms with $\omega_0 = \omega$. Then we have the following definition

DEFINITION 5.4.— *An unfolding ω_μ of ω_0 is called a versal unfolding of ω_0 if all unfoldings of ω_0 are C^0-equivalent to an unfolding C^∞-induced from ω_μ.*

5.1 Preliminaries

The technical lemmas below will be used in next sections.

LEMMA 5.5.— *Let $\omega_\mu = a(x,y,\mu)\,\mathrm{d}y^2 + 2b(x,y,\mu)\,\mathrm{d}x\,\mathrm{d}y + c(x,y,\mu)\,\mathrm{d}x^2$ be an arbitrary smooth family of PQD forms with parameter $\mu \in \mathbb{R}^k$ such that ω_0 has a simple singular points at the origin. Then there exists a change of coordinates of the form $(x,y,\mu) = h(u,v,\mu)$ such that, for all μ, with $|\mu|$ small, the origin is a singular point of*

$$(u,v)^*(\omega_\mu).$$

Proof.— By assumption $(a,b)((0,0),\overline{0}) = (0,0)$ and

$$D_1(a,b)((0,0),\overline{0}) = \begin{pmatrix} 0 & 1 \\ b_1 & b_2 \end{pmatrix}.$$

As the map $(a,b) : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$ is smooth, it follows from the Implicit Function Theorem that there exists a smooth map S defined in a small neighborhood of $\overline{0} \in \mathbb{R}^k$ such that $S(\overline{0}) = (0,0)$ and

$$(a,b)(S(\mu),\mu) = (0,0)$$

for all μ in such neighborhood.

As ω_μ is positive for all μ, it follows that $(a, b, c)(S(\mu), \mu) = (0, 0, 0)$. From this point, the lemma follows by using the change of coordinates

$$(x, y, \mu) = (u, v, \mu) - (S(\mu), \overline{0}) . \ \square$$

LEMMA 5.6.— *Let* $\omega(\mu)$ *be a smooth family of* PQD *forms with parameter* $\mu \in \mathbb{R}^k$. *Suppose that* $\omega(0)$ *has a simple singular point* p_0. *Let* $U \subset M$ *be a neighborhood of* p_0 *and let* $V \subset \mathbb{R}^k$ *be a neighborhood of the origin such that* $\text{Sing}(\omega(\mu)) \cap U = \{p(\mu)\}$, *for every* $\mu \in V$, *where* $p(\mu)$ *is the* C^∞ *continuation of the singular point* $p_0 = p(0)$ *in* U.

Let $(x, y) : (U, p_0) \longrightarrow (\mathbb{R}^2, (0, 0))$ *be a local chart such that*

$$
\begin{aligned}
(x, y)^*(\omega_0) = & (y + M_1(x, y)) \, \mathrm{d}y^2 + 2(b_1 x + b_2 y + M_2(x, y)) \, \mathrm{d}x \, \mathrm{d}y + \\
& + (-y + M_3(x, y)) \, \mathrm{d}x^2
\end{aligned}
$$

with $b_1 \neq 0$ *and* $M_i(x, y) = \mathrm{O}((x^2 + y^2)^{1/2})$, $i = 1, 2, 3$.

Under these conditions, there exists a local chart $\phi : (U_0 \times V_0, (p_0, 0)) \longrightarrow (\mathbb{R}^2 \times \mathbb{R}^k, (0, 0, 0))$ *of the form* $\phi(p, \mu) = (x(p, \mu), y(p, \mu), \mu)$ *with* $\phi(p, 0) = (x(p), y(p), 0)$ *for all* $p \in U$, *and* $\phi(p(\mu), \mu) = (0, 0, \mu)$, *for all* $\mu \in V$, *such that if* $(x, y)^*(\omega(\mu)) = a(x, y, \mu) \, \mathrm{d}y^2 + b(x, y, \mu) \, \mathrm{d}x \, \mathrm{d}y + c(x, y, \mu) \, \mathrm{d}x^2$ *and:*

(a) $b_1 \neq 1/2$, *then*

$$a(x, y, \mu) = y + M_1(x, y, \mu)$$
$$b(x, y, \mu) = (b_1 + B_1(\mu))x + (b_2 + B_2(\mu))y + M_2(x, y, \mu)$$
$$c(x, y, \mu) = -y + M_3(x, y, \mu)$$

with $B_i(0) = 0$, $i = 1, 2$;

(b) $b_1 = 1/2$, *then*

$$a(x, y, \mu) = A_1(\mu)x + y + M_1(x, y, \mu)$$
$$b(x, y, \mu) = \frac{1}{2} x + \left(b_2 - \frac{1}{2} A_1(\mu)\right) y + M_2(x, y, \mu)$$
$$c(x, y, \mu) = C_1(\mu)x + (-1 + C_2(\mu))y + M_3(x, y, \mu)$$

with $A_1(0) = C_1(0) = C_2(0) = 0$;

where (in both cases)

$$M_k(0,0,\mu) = \frac{\partial M_k}{\partial x}(0,0,\mu) = \frac{\partial M_k}{\partial y}(0,0,\mu) = 0$$

for all $(k, \mu) \in \{1, 2, 3\} \times V_0$.

Proof. — By using the previous Lemma and by taking $U \times V$ smaller if necessary, we may assume that $U \times V$ is open and that there exists a local chart $\phi : \big(U \times V, (p_0, 0)\big) \to \big(\mathbb{R}^2 \times \mathbb{R}^k, (0,0,0)\big)$ of the form $\phi(p, \mu) = \big(x(p, \mu), y(p, \mu), \mu\big)$ and such that $\phi(p, 0) = \big(x(p), y(p), 0\big)$, for every $p \in U$, and $\phi\big(p(\mu), \mu\big) = (0, 0, \mu)$, for every $\mu \in V$.

Then if $(x, y)^*\big(\omega(\mu)\big) = a(x, y, \mu)\,\mathrm{d}y^2 + b(x, y, \mu)\,\mathrm{d}x\,\mathrm{d}y + c(x, y, \mu)\,\mathrm{d}x^2$ we have that

$$a(x, y, \mu) = a_1(\mu)x + a_2(\mu)y + M_1(x, y, \mu)$$
$$b(x, y, \mu) = b_1(\mu)x + b_2(\mu)y + M_2(x, y, \mu)$$
$$c(x, y, \mu) = c_1(\mu)x + c_2(\mu)y + M_3(x, y, \mu)$$

with

$$(a_1, b_1, c_1)(\mu) = (0, b_1, 0) + (A_1, B_1, C_1)(\mu)$$
$$(a_2, b_2, c_2)(\mu) = (1, b_2, -1) + (A_2, B_2, C_2)(\mu)$$

and

$$M_k(0,0,\mu) = \frac{\partial M_k}{\partial x}(0,0,\mu) = \frac{\partial M_k}{\partial y}(0,0,\mu) = 0\,,$$

for all $(k, \mu) \in \{1, 2, 3\} \times V$.

Let $A_{(\alpha,\beta,\gamma,\delta,\mu)} : \mathbb{R}^2 \to \mathbb{R}^2$ be the family of linear isomorphisms with parameter $(\alpha, \beta, \gamma, \delta, \mu) \in \mathbb{R}^4 \times V$ such that if $A = A_{(\alpha,\beta,\gamma,\delta,\mu)}$ its inverse is given by

$$A^{-1}(x, y, \mu) = \big((1 + \alpha)x + \beta y\,,\, \gamma x + (1 + \delta)y\,,\, \mu\big)\,.$$

Under these conditions, we have that $\overline{\phi} = A \circ \phi$ is the form $\overline{\phi}(p, \mu) = \big(u(p, \mu), v(p, \mu), \mu\big)$ and so, if

$$(u, v)^*\big(\omega(\mu)\big) = \overline{a}(u, v, \mu)\,\mathrm{d}v^2 + \overline{b}(u, v, \mu)\,\mathrm{d}u\,\mathrm{d}v + \overline{c}(u, v, \mu)\,\mathrm{d}u^2$$

then

$$\overline{a}(u, v, \mu) = \overline{a}_1(\mu)u + \overline{a}_2(\mu)v + \overline{M}_1(u, v, \mu)$$
$$\overline{b}(u, v, \mu) = \overline{b}_1(\mu)u + \overline{b}_2(\mu)v + \overline{M}_2(u, v, \mu)$$
$$\overline{c}(u, v, \mu) = \overline{c}_1(\mu)u + \overline{c}_2(\mu)v + \overline{M}_3(u, v, \mu)$$

where

$$\bar{a}_1(\mu) = (1+\delta)^2\big((1+\alpha)a_1(\mu) + \gamma a_2(\mu)\big) +$$
$$+ 2\beta(1+\delta)\big((1+\alpha)b_1(\mu) + \gamma b_2(\mu)\big) + \beta^2\big((1+\alpha)c_1(\mu) + \gamma c_2(\mu)\big)$$

$$\bar{a}_2(\mu) = (1+\delta)^2\big(\beta a_1(\mu) + (1+\delta)a_2(\mu)\big) +$$
$$+ 2\beta(1+\delta)\big(\beta b_1(\mu) + (1+\delta)b_2(\mu)\big) + \beta^2\big(\beta c_1(\mu) + (1+\delta)c_2(\mu)\big)$$

$$\bar{b}_1(\mu) = \gamma(1+\delta)\big((1+\alpha)a_1(\mu) + \gamma a_2(\mu)\big) +$$
$$+ \big((1+\alpha)(1+\delta) + \beta\gamma\big)\big((1+\alpha)b_1(\mu) + \gamma b_2(\mu)\big) +$$
$$+ (1+\alpha)\beta\big((1+\alpha)c_1(\mu) \dotplus \gamma c_2(\mu)\big)$$

$$\bar{b}_2(\mu) = \gamma(1+\delta)\big(\beta a_1(\mu) + (1+\delta)a_2(\mu)\big) +$$
$$+ \big((1+\alpha)(1+\delta) + \beta\gamma\big)\big(\beta b_1(\mu) + (1+\delta)b_2(\mu)\big) +$$
$$+ (1+\alpha)\beta\big(\beta c_1(\mu) + (1+\delta)c_2(\mu)\big)$$

$$\bar{c}_1(\mu) = \gamma^2\big((1+\alpha)a_1(\mu) + \gamma a_2(\mu)\big) + 2(1+\alpha)\gamma\big((1+\alpha)b_1(\mu) + \gamma b_2(\mu)\big) +$$
$$+ (1+\alpha)^2\big((1+\alpha)c_1(\mu) + \gamma c_2(\mu)\big)$$

$$\bar{c}_2(\mu) = \gamma^2\big(\beta a_1(\mu) + (1+\delta)a_2(\mu)\big) + 2(1+\alpha)\gamma\big(\beta b_1(\mu) + (1+\delta)b_2(\mu)\big) +$$
$$+ (1+\alpha)^2\big(\beta c_1(\mu) + (1+\delta)c_2(\mu)\big)\,.$$

Also, the following is satisfied:

$$(\bar{a}_1, \bar{b}_1, \bar{c}_1)(0) = (0, b_1, 0)$$

and

$$(\bar{a}_2, \bar{b}_2, \bar{c}_2)(0) = (1, b_2, -1)\,.$$

Case (a) $b_1 \neq 1/2$

First we consider the situation $c_1(\mu) \equiv 0$ and $a_1(\mu) \equiv 0$ for $|\mu|$ small. If $A_{(\alpha,\beta,\gamma,\delta,\mu)}$ is taken so that $\beta = \gamma = 0$ we leave these conditions invariant by the coordinate transformation $\overline{\phi}$ and

$$\bar{a}_2(\mu) = (1+\delta)^3\big(1 + A_2(\mu)\big)$$
$$\bar{c}_2(\mu) = (1+\alpha)^2(1+\delta)\big(-1 + C_2(\mu)\big)\,.$$

Then, it is clear that there exists $\alpha = \alpha(\mu)$ and $\delta = \delta(\mu)$ such that $\bar{a}_2(\mu) = 1$ and $\bar{c}_2(\mu) = -1$ for every μ near the origin.

Now we study the case in which if $c_1(\mu) \equiv 0$ for $|\mu|$ small. As above, by taking $\gamma = 0$, we leave this condition invariant by $\bar{\phi}$. Moreover if we take $\alpha = 0$ and $\delta = 0$ too, we obtain

$$\bar{a}_1(t) = A_1(\mu) + 2\beta(b_1 + B_1(\mu))$$

and as $b_1 \neq 0$, we can choose $\beta = \beta(\mu)$ such that $\bar{a}_1(\mu) \equiv 0$ for $|\mu|$ small.

Finally, if $c_1(\mu) \not\equiv 0$, by taking $\alpha = \beta = \delta = 0$ we obtain

$$\bar{c}_1(\mu) =$$
$$= C_1(\mu) + \gamma\big(2b_1 - 1 + 2B_1(\mu) + C_2(\mu)\big) + \gamma^2\big(a_1(\mu) + 2b_2(\mu)\big) + \gamma^3 a_2(\mu)$$

and follows from Implicit Function Theorem, that there exist $\gamma = \gamma(\mu)$ for which the new $\bar{c}_1(\mu) \equiv 0$.

Case (b) $b_1 = 1/2$

First we suppose $a_1(\mu) + 2b_2(\mu) \equiv 2b_2$ for $|\mu|$ small. Taking $\beta = \gamma = 0$ we leave this condition invariant and

$$\bar{a}_2(\mu) = (1 + \delta)^3(1 + A_2(\mu))$$
$$\bar{b}_1(\mu) = (1 + \alpha)^2(1 + \delta)\left(\frac{1}{2} + B_1(\mu)\right).$$

Then it is clear, from Implicit Function Theorem, that there exist $\alpha = \alpha(\mu)$ and $\delta = \delta(\mu)$ with $\alpha(0) = \delta(0) = 0$ such that $\bar{a}_2(\mu) \equiv 1$ and such that $\bar{b}_1(\mu) \equiv 1/2$.

If $\alpha_1(\mu) + 2b_2(\mu) \not\equiv 2b_2$, taking $\alpha = \beta = \delta = 0$ we obtain

$$\bar{a}_1(\mu) + 2\bar{b}_2(\mu) = 2b_2 + A_1(\mu) + 2B_2(\mu) + 3\gamma(1 + A_2(\mu))$$

and by Implicit Function Theorem this situation can be reduced to the one right above.

Thus the proof is complete. $\square$
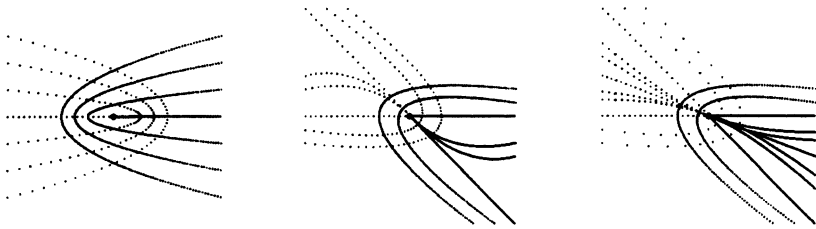
5.2 Versal unfolding of a D_{12} singularity

The object of this subsection is to prove the following result.

THEOREM 5.7. — *A versal unfolding of a singularity of D_{12} type is*

$$y \, dy^2 + 2\big((1 - \lambda)x + y\big) \, dx \, dy - y \, dx^2$$

with $\lambda \in \mathbb{R}$, $\lambda < 1/2$.

Observe first that for this family the origin is of type D_1, D_{12} or D_2 according as λ is negative, zero or positive, respectively.

Fig. 4

Proof of Theorem. — Let

$$\omega_\mu = a(x, y, \mu) \, dy^2 + 2b(x, y, \mu) \, dx \, dy + c(x, y, \mu) \, dx^2$$

be an arbitrary smooth family with parameter $\mu \in \mathbb{R}^k$ such that ω_0 has at the origin a D_{12} singular point.

We may suppose that

$$a(x, y, \mu) = y + M_1(x, y, \mu)$$
$$b(x, y, \mu) = B_1(\mu)x + B_2(\mu)y + M_2(x, y, \mu)$$
$$c(x, y, \mu) = -y + M_3(x, y, \mu)$$

with $2B_1(0) = B_2(0)^2 + 1$, $B_1(0) \neq 0, 1/2$ and $M_k(x, y, \mu) = \mathrm{O}\big((x^2 + y^2)^{1/2}\big)$ for $k = 1, 2, 3$ and $|\mu|$ small (see Lemma 5.6).

It follows from Proposition 4.1 that, for $|\mu|$ small, this family is C^0-equivalent to the family

$$\widetilde{\omega}_\mu = y \, dy^2 + 2(B_1(\mu)x + B_2(\mu)y) \, dx \, dy - y \, dx^2 \, .$$

Let us consider the real valued function ψ defined in a neighorhood of the origin of $\mathbb{R}^k$ by means of

$$\psi(\mu) = B_2(\mu)^2 - 2B_1(\mu) + 1$$

and the family

$$v_\lambda = y\,\mathrm{d}y^2 + 2((1-\lambda)x + y)\,\mathrm{d}x\,\mathrm{d}y - y\,\mathrm{d}x^2$$

with parameter $\lambda \in \mathbb{R}$.

Then the unfolding induced by ψ from the family $(v_\lambda)_{\lambda \in \mathbb{R}}$

$$\overline{\omega}_\mu = v_{\psi(\mu)} = y\,\mathrm{d}y^2 + 2\Big((1-\psi(\mu))x + y\Big)\,\mathrm{d}x\,\mathrm{d}y - y\,\mathrm{d}x^2\,.$$

Finally, as the separatrix polynomials $\widetilde{\omega}_\mu(x,y)(x,y)$ and $\overline{\omega}_\mu(x,y)(x,y)$ have discriminants which only differ by a positive factor, the family $\widetilde{\omega}_\mu$ is C^0-equivalent to $\overline{\omega}_\mu$. This completes the proof.

5.3 Versal unfolding of a $\widetilde{D}_1$-singularity

This subsection is devoted to prove that a versal unfolding of a $\widetilde{D}_1$-singular point is:

$$v(\lambda_1, \lambda_2) = y\,\mathrm{d}y^2 + x\,\mathrm{d}x\,\mathrm{d}y + (\lambda_1 x + (-1 + \lambda_2)y)\,\mathrm{d}x^2$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, small.

Remark 5.8. — When $1 - \lambda_2 - \lambda_1^2 > 0$, the origin is a simple singular point of $v(\lambda_1, \lambda_2)$. For these parameters, the separatrix polynomial is

$$y^3 + \lambda_2 x^2 y + \lambda_1 x^3$$

which has discriminant

$$\Delta(\lambda_1, \lambda_2) = 4\lambda_2^3 + 27\,\lambda_1^2\,.$$

Then:

(a) The origin is a D_1-singular point (resp. a D_2-singular point) of $v(\lambda_1, \lambda_2)$ if $\Delta(\lambda_1, \lambda_2)$ is negative (resp. positive).

(b) The origine is a D_{12}-singular point if $\Delta(\lambda_1, \lambda_2) = 0$ and $(\lambda_1, \lambda_2) \neq (0,0)$.

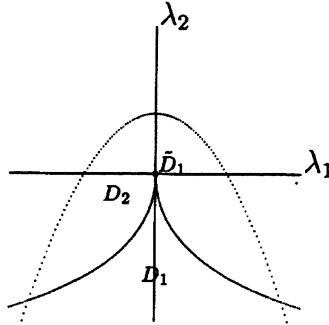(c) The origin is a $\widetilde{D}_1$-singular point if $(\lambda_1, \lambda_2) = (0,0)$.

Fig. 5

THEOREM 5.9. — *The versal unfolding of a singularity of type $\widetilde{D}_1$ is*

$$y \, dy^2 + x \, dx \, dy + \left(\lambda_1 x + (-1 + \lambda_2)y\right) dx^2$$

with λ_1, $\lambda_2 \in \mathbb{R}$, $1 - \lambda_2 - \lambda_1^2 > 0$.

Proof. — Let

$$\omega_\mu = a(x, y, \mu) \, dy^2 + 2b(x, y, \mu) \, dx \, dy + c(x, y, \mu) \, dx^2$$

be an arbitrary smooth family with parameter $\mu \in \mathbb{R}^k$ such that ω_0 has at the origin a $\widetilde{D}_1$ singular point.

By using Lemma 5.6, we may suppose that

$$a(x, y, \mu) = A_1(\mu)x + y + M_1(x, y, \mu)$$
$$b(x, y, \mu) = \frac{1}{2}\left(x - A_1(\mu)y\right) + M_2(x, y, \mu)$$
$$c(x, y, \mu) = C_1(\mu)x + (-1 + C_2(\mu))y + M_3(x, y, \mu)$$

with $A_1(0) = C_1(0) = C_2(0) = 0$ and $M_k(x, y, \mu) = O((x^2 + y^2)^{1/2})$ for $k = 1, 2, 3$ and $|\mu|$ small.

It follows from Proposition 4.1 that, for $|\mu|$ small, this family is C^0-equivalent to the family.

$$\widetilde{\omega}_\mu = \left(A_1(\mu)x + y\right) dy^2 + \left(x - A_1(\mu)\right) dx \, dy + \left(C_1(\mu)x + (-1 + C_2(\mu))y\right) dx^2.$$

Let us consider the real bi-valued function ψ defined in a neighborhood of the origin of $\mathbb{R}^k$ by means of

$$\psi(\mu) = \big(C_1(\mu), C_2(\mu)\big)$$

and the family

$$v_{\lambda_1,\lambda_2} = y\,\mathrm{d}y^2 + x\,\mathrm{d}x\,\mathrm{d}y + \big(\lambda_1 x + (-1 + \lambda_2)y\big)\,\mathrm{d}x^2$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$.

Therefore the unfolding induced by ψ from the family v_{λ_1,λ_2} is

$$\overline{\omega}_\mu = v_{\psi(\mu)} = y\,\mathrm{d}y^2 + x\,\mathrm{d}x\,\mathrm{d}y + \Big(C_1(\mu)x + (-1 + C_2(\mu))y\Big)\,\mathrm{d}x^2\,.$$

Finally, as the polynomials $\widetilde{\omega}_\mu(x,y)(x,y)$ and $\overline{\omega}_\mu(x,y)(x,y)$ are equal for every μ, we have that the family $\widetilde{\omega}_\mu$ is C^0-equivalent to $\overline{\omega}_\mu$. This completes the proof. $\square$

Acknowledgements

References

[Du] DUMORTIER (F.) .— *Singularities of vector fields on the plane*, J. Differential Equation **23**, n° 1 (1977), pp. 53-106.

[DRS] DUMORTIER (F.), ROUSSARIE (R.) and SOTOMAYOR (J.) .— *Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case*, Erg. Theory and Dyn. Syst. **7** (1987), pp. 375-413.

[Gu1] GUÍÑEZ (V.) .— *Positive quadratic differential forms and foliations with singularities on surfaces*, Trans. Amer. Math. Soc. **309**, n° 2 (1988), pp. 477-502.

[Gu2] GUÍÑEZ (V.) .— *Nonorientable polynomial foliations on the plane*, J. Differential Equations **87** (1990), pp. 391-411.

[Gu3] GUÍÑEZ (V.) .— *Locally stable singularities for positive quadratic differential forms*, J. Differential Equations **110** (1994), pp. 1-37.

[GS1] GUTIERREZ (C.) and SOTOMAYOR (J.) .— *An approximation theorem for immersions with stable configurations of line of principal curvature*, Lecture Notes in Math. **1007**, Springer (1983), pp. 332-368.

[GS2] GUTIERREZ (C.) and SOTOMAYOR (J.) .— *Structurally stable configurations of lines of principal curvature*, Astérique **98-99** (1982), pp. 195-215.

[HPS] HIRSCH (M.), PUGH (C.) and SHUB (M.) . — *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer Verlag 1977.

[Kl] KLOTZ (T.) . — *On Bol's proof of Carathéodory's Conjecture*, Comm. Pure Appl. Math. **12** (1959), pp. 277-311.

[Ma] MARTINET (J.) . — *Singularities of differentiable functions*, London Math. Soc. Lecture Notes Series **58**, Cambridge Univ. Press, 1982.

[SX] SMYTH (B.) and XAVIER (F.) . — *A geometric estimate for the index of an umbilic point on a smooth surface*, Bull. London Math. Soc. **24** (1992), pp. 176-180.

[So] SOTOMAYOR (J.) . — *Generic one-parameter families of vector fields on two-dimensional manifolds*, Publ. Math. IHES. **43** (1974), pp. 5-46.

[Ta] TAKENS (F.) . — *Singularities of vector fields*, Publ. Math. IHES **43** (1974), pp. 47-100.