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H^{∞}-extensibility and finite proper holomorphic surjections(*)

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RÉSUMÉ. — Soit $\theta : X \to Y$ une application bornée, propre, holomorphe, et surjective entre deux espaces de Banach analytiques. Si Y possède la propriété de prolongement pour les fonctions H^{∞}, on montre que X la possède également. Réciproquement, si X possède cette propriété et X ne contient pas un ensemble analytique compact de dimension positive, alors toute application holomorphe d'un domaine de Riemann D étalé sur un Banach avec image dans Y peut être prolongée comme une application Gâteaux-holomorphe sur chaque prolongement H^{∞} de D; de surcroît, le prolongement est holomorphe dans le complémentaire d'une hypersurface.

ABSTRACT. — Let $\theta : X \to Y$ be a finite proper holomorphic surjection, where X and Y are Banach analytic spaces. It is shown that if Y has the holomorphic H^{∞}-extension property, so has X. Conversely if X has the holomorphic H^{∞}-extension property, where X does not contain a compact analytic set of positive dimension, then every holomorphic map from a Riemann domain D over a Banach space into Y can be extended Gateaux-holomorphically on every H^{∞}-extension of D. Moreover the extension is holomorphic outside a hypersurface.

The extension of holomorphic maps from a Riemann domain D over a Stein manifold to its envelope of holomorphy $\widehat{D}_{\infty}$ for the Banach algebra of bounded holomorphic functions $H^{\infty}(D)$ has been investigated by some authors.

For holomorphic maps with values in finite dimensional complete C-spaces, the problem was considered by Sibony [6], Hirschowitz [3], and recently by Nguyen van Khue and Bui Dac Tac [4]. The aim of the present paper is to consider the problem in the infinite dimensional case.

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Let X be a Banach analytic space in the sense of Douady [1]. As in the finite dimensional case, we define the Carathéodory pseudodistance C_X on X by the formula

$$C_X(x,y) = \text{Sup}\left\{\left|f(x) - f(y)\right| : |f| \leq 1\,,\ f \in H^\infty(X)\right\}\,.$$

We say that X is a C-space if C_X is a distance defining the topology of X.

Let (D, p, B) and (D', q, B) be Riemann domains over a Banach space B. D' is called a H^∞-extension of D if there is a holomorphic map $e : D \to D'$ such that $p = q \cdot e$ and for every bounded holomorphic function f on D, there exists a bounded holomorphic function f' on D' such that $f = f' \cdot e$.

A Banach analytic space X is said to be a space having the holomorphic H^∞-extension property (for short, the HEH$^\infty$-property) if for every holomorphic map g from a Riemann domain D over a Banach space into X there exists a holomorpnic map g' from D' into X such that $g = g' \cdot e$, where D' is a H^∞-extension of D and D' is a C-space. In this case we say also that g can be extented to a holomorphic map g' on D'.

The main result of this note is the following.

THEOREM 1.— *Let θ be a finite proper holomorphic map from a Banach analytic space X onto a Banach analytic space Y. Then:*

- *(i) if Y has the* HEH$^\infty$*-property and $H^\infty(X)$ separates the points of the fibers of θ, then X has the* HEH$^\infty$*-property;*
- *(ii) if X has the* HEH$^\infty$*-property and X does not contain a compact analytic set of positive dimension, then every holomorphic map from D into Y can be extended Gateaux holomorphically on D', where D is a Riemann domain over a Banach space, D' is a H^∞-extension of D and D' is a C-space.*

Moreover, the extension is holomorphic outside a hypersurface.

Let X be a Banach analytic space. We say that an upper semi-continuous function $\varphi : X \to [-\infty\,, \infty)$ is plurisubharmonic if for every holomorphic map $\sigma : \Delta \to X$ the function $\varphi \circ \sigma$ is subharmonic, where Δ is the unit disc.

Let Z be a Banach analytic space. By $F_c(Z)$ we denote the hyperspace of non-empty compact subsets of Z. An upper semi-continuous multivalued function $K : X \to F_c(Z)$, where X is a Banach analytic space, is called analytic in the sense of Slodkowski [7] if for every plurisubharmonic function

Ψ on a neighbourhood of $\Gamma_{K\restriction_G}$, where G is an open set in X and $\Gamma_{K\restriction_G}$ denote the graph of K on G, the function

$$\varphi(x) = \max\{\Psi(x,z) \mid z \in K(x))\}$$

is plurisubharmonic on G.

LEMMA 1 ([5]).— *Let $K : Y \to F_c(X)$ be an analytic multivalued function such that* $\operatorname{card} K(y) < \infty$ *for all $y \in Y$, where Y is a connected Banach analytic space. Assume that U and V are disjoint open subsets of X such that $K(y) \subset U \cup V$ for all $y \in Y$. Then either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$.*

Proof.— Define Ψ on $Y \times (U \cup V)$ by

$$\Psi(y,z) = \begin{cases} 1 & \text{if } z \in U \\ 0 & \text{if } z \in V. \end{cases}$$

Then Ψ is plurisubharmonic on a neighbourhood of the graph of K, so φ is plurisubharmonic on Y, where

$$\begin{aligned} \varphi(y) &= \max\{\Psi(y,z) \mid z \in K(y))\} \\ &= \begin{cases} 0 & \text{if } K(y) \cap U = \emptyset \\ 1 & \text{if } K(y) \cap U \neq \emptyset. \end{cases} \end{aligned}$$

By the plurisubharmonicity of φ and the connectedness of Y, it implies that either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$. The lemma is proved. □

LEMMA 2.— *Let $K : Y \to F_c(X)$ be an analytic multivalued function such that* $\operatorname{card} K(y) < \infty$ *for all $y \in Y$. Then*

$$V_m = \{y \in Y \mid \operatorname{card} K(y) < m\}$$

is closed in Y for every $m \geq 1$.

Proof.— Given a sequence $\{y_n\}$ in V_m, $y_n \to y^*$, choose disjoint neighbourhoods U_i of x_i, $i = 1, \ldots, \ell$, where $\{x_1, \ldots, x_\ell\} = K(y^*)$. Take a neighbourhood D of y^* such that

$$K(D) \subset \bigcup_{i=1}^{\ell} U_i .$$

Then by lemma 1, $K(y) \cap U \neq \emptyset$ for all $i = 1, \ldots, \ell$ and for all $y \in D$. Hence $m >$ card $K(y_n) \geq 1$ for sufficiently large n. This implies that $y^* \in V_m$. The lemma is proved. □

LEMMA 3.— *Let $\theta : X \to Y$ be a finite proper holomorphic surjection, where X and Y are Banach analytic spaces. Then the multivalued function*

$$K : Y \to F_c(X)$$

given by

$$K(y) = \theta^{-1}(y)$$

is analytic.

Proof

(i) Consider first the case where $Y = \Delta$, the unit disc in $\mathbb{C}$.

Since θ is proper, K is upper semi-continuous. Let Ψ be a plurisubharmonic function on a neighbourhood of $\Gamma_{K\restriction_G}$, where G is an open subset of Δ. Since θ is a branch covering map [2], there exists a discrete sequence A in Δ such that

$$\theta : X \setminus \theta^{-1}(A) \to \Delta \setminus A$$

is an unbranched covering map of order $m < \infty$. Let $y_0 \in \Delta \setminus A$ and

$$\theta^{-1}(y_0) = \{x_1, \ldots, x_m\}.$$

Take a neighbourhood W of y_0 such that

$$\theta^{-1}(W) = U_1 \cup \cdots \cup U_m ,$$

where U_j are disjoint, $x_j \in U_j$ and $\theta : W \cong U_j$, $j = 1, \ldots, m$. Then the function

$$\varphi(y) = \max_j \max\{\Psi(y, x) \mid z \in \theta^{-1}(y) \cap U_j\}$$

is subharmonic on $W \cap G$. Since φ is locally bounded on G, it follows that φ is subharmonic on G.

(ii) Consider now the general case where Y is a Banach analytic space.

Let φ be as in (i) . Obviously φ is upper semi-continuous because of the upper semi-continuity of K and Ψ. It remains to check that $\varphi \circ h$ is subharmonic on Δ for every holomorphic map $h : \Delta \to X$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\Delta} & \xrightarrow{\tilde{h}} & X \\ {\scriptstyle \tilde{\theta}}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\ \Delta & \xrightarrow{h} & Y \end{array}$$

where $\tilde{\Delta} = \Delta \times_Y X$. By (i) and by the relation

$$\varphi \circ h(\lambda) = \max\left\{\Psi\big(h(\lambda), z\big) \;\Big|\; \theta(z) = h(\lambda)\right\}$$

it follows that $\varphi \circ h$ is subharmonic on Δ. The lemma is proved. □

Let X and D be Banach analytic spaces. A finite proper holomorphic surjection $\pi : X \to D$ is called a branch covering map if it satisfies the following:

(i) there is a closed subset A of D which is a removable for bounded holomorphic germs on $D \setminus A$;

(ii) $\pi : X \setminus \pi^{-1}(A) \to D \setminus A$ is a local biholomorphism and $\operatorname{card} \pi^{-1}(z)$ is constant on every connected component of $D \setminus A$.

LEMMA 4.— *Let θ be a finite proper holomorphic map from a Banach analytic space X onto an open set D in a Banach space B. Then θ is a branch covering map.*

Proof.— Without loss of generality we may assume that D is convex. For each $n \geq 1$ put

$$F_n = \left\{y \in D \mid \operatorname{card} \theta^{-1}(y) < n\right\}.$$

By lemma 2 and lemma 3, F_n is closed in D. Applying the Baire theorem to $D = \bigcup_1^\infty F_n$, we can find n_0 such that $\operatorname{Int} F_n \neq \emptyset$. Put

$$m = \max\left\{\operatorname{card} \theta^{-1}(y) \mid y \in \operatorname{Int} F_{n_0}\right\}.$$

Since $\theta : \theta^{-1}(E \cap D) \to E \cap D$ is a branch covering map for every finite dimensional subspace E of B [2], by the connectedness of $D \cap E$ for all subspace E of B, $\dim E < \infty$, we have

$$\begin{aligned}&\sup\left\{\operatorname{card}\theta^{-1}(y) \mid y \in D\right\} = \\&\qquad = \sup\left\{\operatorname{card}\theta^{-1}(y) \mid y \in D \cap E\,,\ E \subset B\,,\ \dim E < \infty\right\} = m\,.\end{aligned}$$

Put

$$Z = \left\{y \in D \mid \operatorname{card}\theta^{-1}(y) < m\right\}.$$

Then Z is closed in D, and from the finiteness and properness of θ it follows that

$$\theta : X \setminus \theta^{-1}(Z) \to D \setminus Z$$

is an unbranched covering map. It remains to show that Z is removable for bounded holomorphic germs. Let h be a bounded holomorphic function on $U \setminus Z$, where U is an open subset of D. Then for every finite dimensional space E of B such that

$$\sup\left\{\operatorname{card}\theta^{-1}(y) \mid y \in E \cap D\right\} = m\,,$$

$h\restriction_{U \setminus Z}$ can be extended holomorphically on U. From the relation

$$D = \bigcup\left\{E \cap D \,\middle|\, E \subset B\,,\ \dim E < \infty\,,\ \sup\left\{\operatorname{card}\theta^{-1}(y) \mid y \in D \cap E\right\} = m\right\},$$

it follows that h can be extended to a bounded Gateaux-holomorphic function $\widehat{h}$ on U. By the boundedness of $\widehat{h}$, we deduce that $\widehat{h}$ is holomorphic on U. The lemma is proved. □

LEMMA 5.— *Let $\theta : X \to D$, where D is a C-manifold, be a branch covering map. Denote by $\mathrm{SH}^\infty(X)$ and $\mathrm{SH}^\infty(D)$ the spectra of Banach algebras $H^\infty(X)$ and $H^\infty(D)$, respectively. Let $\widehat{\theta} : \mathrm{SH}^\infty(X) \to \mathrm{SH}^\infty(D)$ be the map induced by θ. Then*

$$\widehat{\theta} : \widehat{\theta}^{-1}(D) \to D$$

is also a branch covering map.

Proof.— Obviously $\widehat{\theta} : \widehat{\theta}^{-1}(D) \to D$ is finite, proper and surjective, since $H^\infty(X)$ is an integral extension of finite degree of $H^\infty(D)$. By lemma 4, it suffices to prove that $\widehat{\theta}^{-1}(D)$ is a Banach analytic space. Let $B(0,r)$ (resp. $B^*(0,r)$) denote the open ball in $H^\infty(X)$ (resp. $\big(H^\infty(X)\big)^*$) centred at 0 with radius $r > 0$. Consider the holomorphic map

$$F : (D \setminus Z) \times B^*(0,2) \longrightarrow H^\infty\big(B(0,2)\big)$$

given by

$$F(z,w)(h) = w(h)^m + \sigma_{m-1}\big(h \circ p_1(z), \ldots, h \circ p_m(z)\big) w(h)^{m-1} + \\ + \cdots + \sigma_0\big(h \circ p_1(z), \ldots, h \circ p_m(z)\big),$$

where z is the branch locus of θ, m the order of θ and $\sigma_0, \ldots, \sigma_{m-1}$ are elementary symmetric polynomials in m variables and

$$\theta^{-1}(z) = \big(p_1(z), \ldots, p_m(z)\big) \quad \text{for } z \in D \setminus Z.$$

Since $\sigma_0, \ldots, \sigma_{m-1}$ are bounded holomorphic functions on $D \setminus Z$, it follows that F is holomorphic on $D \times B^*(0,2)$. We have

$$F^{-1}(0) = \big\{(z,w) \mid \widehat{\theta}(w) = z\big\} \cong \widehat{\theta}^{-1}(D).$$

Hence $\widehat{\theta} : \widehat{\theta}^{-1}(D) \to D$ is a branch covering map. The lemma is proved. □

LEMMA 6.— *Every Banach space has the* HEH$^\infty$*-property.*

Proof.— Let D be a Riemann domain over a Banach space B and D' a H^∞-extension of D. Let $f : D \to E$ be a holomorphic map, where E is a Banach space.

For each $x^* \in E^*$, by $\widehat{x^* f}$ we denote the holomorphic extension of $x^* f$ on D'. Since D' is a H^∞-extension of D, from the open mapping theorem, it follows that

$$\big\|\widehat{x^* f}\big\| = \big\|x^* f\big\| \quad \text{for all } x^* \in E^*.$$

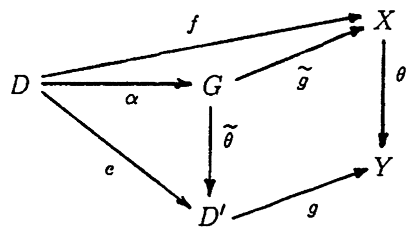
On the other hand, by the uniqueness, $\widehat{x^* f}(z)$ is a continuous linear function on E^* for every $z \in D'$. Thus we can define a bounded map $\widehat{f} : D' \to E^{**}$ by

$$\big(\widehat{f}(z)\big)(x^*) = \widehat{x^* f}(z)$$

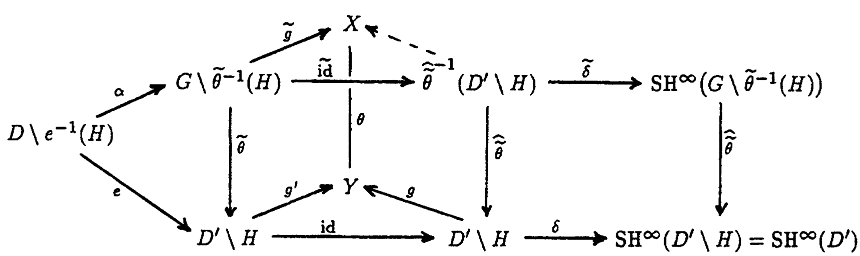
which is separately holomorphic in variables $z \in D'$ and $x^* \in E^*$. From the boundedness of $\widehat{f}(D')$ we deduce that $\widehat{f}$ is holomorphic and $\widetilde{f}(D') \subset E$. Obviously $\widehat{f}$ is a holomorphic extension of f on D'. The lemma is proved. □

Proof of theorem 1

(i) Let first Y have the HEH$^\infty$-property. Let $f : D \to X$ be a holomorphic map, where D is a Riemann domain over a Banach space B. By hypothesis, there is a holomorphic map $g : D' \to Y$ which is a holomorphic extension of θf on D', where D' is a H^∞-extension of D. Consider the commutative diagram



where $G = D' \times_Y X$, $\widetilde{\theta}$ and $\widetilde{g}$ are the canonical projections, α and e are the canonical maps. By lemma 4, $\widetilde{\theta}$ is a branch covering map. Let H denote the branch locus of $\widetilde{\theta}$. Consider the commutative diagram



where
$$\widehat{\widetilde{\theta}} : \mathrm{SH}^\infty(G \setminus \widetilde{\theta}^{-1}(H)) \longrightarrow \mathrm{SH}^\infty(D' \setminus H) \cong \mathrm{SH}^\infty(D')$$

is induced by $\widetilde{\theta} : G \setminus \widetilde{\theta}^{-1}(H) \to D' \setminus H$. From lemma 5, it follows that

$$\widetilde{\widetilde{\theta}} : \widetilde{\widetilde{\theta}}^{-1}(D' \setminus H) \longrightarrow D' \setminus H$$

is a branch covering map. By lemma 6, $\left(H^\infty(G \setminus \widetilde{\theta}^{-1}(H))\right)^*$ has the HEH$^\infty$-property.

Since $D' \setminus H$ is also a H^∞-extension of $D \setminus e^{-1}(H)$, there exists

$$h : D' \setminus H \longrightarrow \Big(H^\infty\big(G \setminus \widetilde{\theta}^{-1}(H)\big)\Big)^*$$

which is a holomorphic extension of

$$\widetilde{\mathrm{id}}\,\alpha : D \setminus e^{-1}(H) \longrightarrow \Big(H^\infty\big(G \setminus \widetilde{\theta}^{-1}(H)\big)\Big)^* .$$

From the relation $\widehat{\widetilde{\theta}} h = \delta$, where $\delta : D' \setminus H \to \mathrm{SH}^\infty(D' \setminus H)$ is the canonical map, we have $h(D' \setminus H) \subset \widehat{\widetilde{\theta}}^{-1}(D' \setminus H)$. Since $H^\infty(X)$ separates the points of the fibers of θ, there exists a holomorphic mapping $\widehat{\widetilde{g}} : \widehat{\widetilde{\theta}}^{-1}(D' \setminus H) \to X$ such that $g\,\widehat{\widetilde{\theta}} = \theta\,\widehat{\widetilde{g}}$. Put

$$f_1 = \widehat{\widetilde{g}}\,h .$$

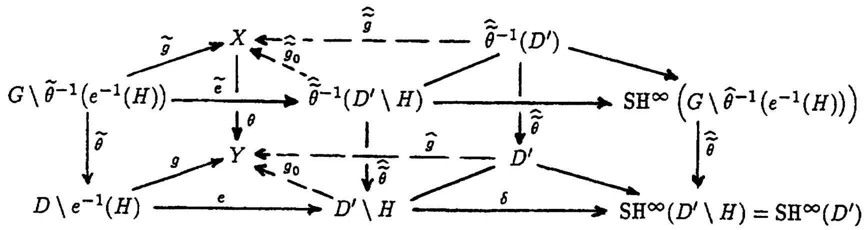
Assume now $z \in H$. Take two neighbourhoods U and V of z and $g(z)$, respectively, such that $g(U) \subset V$ and $\theta^{-1}(V)$ is an analytic set in a finite union W of balls in a Banach space. Then $f_1 : U \setminus H \to W$ can be extended holomorphically on U. This implies that f_1 and hence f can be extended holomorphically on D'.

(ii) Let X be a space having the HEH$^\infty$-property and let $g : D \to Y$ be a holomorphic map, where D is a Riemann domain over a Banach space B. Let D' be a H^∞-extension of D which is a C-space. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \widetilde{g}\ } & X \\ \Big\downarrow{\scriptstyle \widetilde{\theta}} & & \Big\downarrow{\scriptstyle \theta} \\ D & \xrightarrow{\ g\ } & Y \end{array}$$

where $G = D \times_Y X$, $\widetilde{\theta}$ and $\widetilde{g}$ are the canonical projections.

Obviously $\widehat{\widetilde{\theta}} : \mathrm{SH}^\infty(G) \to \mathrm{SH}^\infty(D')$ is finite, proper and surjective, since $H^\infty(G)$ is an integral extension of finite degree of $H^\infty(D)$ and every bounded holomorphic function on D can be extended to a bounded holomorphic function on D'. By lemmas 4 and 5, $\widetilde{\theta}$ and $\widehat{\widetilde{\theta}} : \widehat{\widetilde{\theta}}^{-1}(D') \to D'$ are branch covering maps. Let H denote the branch locus of $\widehat{\widetilde{\theta}} : \widehat{\widetilde{\theta}}^{-1}(D') \to D'$. Consider the commutative diagram



where δ is the canonical map. Since every bounded holomorphic function on $G \setminus \widetilde{\theta}^{-1}(e^{-1}(H))$ can be extended to a bounded holomorphic function on $\mathrm{SH}^\infty\left(G \setminus \widetilde{\theta}^{-1}(e^{-1}(H))\right)$ and the topology of $\widehat{\widehat{\theta}}^{-1}(D' \setminus H)$ is defined by bounded holomorphic functions, it follows that $\widehat{\widehat{\theta}}^{-1}(D' \setminus H)$ is a H^∞-extension of $G \setminus \widetilde{\theta}^{-1}(e^{-1}(H))$ and it is a C-space. By hypothesis, $\widetilde{g}$ can be extended to a holomorphic map

$$\widehat{\widehat{g}}_0 : \widehat{\widehat{\theta}}^{-1}(D' \setminus H) \longrightarrow X .$$

It is easy to see that $\widetilde{e}\,\widetilde{\theta}^{-1}(x) = \widehat{\widehat{\theta}}^{-1}\big(e(x)\big)$ for every $x \in D \setminus e^{-1}(H)$. This yields

$$\widehat{\widehat{g}}_0 \restriction_{\widehat{\widehat{\theta}}^{-1}(e(x))} = \text{const} \quad \text{for all } x \in D \setminus e^{-1}(H) .$$

Since $\widehat{\widehat{\theta}} : \widehat{\widehat{\theta}}^{-1}(D' \setminus H) \to D' \setminus H$ is a branch covering map, it follows that the exists a holomorphic map $\widehat{g}_0 : D' \setminus H \to Y$ such that $\theta\, \widehat{\widehat{g}}_0 = \widehat{g}\, \widehat{\widehat{\theta}}$.

X does not contain a compact set of positive dimension. By the Hironaka singular resolution theorem, for every finite dimensional subspace E of B such that $q^{-1}(E) \not\subset e(H)$,

$$\widehat{\widehat{g}}_0 \restriction_{\widehat{\widehat{\theta}}^{-1}(q^{-1}(E) \setminus H)}$$

can be extended to a holomorphic map $\widehat{\widehat{g}}_E : \widehat{\widehat{\theta}}^{-1}\big(q^{-1}(E)\big) \to X$. This yields that $\widehat{g}_0 \restriction_{q^{-1}(E) \setminus H}$ can be extended to a holomorphic map $\widehat{g}_E : q^{-1}(E) \to Y$. Thus $\widehat{g}_0$ and hence g can be extended to a Gateaux holomorphic map $\widehat{g} : D' \to Y$ which is holomorphic on $D' \setminus H$. The theorem is proved. □

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