ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

MANUEL DE LEÓN
EUGENIC MERINO
JOSÉ A. OUBIÑA
MODESTO SALGADO

A global characterization of jet bundles of p^1-velocities and covelocities

Annales de la faculté des sciences de Toulouse 6e série, tome 4, nº 1 (1995), p. 61-75

<http://www.numdam.org/item?id=AFST_1995_6_4_1_61_0>

Annales de la Faculté des Sciences de Toulouse Vol. IV, n° 1, 1995

A global characterization of jet bundles of p^1-velocities and covelocities(*)

MANUEL DE LEÓN[1], EUGENIO MERINO[2],
JOSÉ A. OUBIÑA[3] and MODESTO SALGADO[3]

RÉSUMÉ. — Le but de ce travail est de montrer des caractérisations globales du fibré tangent des p^1-vitesses et du fibré cotangent des p^1-covitesses. Dans le premier cas, la caractérisation que l'on exhibe généralise des résultats précédents des auteurs. La deuxième caractérisation étend les résultats de Nagano sur le fibré cotangent. Les démonstrations s'appuient dans la caractérisation de Nagano des fibrés vectoriels à l'aide des propriétés des champs de vecteurs canoniques.

ABSTRACT. — We give global characterizations of tangent and cotangent bundles of p^1-velocities and covelocities. The first one generalizes the previous results of the authors for tangent bundles and the second one extends the results of Nagano for cotangent bundles. The Nagano's characterization of vector bundles by the properties of its canonical vector fields is widely used.

KEY-WORDS : p-symplectic manifolds, p-tangent manifolds, jet bundles, p^1-velocities, p^1-covelocities.

MS Classification (1991) : 58A20, 53C15.

(*) Reçu le 1 juillet 1993
Partially supported by DGICYT-Spain, Proyecto PB91-0142, Xunta de Galícia, Proxecto XUGA8050189 and Universidade de Coruña.

(1) Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid (Spain)

(2) Departamento de Matemáticas, Escola Politécnica Superior de Enxeñería, Esteiro, 15403 Ferrol, Universidade de Coruña (Spain)

(3) Departamento de Xeometría e Topoloxía, Facultade de Matemáticas, Universidade de Santiago de Compostela (Spain).

1. Introduction

The problem of the characterization of tangent and cotangent bundles has been studied by several authors [4], [5], [10], [18], [19], [20]. In the case of the characterization of cotangent bundles, Nagano [18] has proved that if M is a differentiable manifold endowed with a vector field C satisfying the same properties of those satisfied by the canonical vector field of a vector bundle, then there exists a unique bundle structure on M over the singular submanifold S of C such that C is the canonical vector field. If, moreover, M is an exact symplectic manifold then M is isomorphic to the cotangent bundle T^*S, indeed as symplectic vector bundles. A different approach was used for the case of tangent bundles [4]. Since the tangent bundle of an arbitrary manifold possesses a canonical almost tangent structure, the starting point is to consider an almost tangent manifold M. If M is integrable and satisfies some global hypothesis, then it is possible to prove that M is an affine bundle modelled on the tangent bundle TN of some manifold N and hence diffeomorphic to it. Moreover, if the affine bundle admits a global section, then M is isomorphic to TN via the isomorphism induced by the section. This result can be extended to cotangent bundles ([19], [20]). Recently we have used the ideas of Nagano to give a characterization of tangent bundles [10].

In [6], [7] we have extended these results to the global characterization of tangent and cotangent bundles of p^1-velocities and covelocities by using similar procedures to thoses of Crampin, Thompson et al. Thus, we have proved that an integrable p-almost tangent (cotangent) manifold satisfying some global hypotheses is an affine bundle modelled in a tangent (cotangent) bundle of p^1-velocities. In this paper, we return to the idea of Nagano and prove that an integrable p-almost tangent (resp. cotangent) manifold endowed with a family of p vector fields $C_1, \ldots, C_p$ satisfying some hypotheses is globally isomorphic as a vector bundle over S to $T_p^1 S$ (resp. $(T_p^1)^* S$) where S is the singular submanifold defined by the vector fields $C_1, \ldots, C_p$.

All these problems are interesting for Mechanics and Classical Field theories. In fact, tangent and cotangent bundles are the natural framework where the Lagrangian and Hamiltonian formalisms are developed [15]. Also classical field theories may be formulated in tangent bundles of p^1-velocities and covelocities ([11], [12], [13], [14]). Then it is relevant to have some criteria to decide when a manifold is globally a jet bundle.

The paper is structured as follows. In section 2 we recall the results of Nagano. Sections 3 and 4 are devoted to give a global characterization of tangent bundles of p^1-velocities, and in sections 5 and 6 we consider the case of the cotangent bundles of p^1-covelocities.

2. Characterization of vector bundles

Let M be a differentiable manifold and X a vector field on M. If $x \in M$ is a singular point of X, i.e. $X_x = 0$, then we define the *characteristic operator* $(A_X)_x$ of X at x as the linear endomorphism $(A_X)_x : T_xM \to T_xM$ given by

$$(A_X)_x(Y) = \nabla_Y X ,$$

where ∇ is an arbitrary linear connection on M. If we choose local coordinates (x^i) on M and put

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^i \frac{\partial}{\partial x^i}, \quad \nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

then we have

$$(A_X)_x Y = Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}, \tag{2.1}$$

since x is a singular point. Hence $(A_X)_x$ does not depend on ∇.

Now, let M be the total space of a vector bundle $M \to N$. Then the *canonical vector field of the vector bundle* M is the infinitesimal generator C of the global flow on M induced by the scalar multiplication on each fibre. This vector field satisfies the following properties :

(i) C is complete, i.e. it generates a global one-parameter transformation group on M;

(ii) for each point $x \in M$, there exists a unique $\lim_{t\to-\infty}(\exp tC)(x)$, where $\exp tC$ denotes the flow of C;

(iii) the characteristic operator $(A_C)_x$ associated to C satisfies

$$((A_C)_x)^2 = (A_C)_x$$

for each singular point x of C;

(iv) the set S of the singular points of C is a submanifold of M of codimension $= \operatorname{rank}(A_C)_x$ for all $x \in S$.

In fact, choose bundle coordinates (x^i, y^a) on M, where (x^i) are coordinates in N and (y^a) are coordinates in the fibre. Then C is locally expressed by

$$C = y^a \frac{\partial}{\partial y^a}.$$

Hence the singular set S of C is the zero section of M, and so, it is diffeomorphic to N.

Nagano [8] has proved the converse:

THEOREM 2.1.— *Suppose that there exists a vector field C on a manifold M satisfying the above conditions (i)-(iv). Then there exists a unique vector bundle structure on M such that C is the canonical vector field.*

We give a sketch of the proof. If S is the singular submanifold, we put

$$N(S)_x = \{X \in T_x M \mid (A_C)_x(X) = X\}, \tag{2.2}$$

for each $x \in S$. Then $N(S)$ is the normal bundle of S in M, i.e.

$$(TM)|_S = TS \oplus N(S).$$

Moreover we have

$$T_x S = \{X \in T_x M \mid (A_C)_x(X) = 0\}. \tag{2.3}$$

Then we can define a map $\phi : N(S) \to M$ as follows. We first define the exponential map $\exp : E \to M$ with respect to some linear connection, where E is a sphere bundle $E \subset N(S)$ and then we extend ϕ to $N(S)$. This construction is possible from the properties of C. Moreover ϕ becomes a diffeomorphism and then the vector bundle structure on $N(S) \to S$ is transferred to $M \to S$ in such a way that C becomes the canonical vector field of $M \to S$.

As a direct consequence we have the following.

COROLLARY 2.1.— *Two vector bundles are isomorphic if and only if there exists a diffeomorphism which preserves the canonical vector fields.*

3. p-Almost tangent structures

Let M be a $(p+1)n$-dimensional manifold and let there be given a p-tupla of tensor fields $(J_1, \ldots, J_p)$ of type $(1,1)$ such that

(1) $J_a \circ J_b = J_b \circ J_a = 0$,

(2) rank $J_a = n$,

(3) $\mathrm{Im}\, J_a \cap \left(\bigoplus_{b\neq a} \mathrm{Im}\, J_b\right) = 0$, for each a, $1 \leq a \leq p$.

The p-tupla $(J_1, \ldots, J_p)$ is called a *p-almost tangent structure* on M and M is said to be a *p-almost tangent manifold* ([8], [16]).

Let N be a differentiable manifold and $T^1_p N$ its tangent bundle of p^1-velocities, i.e. the manifold of all 1-jets of mappings from $\mathbb{R}^p$ to N at the origin $0 \in \mathbb{R}^p$ (see [17]). $T^1_p N$ is a manifold of dimension $(p+1)n$. We denote by $\pi : T^1_p N \to N$ the canonical projection defined by $\pi(j^1_{0,x} f) = x$. We have a canonical diffeomorfism

$$\Gamma : T^1_p N \longrightarrow TN \oplus \cdots \oplus TN ,$$

of $T^1_p N$ onto the Whitney sum of TN with itself p times, defined by

$$\Gamma(j^1_{0,x} f) = (j^1_{0,x} f_1, \ldots, j^1_{0,x} f_p) ,$$

where f_a $(a = 1, \ldots, p)$ is the curve on N given by

$$f_a(t) = f(0, \ldots, t, \ldots, 0) ,$$

where t is placed at the a-th position. Then, each element $X \in \left(T^1_p N\right)x = \pi^{-1}(x)$, $x \in N$, may be identified via Γ with a p-tupla $(X_1, \ldots, X_p)$ of vectors $X_a \in T_x N$, $1 \leq a \leq p$, and $\pi : T^1_p N \to N$ has a unique vector bundle structure such that Γ is a vector bundle isomorphism.

Now, we may define p tensor fields J_a $(a = 1, \ldots, p)$ of type $(1,1)$ on $T^1_p N$ as the respective (a)-lifts of the identity tensor of N to $T^1_p N$ [17]. If we consider fibred coordinates (x^i, x^i_a) on $T^1_p N$ then we have

$$J_a \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i_a} , \quad J_a \left(\frac{\partial}{\partial x^i_b}\right) = 0 ,$$

and $(J_1, \ldots, J_p)$ defines the canonical p-almost tangent structure on $T^1_p N$.

For each a, $1 \leq a \leq p$, we have a canonical projection

$$\mu^a : T_p^1 N \longrightarrow T_{p-1}^1 N ,$$

defined by $\mu^a(X_1, \ldots, X_p) = (X_1, \ldots, \widehat{X}_a, \ldots, X_p)$, for $p > 1$, where the hat over a term means that it is to be omitted, and the tangent bundle projection $\mu^1 : T_p^1 N = TN \to N$ for $p = 1$. Denote by C_a, $1 \leq a \leq p$, the canonical vector field of the vector bundle $\mu^a : T_p^1 N \to T_{p-1}^1 N$, which, in fibred coordinates, is expressed by

$$C_a = x_a^i \frac{\partial}{\partial x_a^i} .$$

Then we obtain the following identities:

$$[\, C_a \, , \, C_b \,] = 0 \, , \quad L_{C_a} J_b = -\delta_{ab} J_b \, , \quad J_a C_b = 0 \, , \quad 1 \leq a \, , \, b \leq p \, . \tag{3.1}$$

A p-almost tangent structure $(J_1, \ldots, J_p)$ on a manifold M is integrable if and only if M is locally isomorphic to a tangent bundle of p^1-velocities. In [8] we have proved that a such structure $(J_1, \ldots, \dot{J}_p)$ on M is integrable if and only if $\{J_a, J_b\} = 0$, $1 \leq a, b \leq p$, where $\{J_a, J_b\}$ is the $(1,2)$-type tensor field defined by

$$\{J_a, J_b\}(X,Y) = [J_a X \, , \, J_b Y] - J_a[X \, , \, J_b Y] - J_b[J_a X \, , \, Y] \, .$$

In the next section we shall prove that an integrable p-almost tangent manifold satisfying some additional hypotheses is globally a tangent bundle of p^1-velocities.

4. Characterization ot tangent bundles of p^1-velocities

THEOREM 4.1.— *Let M be a $(p+1)n$-dimensional manifold endowed with an integrable p-almost tangent structure $(J_1, \ldots, J_p)$ and p vector fields $C_1, \ldots, C_p$ on M satisfying (3.1), i.e.*

$$[\, C_a \, , \, C_b \,] = 0 \, , \quad L_{C_a} J_b = -\delta_{ab} J_b \, , \quad J_a C_b = 0 \, , \quad 1 \leq a \, , \, b \leq p \, .$$

If $C_1, \ldots, C_p$ also satisfy the condition (i)-(ii), then there exists a unique vector bundle structure on M which is isomorphic to the tangent bundle $T_p^1 S$

of p^1-velocities of the singular submanifold S of $C = C_1 + \cdots + C_p$. Moreover this isomorphism transport the canonical p-almost tangent structure and the canonical vector fields of $T^1_p S$ to $(J_1, \ldots, J_p)$ and $C_1, \ldots, C_p$, respectively.

Proof.— Since $(J_1, \ldots, J_p)$ is integrable then there exist adapted local coordinates $(\overline{x}^i, \overline{x}^i_a)$ in such a way that $J_1, \ldots, J_p$ are locally given by

$$J_a\left(\frac{\partial}{\partial \overline{x}^i}\right) = \frac{\partial}{\partial \overline{x}^i_a}, \quad J_a\left(\frac{\partial}{\partial \overline{x}^i_b}\right) = 0 .$$

Suppose that C_a is locally written by

$$C_a = (A_a)^i \frac{\partial}{\partial \overline{x}^i} + (B_a)^i_b \frac{\partial}{\partial \overline{x}^i_b},$$

where

$$(A_a)^i = (A_a)^i(\overline{x}, \overline{x}_1, \ldots, \overline{x}_p), \quad (B_a)^i_b = (B_a)^i_b(\overline{x}, \overline{x}_1, \ldots, \overline{x}_p) .$$

From $J_c C_a = 0$ we deduce $(A_a)^i = 0$, and from $L_{C_a} J_c = -\delta_{ac} J_c$ we obtain

$$\frac{\partial (B_a)^i_c}{\partial \overline{x}^j_d} = \delta^d_a \delta^d_c \delta^i_j .$$

Then we can write C_a as follows:

$$C_a = (B_a)^i_a(\overline{x}, \overline{x}_1, \ldots, \overline{x}_p) \frac{\partial}{\partial \overline{x}^i_a} + \sum_{b \neq a} (B_a)^i_b(\overline{x}) \frac{\partial}{\partial \overline{x}^i_b} .$$

Now, using that $[C_a, C_b] = 0$ we obtain $(B_a)^i_b = (B_b)^i_a = 0$ whenever $a \neq b$. Hence we have

$$C_a = (B_a)^i_a(\overline{x}, \overline{x}_1, \ldots, \overline{x}_p) \frac{\partial}{\partial \overline{x}^i_a} .$$

Define a new system of local coordinates (x^i, x^i_a) by

$$x^i = \overline{x}^i, \quad x^i_a = (B_a)^i_a(\overline{x}, \overline{x}_c) .$$

Thus we obtain

$$C_a = x^i_a \frac{\partial}{\partial x^i_a}, \tag{4.1}$$

and moreover (x^i, x^i_c) are also coordinates adapted to $(J_1, \ldots, J_p)$. Then the singular submanifold S_a defined by C_a is locally defined by the vanishing of the coordinates $x^i_a = 0$, and then it has dimension pn. Further, if we consider the vector field $C = C_1 + \cdots + C_p$ then

$$C = x^i_1 \frac{\partial}{\partial x^i_1} + \cdots + x^i_p \frac{\partial}{\partial x^i_p}, \tag{4.2}$$

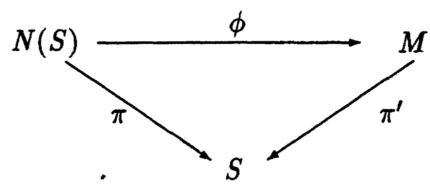
and hence the singular submanifold S defined by C has dimension n. In fact, we have $S = S_1 \cap \ldots \cap S_p$.

From (4.2) we directly deduce that C satisfies the conditions (iii) and (iv). The conditions (i) and (ii) are easily deduced as follows. Since $[C_a\,,\,C_b] = 0$ then

$$\exp tC = \exp tC_1 \circ \cdots \circ \exp tC_p\,,$$

from which we deduce that C is complete and for each $x \in M$, there exists a unique $\lim_{t\to-\infty}(\exp tC)(x)$.

Now, by Nagano's theorem we obtain a unique vector bundle structure on M over S such that C is the canonical vector field and we have an isomorphism ϕ:



where π is the canonical projection and π' is the induced projection via ϕ.

Note that in coordinates (x^i, x^i_a) the characteristic operator A_C is given by

$$(A_C)_x\left(\frac{\partial}{\partial x^i}\right) = 0\,, \quad (A_C)_x\left(\frac{\partial}{\partial x^i_a}\right) = \frac{\partial}{\partial x^i_a}\,,$$

at each point $x \in S$. Therefore, we have

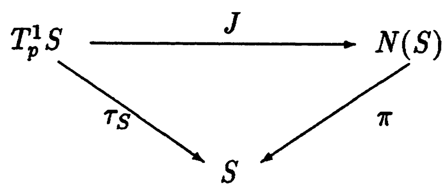
$$N(S)_x = \left\{X \in T_xM \;\middle|\; X = \sum_{a=1}^{p} X^i_a \frac{\partial}{\partial x^i_a}\right\},$$

$$T_xS = \left\{X \in T_xM \;\middle|\; X = X^i \frac{\partial}{\partial x^i}\right\}.$$

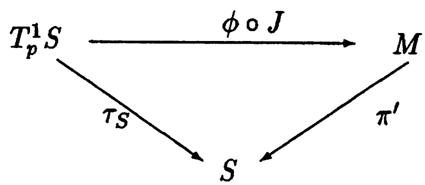
Then we can consider J_a as a vector bundle homomorphism $J_a : TS \to N(S)$ and define a vector bundle isomorphism $J : TS \oplus \cdots \oplus TS \cong T^1_p S \to N(S)$ by

$$J(X_1, \ldots, X_p) = \sum_{a=1}^{p} J_a X_a ,$$

Thus we obtain the following commutative diagram:



where $\tau_S : T^1_p S \to S$ is the canonical projection. Combining both results we obtain a vector bundle isomorphism



which applies the canonical p-almost tangent structure and the canonical vector fields of $T^1_p S$ to $(J_1, \ldots, J_p)$ and $C_1, \ldots, C_p$, respectively.

Finally, the unicity is a direct consequence of Corollary 2.1. □

5. p-Almost cotangent structures

Suppose that M is a $(p+1)n$-dimensional manifold endowed with a family $\{\omega_a, V_a, \ 1 \leq a \leq p\}$ of p 2-forms ω_a of rank $2n$ and p n-dimensional distributions V_a such that

(1) $V_{a_1 a_2 \ldots a_r} \cap V_a = 0$, for each $a \neq a_1, \ldots, a_r$, $1 \leq a_1 < \cdots < a_r \leq p$,

(2) $K_a = \mathrm{Ker}\, s_{\omega_a} = \bigoplus_{b=1, b\neq a}^{p} V_b$,

(3) $\omega_a|_{V_a \times V_a} = 0$,

where $V_{a_1 a_2 \ldots a_r} = V_{a_1} + \cdots + V_{a_r}$ and $s_{\omega_a} : TM \to T^*M$ is the bundle morphism defined by $s_{\omega_a}(X) = i_X \omega_a$. From (2) we have rank

$\omega_a = \operatorname{codim} K_a = 2n$. The family $\{\omega_a, V_a, 1 \leq a \leq p\}$ is called a *p-almost cotangent structure*, and such a manifold M is called a *p-almost cotangent manifold* ([9], [16]).

Remark 5.1.— These kind of geometric structures was independently introduced by Awane ([1]-[3]) and named p-symplectic structures. A p-almost cotangent structure is in fact a reduction of $\mathrm{Gl}\big((p+1)n, \mathbb{R}\big)$ to the p-symplectic group $\mathrm{SP}(p, n; \mathbb{R})$ ([1], [2], [9], [16]).

Let N be a manifold of dimension n and denote by $(T_p^1)^* N$ the *cotangent bundle of p^1-covelocities of N*, i.e. the manifold of all 1-jets of mappings from N to $\mathbb{R}^p$ with target $0 \in \mathbb{R}^p$. $(T_p^1)^* N$ is a manifold of dimension $(p+1)n$. We denote by $\pi : (T_p^1)^* N \to N$ the canonical projection defined by $\pi(j^1_{x,0} f) = x$. Now, we have a canonical diffeomorphism

$$\Lambda : (T_p^1)^* N \longrightarrow T^*N \oplus \cdots \oplus T^*N ,$$

of $(T_p^1)^* N$ with the Whitney sum of T^*N with itself p times. Λ is given by

$$\Lambda(j^1_{x,0} f) = (j^1_{x,0} f^1, \ldots, j^1_{x,0} f^p) ,$$

where $f(x) = \big(f^1(x), \ldots, f^p(x)\big) \in \mathbb{R}^p$. Then each element

$$\theta \in \Big((T_p^1)^* N\Big)_x = \pi^{-1}(x) , \quad x \in N ,$$

may be identified, via Λ with a p-tupla $(\theta^1, \ldots, \theta^p)$ of 1-forms $\theta^a \in T_x^* N$, $1 \leq a \leq p$, and we consider $\pi : (T_p^1)^* N \to N$ as a vector bundle over N isomorphic to the Whitney sun of T^*N with itself p times. Obviously, when $p = 1$, then $(T_1^1)^* N = T^*N$. Moreover, for each a, $1 \leq a \leq p$, we have a canonical projection

$$\rho^a : (T_1^p)^* N \longrightarrow (T_{p-1}^1)^* N$$

defined by $\rho^a(\theta^1, \ldots, \theta^p) = (\theta^1, \ldots, \widehat{\theta^a}, \ldots, \theta^p)$, for $p > 1$, and the cotangent bundle projection $\rho^1 : T^*N \to N$ for $p = 1$. Then we have p canonical vertical distributions $V_a = \operatorname{Ker} T\rho^a$, $1 \leq a \leq p$. Furthermore we have

$$V = \bigoplus_{a=1}^{p} V_a = \operatorname{Ker} T\pi .$$

Now, we may define p presymplectic forms ω_a, $1 \leq a \leq p$ as follows. First, we define a canonical 1-form λ_a for each a, $1 \leq a \leq p$, by setting

$$\lambda_a(\theta)(X) = \theta^a(x)\big(T\pi(X)\big)\,, \quad X \in T_\theta\left(\left(T^1_p\right)^* N\right), \quad \pi(\theta) = x\,.$$

If we consider fibred coordinates (x^i, x^a_i) on $\left(T^1_p\right)^* N$ then we have

$$\lambda_a = x^a_i dx^i\,, \quad \omega_a = -d\lambda_a = dx^i \wedge dx^a_i\,, \quad V_a = \left\langle \frac{\partial}{\partial x^a_i} \right\rangle .$$

Denote by $C_1, \ldots, C_p$ the canonical vector fields of the vector bundles $\rho^a : \left(T^1_p\right)^* N \to \left(T^1_{p-1}\right)^* N$. In fibred coordinates we have

$$C_a = x^a_i \frac{\partial}{\partial x^a_i}\,.$$

Then C_a satisfies (i)-(iv). We also have the following identities

$$(\lambda_a)\big|_{V_b} = 0\,, \quad \mathrm{i}_{C_a}\omega_b = -\delta_{ab}\lambda_b\,, \quad 1 \leq a, b \leq p\,.$$

In terms of G-structures, a p-almost cotangent structure $\{\omega_a, V_a\}$ on M is integrable if and only if the p-almost cotangent manifold M is locally isomorphic to a cotangent bundle of p^1-covelocities. In such a case, we can choose local coordinates (x^i, x^a_i) around each point such that

$$\omega_a = dx^i \wedge dx^a_i\,, \quad V_a = \left\langle \frac{\partial}{\partial x^a_i} \right\rangle .$$

In [9] we have proved that $\{\omega_a, V_a\}$ in integrable if and only if each 2-form ω_a is closed and the distribution $V_1 \oplus \cdots \oplus V_p$ is involutive.

Moreover we can prove that the mapping $X \to \mathrm{i}_X\omega_a$ defines an isomorphism of vector bundles over M of V_a onto $\nu^* V$, where $\nu^* V$ denotes the dual vector bundle of the transverse bundle νV determined by the foliation V (see [3]).

6. Characterization of cotangent bundles of p^1-covelocities

In this section we shall prove that an integrable p-almost cotangent manifold with some additional hypothesis is globally a cotangent bundle of p^1-covelocities.

THEOREM 6.1.— *Let M be a $(p+1)n$-dimensional manifold endowed with an integrable p-almost cotangent structure $\{\omega_a, V_a, 1 \leq a \leq p\}$ such that the presymplectic forms ω_a are globally exact, i.e. $\omega_a = -d\lambda_a$ with $(\lambda_a)|_{V_b} = 0$. Consider the vector fields $C_1, \ldots, C_p$ on M defined by*

$$\mathrm{i}_{C_a}\omega_b = -\delta_{ab}\lambda_b\,.$$

If the vector fields $C_1, \ldots, C_p$ satisfy (i)-(ii) then there exists a unique vector bundle structure on M which is isomorphic to the cotangent bundle $(T_p^1)^*S$ of p^1-covelocities of the singular submanifold S of $C = C_1 + \cdots + C_p$. Moreover this isomorphism transports the canonical p-almost cotangent structure and the canonical vector fields of $(T_p^1)^*S$ to $\{\omega_a, V_a\}$ and $C_1, \ldots, C_p$, respectively.

Proof.— Since $\{\omega_a, V_a\}$ is integrable, then there exist adapted coordinates $\overline{x}^i, \overline{x}_i^1, \ldots, \overline{x}_i^p$ such that

$$\omega_a = d\overline{x}^i \wedge d\overline{x}_i^a\,, \quad V_a = \left\langle \frac{\partial}{\partial \overline{x}_i^a} \right\rangle.$$

Hence, from $\omega_a = -d\lambda_a$ we deduce

$$\lambda_a = \overline{x}_i^a d\overline{x}^i + df_a = \left(\overline{x}_i^a + \frac{\partial f_a}{\partial \overline{x}^i}\right) d\overline{x}^i + \sum_{b=1}^{p} \frac{\partial f_a}{\partial \overline{x}_b^i}\, d\overline{x}_b^i\,.$$

But $(\lambda_a)|_{V_b} = 0$ implies that

$$\lambda_a = \left(\overline{x}_i^a + \frac{\partial f_a}{\partial \overline{x}^i}\right) d\overline{x}^i\,.$$

If we define new coordinates (x^i, x_i^a) by

$$x^i = \overline{x}^i\,, \quad x_i^a = \overline{x}_i^a + \frac{\partial f_a}{\partial \overline{x}^i}\,,$$

then we have $\lambda_a = x_i^a dx^i$. Consider on M the vector fields $C_1, \ldots, C_p$ defined by

$$\mathrm{i}_{C_a}\omega_b = -\delta_{ab}\lambda_b\,.$$

From a direct computation we obtain

$$C_a = x_i^a \frac{\partial}{\partial x_i^a}\,, \quad 1 \leq a \leq p\,,$$

and hence

$$C = \sum_{a=1}^{p} C_a = x_i^1 \frac{\partial}{\partial x_i^1} + \cdots + x_i^p \frac{\partial}{\partial x_i^p} .$$

Then the singular set S defined by C is an n-dimensional submanifold of M and thus C satisfies (iv). Moreover, the vector field C also satisfies (i) and (ii). Clearly, (iii) follows directly from the local expression of C (see below). Therefore, from the Nagano's theorem we deduce that M has a unique vector bundle structure over S such that C is the canonical vector field. In fact, we have an isomorphism $\phi : N(S) \to M$ such that the diagram

$$\begin{array}{ccc} N(S) & \xrightarrow{\quad \phi \quad} & M \\ & \searrow^{\pi} \quad \swarrow_{\pi'} & \\ & S & \end{array}$$

is commutative, where π is the canonical projection and π' is the induced projection via ϕ.

The characteristic operator A_C is given by

$$\left(A_C\right)_x \left(\frac{\partial}{\partial x^i}\right) = 0 , \quad \left(A_C\right)_x \left(\frac{\partial}{\partial x_i^a}\right) = \frac{\partial}{\partial x_i^a} ,$$

at each point $x \in S$. Therefore, we have

$$N(S)_x = \left\{ X \in T_x M \mid X = \sum_{a=1}^{p} X_i^a \frac{\partial}{\partial x_i^a} \right\} ,$$

$$T_x S = \left\{ X \in T_x M \mid X = X^i \frac{\partial}{\partial x^i} \right\} .$$

Therefore we deduce $N(S)_x = \left(V_1\right)_x \oplus \cdots \oplus \left(V_p\right)_x$.

Now, consider the bilinear form

$$\left(V_a\right)_x \times T_x S \longrightarrow \mathbb{R} , \quad (X, Y) \longrightarrow -\omega_a(X, Y) .$$

Since $-\omega_a$ is non-degenerate on $\left(V_a\right)_x \times T_x S$ then it makes $\left(V_a\right)_x$ the dual space of $T_x S$ and hence we have

$$N(S)_x \cong T_x^* S \oplus \cdots \oplus T_x^* S \cong (T_p^1)_x^* S .$$

Combining this isomorphism with ϕ we obtain a vector bundle isomorphism

$$\begin{array}{ccc} (T^1_p)^*S & \longrightarrow & M \\ & \searrow^{\tau^*_S} \quad \swarrow^{\pi'} & \\ & S & \end{array}$$

where $\tau^*_S : \left(T^1_p\right)^* S \to S$ is the canonical projection.

This ends the proof since the unicity is a direct consequence of Corollary 2.1. □

Acknowledgments

One of the authors (M. De León) acknowledges invitation to the Departamento de Xeometría e Topoloxía (Universidade de Santiago de Compostela) where part of this work was conceived.

References

[1] AWANE (A.) .— *Sur une généralisation des structures symplectiques*, Thèse de $3^{ème}$ cycle, Publications de l'Institut de Recherche Mathématique Avancée, Strasbourg, 1984.

[2] AWANE (A.) .— *k-Symplectic structures*, J. Math. Phys. **33**, n° 12 (1992), pp. 4046-4052.

[3] AWANE (A.) .— *Some affine properties of k-symplectic structures*, Preprint.

[4] CRAMPIN (M.) and THOMPSON (G.) .— *Affine bundles and integrable almost tangent structures*, Math. Proc. Camb. Phil. Soc. **98** (1985), pp. 61-71.

[5] DE FILIPPO (S.), LANDI (G.), MARMO (G.) and VILASI (G.) .— *Tensor fields defining a tangent bundle structure*, Ann. Inst. Henri-Poincaré, Physique théorique **50**, n° 2 (1989), pp. 205-218.

[6] DE LEÓN (M.), MÉNDEZ (I.) and SALGADO (M.) .— *Regular p-almost cotangent structure*, J. Korean Math. Soc. **25**, n° 2 (1988), pp. 273-287.

[7] DE LEÓN (M.), MÉNDEZ (I.) and SALGADO (M.) .— *Integrable p-almost tangent manifolds and tangent bundles of p^1-velocities*, Acta Mathematica Hungarica **58**, n° 1-2 (1991), pp. 45-54.

[8] DE LEÓN (M.), MÉNDEZ (I.) and SALGADO (M.) .— *p-Almost tangent structures*, Rend. Circ. Mat. Palermo, Serie II, XXXVII (1988), pp. 282-294.

[9] DE LEÓN (M.), MÉNDEZ (I.) and SALGADO (M.) .— *p-Almost cotangent structures*, Bollettino U. M. I. **7-A**, n° 7 (1993), pp. 97-107.

[10] DE LEÓN (M.), MERINO (E.), OUBIÑA (J.A.) and SALGADO (M.) .— *A characterization of tangent and stable tangent bundles*, to appear in Ann. Inst. Henri-Poincaré, Physique Théorique, **61**, n° 1 (1994).

[11] De León (M.) and Rodrigues (P.R.) .— *Generalized Classical Mechanics and Field Theory*, North-Holland Math., Serie 112, Amsterdam (1985).

[12] De León (M.) and Rodrigues (P.R.) .— *A contribution to the global formulation of the higher Poincaré-Cartan form*, Letters in Mathematical Physics **14**, n° 4 (1987), pp. 353-362.

[13] De León (M.) and Rodrigues (P.R.) .— *Hamiltonian structures and Lagrangian field theories on jet bundles*, Bol. Acad. Galega de Ciencias VII (1988), pp. 69-81.

[14] De León (M.) and Rodrigues (P.R.) .— *n^k-Almost Tangent Structures and the Hamiltonization of Higher Order Field Theories*, J. Math. Phys. **30**, n° 6 (1989), pp. 1351-1353.

[15] De León (M.) and Rodrigues (P.R.) .— *Methods of Differential Geometry in Analytical Mechanics*, North-Holland Math., Serie 152, Amsterdam (1989).

[16] Méndez (I.) .— *p-Estructuras casi tangentes y p-estructuras casi cotangentes*, Publicaciones del Departamento de Geometría y Topología **78**, Universidad de Santiago de Compostela (1990).

[17] Morimoto (A.) .— *Prolongations of Geometric Structures*, Lecture Note, Math. Inst. Nagoya Univ. (1969).

[18] Nagano (T.) .— *1-Forms with the exterior derivative of maximal rank*, J. Differential Geometry **2** (1968), pp. 253-264.

[19] Thompson (G.) .— *Integrable almost cotangent structures and Legendrian bundles*, Math. Proc. Camb. Phil. Soc. **101** (1987), pp. 61-78.

[20] Thompson (G.) and Schwardmann (U.) .— *Almost tangent and cotangent structures in the large*, Transactions of the Amer. Math. Soc. **327**, n° 1 (1991), pp. 313-328.