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Axioms for topological quantum field theories[*]

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RÉSUMÉ. — Nous donnons une définition axiomatique des théories topologiques quantiques des champs.

ABSTRACT. — We give an axiomatic definition of topological quantum field theories.

Introduction

The objective of this paper is to give an axiomatic definition of modular functors and topological quantum field theories (TQFT's). Modular functors emerged recently in the context of 2-dimensional conformal field theories (see G. Segal [Se], G. Moore and N. Seiberg [MS1], [MS2], and references therein). The notion of TQFT was introduced by E. Witten [Wi] who interpreted the Jones polynomial of knots in terms of a 3-dimensional TQFT closely related to the 2-dimensional modular functor.

It was emphasized by M. Atiyah that the notions of modular functor and TQFT have a more general range of applications and may be formalized in the framework of an axiomatic approach. Axioms for modular functors were first given (in the setting of 2-dimensional conformal field theory) by G. Segal [Se] and G. Moore and N. Seiberg [MS1]. Axioms for topological quantum field theories were put forward by M. Atiyah [At] (see also K. Walker [Wa]). A systematic study of axiomatic foundations of TQFT's was carried out by F. Quinn [Qu1], [Qu2]. The major difference between the axiomatic system of Quinn and the one given in this paper is that as the basic notion of the theory we use space-structures rather than domain

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categories as in [Qu1], [Qu2]. This allows us to make the exposition short and straightforward.

The problem with any axiomatic definition is that it should be sufficiently general but not too abstract. It is especially hard to find the balance in axiomatic systems for TQFT's because our stock of non-trivial examples is very limited. There is no doubt that further experiments with axioms for TQFT's will follow.

The reader will notice that our definitions and results have a definite flavour of abstract nonsense. However, they form a natural background for more concrete 3-dimensional theories.

Fix up to the end of the paper a commutative ring with unit K which will play the role of the ground ring.

1. Space-structures and modular functors

1.1. Structures on topological spaces

We introduce a general notion of additional structure on topological spaces. For such additional structures we will use the term space-structures.

A space-structure is a covariant functor from the category of topological spaces and their homeomorphisms into the category of sets with involutions and their equivariant bijections such that the value of this functor on the empty space is a one-element set. Such a functor $\mathfrak{A}$ assigns to every topological space X a set with involution $\mathfrak{A}(X)$ and to every homeomorphism $f: X \to Y$ an equivariant bijection $\mathfrak{A}(f): \mathfrak{A}(X) \to \mathfrak{A}(Y)$ such that the set $\mathfrak{A}(\emptyset)$ has one element, $\mathfrak{A}(\mathrm{id}_X) = \mathrm{id}_{\mathfrak{A}(X)}$ for any topological space X, and $\mathfrak{A}(ff') = \mathfrak{A}(f)\mathfrak{A}(f')$ for any composable homeomorphisms f, f'. Elements of the set $\mathfrak{A}(X)$ are called $\mathfrak{A}$-structures in X. Any pair $(X, \alpha \in \mathfrak{A}(X))$ is said to be an $\mathfrak{A}$-space. By an $\mathfrak{A}$-homeomorphism of an $\mathfrak{A}$-space (X, α) onto an $\mathfrak{A}$-space (X', α') we mean a homeomorphism $f: X \to X'$ such that $\mathfrak{A}(f)(\alpha) = \alpha'$. It is clear that the composition of $\mathfrak{A}$-homeomorphisms is an $\mathfrak{A}$-homeomorphism and that the identity self-homeomorphisms of $\mathfrak{A}$-spaces are $\mathfrak{A}$-homeomorphisms. By abuse of notation we will often denote $\mathfrak{A}$-spaces by the same symbols as their underlying topological spaces.

The $\mathfrak{A}$-structure in X which is the image of $\alpha \in \mathfrak{A}(X)$ under the given involution in $\mathfrak{A}(X)$ is said to be opposite to α. For any $\mathfrak{A}$-space X we denote by $-X$ the same space with the opposite $\mathfrak{A}$-structure. Clearly $-(-X) = X$.

A space-structure $\mathfrak{A}$ is said to be connected if for any disjoint topological spaces X and Y there is an equivariant identification

$$\mathfrak{A}(X \amalg Y) = \mathfrak{A}(X) \times \mathfrak{A}(Y)$$

so that the following conditions hold true.

(1.1a).— The diagram

$$\begin{tikzcd}
\mathfrak{A}(X \amalg Y) \arrow[r, equal] \arrow[d, "="'] & \mathfrak{A}(X) \times \mathfrak{A}(Y) \arrow[d, "\mathrm{Perm}"] \\
\mathfrak{A}(Y \amalg X) \arrow[r, equal] & \mathfrak{A}(Y) \times \mathfrak{A}(X)
\end{tikzcd}$$

is commutative (here Perm is the flip $x \times y \mapsto y \times x$).

(1.1b).— For arbitrary $\mathfrak{A}$-homeomorphisms $f : X \to X'$ and $g : Y \to Y'$ the diagram

$$\begin{tikzcd}
\mathfrak{A}(X \amalg Y) \arrow[r, equal] \arrow[d, "\mathfrak{A}(f \amalg g)"'] & \mathfrak{A}(X) \times \mathfrak{A}(Y) \arrow[d, "\mathfrak{A}(f) \times \mathfrak{A}(g)"] \\
\mathfrak{A}(X' \amalg Y') \arrow[r, equal] & \mathfrak{A}(X') \times \mathfrak{A}(Y')
\end{tikzcd}$$

is commutative.

(1.1c).— For any three topological spaces X, Y, Z the diagram of identifications

$$\begin{tikzcd}
\mathfrak{A}(X \amalg Y \amalg Z) \arrow[r, equal] \arrow[d] & \mathfrak{A}(X \amalg Y) \times \mathfrak{A}(Z) \arrow[d] \\
\mathfrak{A}(X) \times \mathfrak{A}(Y \amalg Z) \arrow[r, equal] & \mathfrak{A}(X) \times \mathfrak{A}(Y) \times \mathfrak{A}(Z)
\end{tikzcd}$$

is commutative.

(1.1d).— If $X = \emptyset$ then the identification $\mathfrak{A}(X \amalg Y) = \mathfrak{A}(X) \times \mathfrak{A}(Y)$ is induced by the identity $\mathrm{id} : \mathfrak{A}(Y) \to \mathfrak{A}(Y)$.

These axioms may be briefly reformulated by saying that the disjoint union of a finite family of $\mathfrak{A}$-spaces acquires the structure of an $\mathfrak{A}$-space in a natural way so that varying the $\mathfrak{A}$-structures on these spaces we get every $\mathfrak{A}$-structure on the disjoint union exactly once.

We may also consider more general space-structures which are defined as above with the only difference that the values of $\mathfrak{A}$ on topological spaces are not sets but rather classes. The concept of class is in fact more convenient in this abstract setting. For example, vector bundles on a given topological space form a class and not a set (unless the space is empty). Another simple example of a space-structure involving classes is the structure of a topological space with a distinguished point endowed with an object of a certain category.

Replacing everywhere the word "space" by the word "pair", or "triple", or "tuple" etc., we get the notions of (connected) space-structures on topological pairs, triples, tuples, etc.

A prototypical example of a space-structure is the orientation of n-dimensional topological manifolds (with a fixed n). The corresponding functor $\mathfrak{A}$ is defined as follows. If a topological space X is homeomorphic to an orientable n-dimensional topological manifold then $\mathfrak{A}(X)$ is the set of orientations in X with the involution induced by inversing of orientation. For other non-empty X we have $\mathfrak{A}(X) = \emptyset$. (The empty set should be treated as an orientable n-dimensional manifold with exactly one orientation.) It is clear that orientation is a connected space-structure. Another useful example of space-structures is provided by the cell structure (also called the CW-structure). It assigns to every topological space X the set of equivalence classes of homeomorphisms of X onto cell spaces, two such homeomorphisms $g_1 : X \to C_1$, $g_2 : X \to C_2$ being equivalent if $g_2 g_1^{-1}$ is a cell homeomorphism. These sets of equivalence classes are endowed with identity involutions. The action of homeomorphisms on these sets is induced by composition in the obvious way. One may similarly formalize the structures of smooth manifold, piecewise-linear manifold, and other standard structures.

1.2. Modular functors

A modular functor assigns modules to topological spaces with a certain space-structure and isomorphisms to the structure preserving homeomorphisms of these spaces. Here are the details. Assume that we have a connected space-structure $\mathfrak{A}$. A modular functor $\mathcal{T}$ based on $\mathfrak{A}$ assigns to every $\mathfrak{A}$-space X a projective K-module $\mathcal{T}(X)$ and to every $\mathfrak{A}$-homeomorphism $f : X \to Y$ a K-isomorphism $f_\sharp : \mathcal{T}(X) \to \mathcal{T}(Y)$ satisfying the following axioms (1.2a)-(1.2c). (By projective K-module we mean a projective K-

module of finite type, *i.e.* a direct summand of the free module K^n with finite n.)

(1.2a).— For any $\mathfrak{A}$-space X we have $(\mathrm{id}_X)_\sharp = \mathrm{id}_{\mathcal{T}(X)}$. For any $\mathfrak{A}$-homeomorphisms $f : X \to Y$, $g : Y \to Z$ we have $(gf)_\sharp = g_\sharp f_\sharp$.

(1.2b).— For each pair of disjoint $\mathfrak{A}$-spaces X, Y there is a fixed identification isomorphism $\mathcal{T}(X \amalg Y) \to \mathcal{T}(X) \otimes \mathcal{T}(Y)$. These identifications are natural, commutative, and associative.

Naturality in this axiom means that $(f \amalg g)_\sharp = f_\sharp \otimes g_\sharp$ for any homeomorphisms of $\mathfrak{A}$-spaces f, g. Commutativity means that the diagram

$$\begin{tikzcd} \mathcal{T}(X \amalg Y) \arrow[r, equal] \arrow[d, "="'] & \mathcal{T}(X) \otimes \mathcal{T}(Y) \arrow[d, "\mathrm{Perm}"] \\ \mathcal{T}(Y \amalg X) \arrow[r, equal] & \mathcal{T}(Y) \otimes \mathcal{T}(X) \end{tikzcd}$$

is commutative. Here Perm is the flip $x \otimes y \mapsto y \otimes x$. Associativity means that for any $\mathfrak{A}$-spaces X, Y, Z the composition of identifications

$$\begin{aligned} \big(\mathcal{T}(X) \otimes \mathcal{T}(Y)\big) \otimes \mathcal{T}(Z) &= \mathcal{T}(X \amalg Y) \otimes \mathcal{T}(Z) \\ &= \mathcal{T}(X \amalg Y \amalg Z) = \mathcal{T}(X) \otimes \mathcal{T}(Y \amalg Z) \\ &= \mathcal{T}(X) \otimes \big(\mathcal{T}(Y) \otimes \mathcal{T}(Z)\big) \end{aligned}$$

is the standard isomorphism identifying $(x \otimes y) \otimes z$ with $x \otimes (y \otimes z)$.

(1.2c).— $\mathcal{T}(\emptyset) = K$ and for any $\mathfrak{A}$-space Y the identification

$$\mathcal{T}(\emptyset \amalg Y) = \mathcal{T}(\emptyset) \otimes \mathcal{T}(Y)$$

is induced by the identification $\mathcal{T}(\emptyset) = K$.

The simplest example of a modular functor is the "trivial" modular functor which assigns K to all topological spaces and id_K to all homeomorphisms. (The underlying space-structure assigns a one-point set to all topological spaces.) For further examples of modular functors see Section 3.

It is clear that every modular functor $\mathcal{T}$ assigns to the disjoint union of a finite family of $\mathfrak{A}$-spaces $\{X_j\}_j$ the tensor product of the modules $\{\mathcal{T}(X_j)\}_j$. There is one subtle point here. Namely, if the family $\{X_j\}_j$ is not ordered then the tensor product in question is the non-ordered tensor product. Recall briefly its definition. Consider all possible total orders in the given (finite) family of modules, form the corresponding tensor products over K and identify them via the canonical isomorphisms induced by permutations of the factors. This results in a module canonically isomorphic to any of these ordered tensor products but itself independent of the choice of ordering. For instance, for any two K-modules G, H their non-ordered tensor products is the K-module F consisting of all pairs $(f \in G \otimes H,\ f' \in H \otimes G)$ such that $f' = \mathrm{Perm}(f)$. The formulas $(f, f') \mapsto f$ and $(f, f') \mapsto f'$ define isomorphisms $F \to G \otimes H$ and $F \to H \otimes G$. For any $g \in G$, $h \in H$ the pair $(g \otimes h,\ h \otimes g)$ is an element of F. The fact that under any modular functor the disjoint union corresponds to non-ordered tensor product of modules follows from the axiom (1.2b). As is customary in algebra, we denote both the ordered and non-ordered tensor products by the same symbol $\otimes$.

1.3. Self-dual modular functors

We say that a modular functor $\mathcal{T}$ is self-dual if it satisfies the following condition.

(1.3a).— For any $\mathfrak{A}$-space X there is a non-degenerate bilinear pairing

$$d_X : \mathcal{T}(X) \otimes \mathcal{T}(-X) \to K\,.$$

The system of pairings $\{d_X\}_X$ is natural with respect to $\mathfrak{A}$-homeomorphisms, multiplicative with respect to disjoint union, and symmetric in the sense that $d_{-X} = d_X \circ \mathrm{Perm}_{\mathcal{T}(X),\mathcal{T}(-X)}$ for any $\mathfrak{A}$-space X.

Here non-degeneracy of a bilinear pairing of K-modules $d : P \otimes_K Q \to K$ means that the adjoint homomorphisms $Q \to P^* = \mathrm{Hom}_K(P, K)$ and $P \to Q^* = \mathrm{Hom}_K(Q, K)$ are isomorphisms.

For instance the trivial modular functor defined in Section 1.2 is self-dual.

Recall that for projective K-modules (of finite type) there is a notion of dimension generalizing the usual dimension of free modules. Namely, for a projective K-module P we set $\mathrm{Dim}(P) = \mathrm{tr}(p)$ where p is the projection of a free K-module of finite type onto its direct summand isomorphic to P. It is well known (and easy to show) that this dimension is correctly defined and satisfies the usual properties of dimension:

$$\mathrm{Dim}(K^n) = n\,, \quad \mathrm{Dim}(P^*) = \mathrm{Dim}(P)\,,$$

$$\mathrm{Dim}(P \oplus P') = \mathrm{Dim}(P) + \mathrm{Dim}(P')\,, \quad \mathrm{Dim}(P \otimes P') = \mathrm{Dim}(P)\,\mathrm{Dim}(P')\,.$$

It is clear that for any self-dual modular functor $\mathcal{T}$ and any $\mathfrak{A}$-space X we have

$$\operatorname{Dim}(\mathcal{T}(-X)) = \operatorname{Dim}(\mathcal{T}(X)) .$$

2. Cobordisms and topological quantum field theories

2.1. Cobordisms of $\mathfrak{A}$-spaces

The notion of cobordism for $\mathfrak{A}$-spaces is motivated by the needs of topological quantum field theories. In the realm of manifolds such a theory associates modules to closed manifolds and homomorphisms to cobordisms. Similar ideas may be applied to arbitary topological spaces whenever there is a suitable notion of cobordism. We describe here a general set up for cobordism theories which will serve as a ground for topological field theories.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be connected space-structures. Assume that any $\mathfrak{B}$-structure on a topological space M gives rise in a certain canonical way to a subspace of M equipped with an $\mathfrak{A}$-structure. We will call this subspace with this $\mathfrak{A}$-structure the boundary of the $\mathfrak{B}$-space M and denote it by ∂M. Thus, ∂M is an $\mathfrak{A}$-space. Warning: in general both the underlying topological space of ∂M and the $\mathfrak{A}$-structure in this space may depend on the choice of the $\mathfrak{B}$-structure in M. We assume that the boundary is natural with respect to $\mathfrak{B}$-homeomorphisms, *i.e.* that any $\mathfrak{B}$-homeomorphism $M_1 \to M_2$ restricts to an $\mathfrak{A}$-homeomorphism $\partial(M_1) \to \partial(M_2)$. We also assume that the passage to the boundary commutes with disjoint union and negation of $\mathfrak{B}$-spaces, *i.e.* that

$$\partial(M_1 \amalg M_2) = \partial(M_1) \amalg \partial(M_2) \quad \text{and} \quad \partial(-M) = -\partial M .$$

We say that the space-structures $(\mathfrak{B}, \mathfrak{A})$ form a cobordism theory if the following four axioms hold true.

(2.1a).— The $\mathfrak{B}$-spaces are subject to gluing as follows. Let M be a $\mathfrak{B}$-space and let ∂M be a disjoint union of $\mathfrak{A}$-spaces X_1, X_2, Y such that X_1 is $\mathfrak{A}$-homeomorphic to $-X_2$. Let M' be the topological space obtained from M by gluing X_1 to X_2 along an $\mathfrak{A}$-homeomorphism $f : X_1 \to -X_2$. Then the $\mathfrak{B}$-structure in M gives rise in a certain canonical way to a $\mathfrak{B}$-structure in M' such that $\partial M' = Y$.

We say that the $\mathfrak{B}$-space M' is obtained from M by gluing X_1 to X_2 along f.

(2.1b).— The gluings of $\mathfrak{B}$-spaces are natural with respect to $\mathfrak{B}$-homeomorphisms and commute with disjoint union and negation. The gluings corresponding to disjoint $\mathfrak{A}$-subspaces of the boundary commute.

(2.1c).— Each $\mathfrak{A}$-structure α in a topological space X gives rise in a certain canonical way to a $\mathfrak{B}$-structure $\alpha \times [0,1]$ in $X \times [0,1]$ such that

$$\partial\big(X \times [0,1], \alpha \times [0,1]\big) = (X \times 0, -\alpha \times 0) \amalg (X \times 1, \alpha \times 1).$$

The homeomorphism $(x,t) \mapsto (x, 1-t) : X \times [0,1] \to X \times [0,1]$ inverts this structure. The correspondence $\alpha \to \alpha \times [0,1]$ is natural with respect to homeomorphisms and commutes with disjoint union and with negation.

Thus for any $\mathfrak{A}$-space X we may form the cylinder $X \times [0,1]$ over X which is a $\mathfrak{B}$-space with the boundary $(-X) \times 0 \amalg X \times 1$.

(2.1d).— Let X, X' be two copies of the same $\mathfrak{A}$-space. Gluing the $\mathfrak{B}$-spaces $X \times [0,1]$, $X' \times [0,1]$ along the identity

$$x \times 1 \mapsto x \times 0 : X \times 1 \to X' \times 0$$

we get a $\mathfrak{B}$-space homeomorphic to the same cylinder $X \times [0,1]$ via a $\mathfrak{B}$-homeomorphism identical on the bases.

Example.— $\mathfrak{B}$ is the structure of oriented $n+1$-dimensional smooth manifolds with boundary, $\mathfrak{A}$ is the structure of oriented n-dimensional smooth manifolds, and ∂ is the usual boundary. This pair of space-structures forms a cobordism theory in the obvious way.

Assume that the space-structures $(\mathfrak{B}, \mathfrak{A})$ form a cobordism theory. By a $(\mathfrak{B}, \mathfrak{A})$-cobordism we mean an arbitrary triple (M, X, Y) where M is a $\mathfrak{B}$-space, X and Y are $\mathfrak{A}$-spaces, and $\partial M = (-X) \amalg Y$. The $\mathfrak{A}$-spaces X and Y are called the (bottom and top) bases of this cobordism and denoted by $\partial_- M$ and $\partial_+ M$ respectively. We say that M is a cobordism between X and Y. For instance, for any $\mathfrak{A}$-space X the cylinder $\big(X \times [0,1], X \times 0, X \times 1\big)$ with the $\mathfrak{B}$-structure in $X \times [0,1]$ induced by the $\mathfrak{A}$-structure in X is a cobordism between two copies of X. This cobordism will be simply denoted by $X \times [0,1]$.

It is clear that we may form disjoint unions of cobordisms. We may also glue cobordisms M_1, M_2 along any $\mathfrak{A}$-homeomorphism $\partial_+(M_1) \to -\partial_-(M_2)$.

2.2. Definition of TQFT's

Let $\mathfrak{B}$ and $\mathfrak{A}$ be connected space-structures forming a cobordism theory.

A topological quantum field theory (briefly, TQFT) based on $(\mathfrak{B}, \mathfrak{A})$ consists of a modular functor $\mathcal{T}$ on $\mathfrak{A}$-spaces and a map τ assigning to every $(\mathfrak{B}, \mathfrak{A})$-cobordism (M, X, Y) a K-homomorphism

$$\tau(M) = \tau(M, X, Y) : \mathcal{T}(x) \to \mathcal{T}(Y)$$

which satisfy the following axioms (2.2a)-(2.2d).

(2.2a) Naturality.— If M_1 and M_2 are $(\mathfrak{B}, \mathfrak{A})$-cobordisms and $f : M_1 \to M_2$ is a $\mathfrak{B}$-homeomorphism preserving the bases then

$$\tau(M_2)\big(f\restriction_{\partial_-(M_1)}\big)_\sharp = \big(f\restriction_{\partial_+(M_1)}\big)_\sharp \tau(M_1) : \mathcal{T}\big(\partial_-(M_1)\big) \to \mathcal{T}\big(\partial_+(M_2)\big)\,.$$

(2.2b) Multiplicativity.— If a cobordism M is the disjoint union of cobordisms M_1, M_2 then under the identifications (1.2b) we have

$$\tau(M) = \tau(M_1) \otimes \tau(M_2)\,.$$

(2.2c) Functoriality.— If a $(\mathfrak{B}, \mathfrak{A})$-cobordism M is obtained from the disjoint union of two $(\mathfrak{B}, \mathfrak{A})$-cobordisms M_1 and M_2 by gluing along an $\mathfrak{A}$-homeomorphism $f : \partial_+(M_1) \to \partial_-(M_2)$ then for some invertible $k \in K$

$$\tau(M) = k\tau(M_2) \circ f_\sharp \circ \tau(M_1)\,.$$

Here the factor k may depend on the choice of M_1 and M_2.

(2.2d) Normalization.— For any $\mathfrak{A}$-space X we have

$$\tau\big(X \times [\,0\,,\,1\,]\big) = \mathrm{id}_{\mathcal{T}(X)}\,.$$

We say that the TQFT $(\mathcal{T}, \tau)$ is anomaly-free if the factor k in (2.2c) may be always taken to be 1.

Each $\mathfrak{B}$-space M gives rise to a cobordism $(M, \emptyset, \partial M)$. The corresponding K-homomorphism $\mathcal{T}(\emptyset) \to \mathcal{T}(\partial M)$ is completely determined by its value on the unity $1 \in K = \mathcal{T}(\emptyset)$. This value is denoted by $\tau(M)$. It follows from the axioms that $\tau(M) \in \mathcal{T}(\partial M)$ is natural with respect to $\mathfrak{B}$-homeomorphisms and multiplicative with respect to disjoint union. In the case $\partial M = \emptyset$ we have $\tau(M) \in \mathcal{T}(\emptyset) = K$ so that $\tau(M)$ is a numerical $\mathfrak{B}$-homeomorphism invariant of M.

It is straightforward to define homomorphisms and isomorphisms of TQFT's. Namely, let $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$ be two TQFT's based on the same pair $(\mathfrak{B}, \mathfrak{A})$. A homomorphism $(\mathcal{T}_1, \tau_1) \to (\mathcal{T}_2, \tau_2)$ assigns to every $\mathfrak{A}$-space X a K-homomorphism $\mathcal{T}_1(X) \to \mathcal{T}_2(X)$ which commutes with the action of homeomorphisms, with the identification isomorphisms for disjoint unions, and with the operators corresponding to cobordisms. A homomorphism $g : (\mathcal{T}_1, \tau_1) \to (\mathcal{T}_2, \tau_1)$ is said to be an isomorphism if for any $\mathfrak{A}$-space X the homomorphism $g(X) : \mathcal{T}_1(X) \to \mathcal{T}_2(X)$ is an isomorphism.

Note two natural operations on TQFT's, the negation and the tensor product. For any TQFT$(\mathcal{T}, \tau)$ we define the opposite TQFT$(-\mathcal{T}, -\tau)$ by the formulas $-\mathcal{T}(X) = \mathcal{T}(-X)$ for any $\mathfrak{A}$-space X and $-\tau(M) = \tau(-M)$ for any $(\mathfrak{B}, \mathfrak{A})$-cobordism M, the action of homeomorphisms and the identification isomorphisms for disjoint unions being determined in the obvious way by the corresponding data for $(\mathcal{T}, \tau)$. For any TQFT's $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$ based on the same pair $(\mathfrak{B}, \mathfrak{A})$ we define their tensor product by the formulas

$$(\mathcal{T}_1 \otimes \mathcal{T}_2)(X) = \mathcal{T}_1(X) \otimes_K \mathcal{T}_2(X)$$

and

$$(\tau_1 \otimes \tau_2)(M) = \tau_1(M) \otimes \tau_2(M)\,.$$

As above the action of homeomorphisms and the identification isomorphisms for disjoint unions are determined by the corresponding data for $(\mathcal{T}_1, \tau_1)$, $(\mathcal{T}_2, \tau_2)$.

3. Properties and examples

3.1. Fundamental properties of TQFT's

Fix connected space-structures $\mathfrak{B}$, $\mathfrak{A}$ forming a cobordism theory and a TQFT $(\mathcal{T}, \tau)$ based on $(\mathfrak{B}, \mathfrak{A})$. We formulate here three fundamental properties of $(\mathcal{T}, \tau)$. The first of these properties apply to an arbitrary TQFT whereas the second and third ones hold only for anomaly-free TQFT's.

THEOREM 3.1.1.— *The modular functor $\mathcal{T}$ is self-dual.*

In the proof of Theorem 3.1.1 given in Section 4 we will explicitly construct for any $\mathfrak{A}$-space X a non-degenerate bilinear

$$d_X : \mathcal{T}(X) \otimes_K \mathcal{T}(-X) \to K$$

satisfying (1.3a).

Theorem 3.1.1 shows that any modular functor which may be extended to a TQFT is self-dual.

The next theorem shows that when a $\mathfrak{B}$-space splits into two pieces M_1 and M_2 with $M_1 \cap M_2 = \partial(M_1) = \partial(M_2)$ then for any anomaly free TQFT $(\mathcal{T}, \tau)$ we may compute $\tau(M)$ from $\tau(M_1)$ and $\tau(M_2)$. Here we need the pairing d provided by the previous theorem.

THEOREM 3.1.2.— *Let M be a $\mathfrak{B}$-space with void boundary obtained by gluing two $\mathfrak{B}$-spaces M_1 and M_2 along an $\mathfrak{A}$-homeomorphism $g : \partial(M_1) \to -\partial(M_2)$. Let $\overline{g} = -g$ be the same mapping g considered as an $\mathfrak{A}$-homeomorphism $-\partial(M_1) \to \partial(M_2)$. If $(\mathcal{T}, \tau)$ is anomaly-free then*

$$\tau(M) = d_{\partial(M_1)}\Big(\tau(M_1) \otimes \overline{g}_\sharp^{-1}\big(\tau(M_2)\big)\Big) = d_{\partial(M_2)}\big(\tau(M_2) \otimes g_\sharp\big(\tau(M_1)\big)\big)\,.$$

Finally, for any $\mathfrak{A}$-space X we compute the dimension $\mathrm{Dim}\big(\mathcal{T}(X)\big)$ of $\mathcal{T}(X)$ in terms of τ. With this view consider the cylinder $X \times [\,0\,,\,1\,]$ with its $\mathfrak{B}$-structure and glue its bases along the homeomorphism

$$x \times 0 \mapsto x \times 1 : X \times 0 \to X \times 1\,.$$

This yields a $\mathfrak{B}$-structure in $X \times S^1$. The resulting $\mathfrak{B}$-space is also denoted by $X \times S^1$. It follows from the axiom (2.1a) that $\partial(X \times S^1) = \emptyset$.

THEOREM 3.1.3.— *If $(\mathcal{T}, \tau)$ is anomaly-free then for any $\mathfrak{A}$-space X we have*

$$\mathrm{Dim}\big(\mathcal{T}(X)\big) = \tau(X \times S^1) \in K\,.$$

The latter two theorems do not extend directly to TQFT's with anomalies, as is clear from the examples given in Section 1.1.2.

Note that Theorems 3.1.2 and 3.1.3 are well known in physical literature (Theorem 3.1.1 is also known though the self-duality of the modular functor is often confused with its unitarity properties which we do not discuss in this paper). Our ability to state and to prove these theorems in a quite abstract set up confirms that our mathematics adequately formalizes physical ideas.

Theorems 3.1.1-3.1.3 are proven in Section 4.

3.2. Examples of TQFT's

We consider here a few elementary examples of TQFT's (for more elaborate examples see [At], [Wa], [Tu3]).

Let $\mathfrak{A}$ be the structure of finite cell space and let $\mathfrak{B}$ be the structure of finite cell space with a fixed cell subspace (which plays the role of the boundary). Thus, $(\mathfrak{B},\mathfrak{A})$-cobordisms are just finite cell triples (M,X,Y) with $X \cap Y = \emptyset$. The gluing of cell spaces and the cell structures in cylinders are defined in the standard way. The examples of TQFT's to follow are based on $(\mathfrak{B},\mathfrak{A})$.

Example 1.— Let $\mathcal{T}$ be the trivial modular functor restricted to finite cell spaces. Fix an invertible element $q \in K$. For any finite cell triple (M,X,Y) define the operator $\tau(M,X,Y) : K \to K$ to be the multiplication by $q^{\chi(M,X)}$ where χ is the Euler characteristic. It is obvious that the pair $(\mathcal{T},\tau)$ is an anomaly-free TQFT. This example is borrowed from [Qu1].

Example 2.— Fix an integer $i \geq 0$ and a finite abelian group G whose order is invertivle in K. For any finite cell space X set $\mathcal{T}(X) = K\big[H_i(X;G)\big]$. Thus, $\mathcal{T}(X)$ is the module of formal linear combinations of the elements of $H_i(X;G)$ with coefficients in K. The action of cell homeomorphisms is induced by their action in H_i. Additivity of homologies with respect to disjoint union yields the identifications satisfying the axioms (1.2b), (1.2c). The operator invariant $\tau = \tau(M,X,Y)$ of a finite cell triple (M,X,Y) transforms any $g \in H_i(X;G)$ into the formal sum of those $h \in H_i(Y;G)$ which are homological to g in M (if there is no such h then $\tau(g) = 0$). The axioms of topological field theory are straightforward. (In the axiom (2.2c) k^{-1} is the order of the group $f_*(F_1) \cap F_2$ where F_1 and F_2 are the kernels of the inclusion homomorphisms $H_i\big(\partial_+(M_1)\,;\,G\big) \to H_i(M_1;G)$ and $H_i\big(\partial_-(M_2)\,;\,G\big) \to H_i(M_2;G)$ respectively.) The resulting TQFT is easily seen to be non-anomaly-free.

Instead of homologies with coefficients in G we may use any homology theory, provided some finiteness assumptions are imposed to assure that the modules $\big\{\mathcal{T}(X)\big\}$ are finitely generated. As an exercise the reader may construct similar TQFT's using cohomology groups or homotopy groups.

3.3. Remarks

Remark 1.— The axioms of TQFT's given above are subject to further generalizations. First of all the axioms may be extended to cover the case of $\mathfrak{A}$-spaces with boundary, which should be preserved under the $(\mathfrak{B}, \mathfrak{A})$-cobordisms. Such an extension could be useful though it makes the exposition more heavy. In dimension 3 such an extension may be avoided because we may always eliminate the boundary of surfaces and 3-cobordisms by gluing in 2-disks and solid tubes respectively. Another possible generalization of TQFT's abandons the condition that the operator invariant τ is defined for all $(\mathfrak{B}, \mathfrak{A})$-cobordisms. A guiding example in this direction could be the "Reidemeister TQFT" defined only for acyclic cobordisms. This "TQFT" involves the trivial modular functor and assigns to every finite cell triple (M, X, Y) equipped with a flat K-linear bundle ξ with $H_*(M, X\,;\,\xi) = 0$ the multiplication by the Reidemeister torsion $\tau(M, X\,;\,\xi) \in K$. To eliminate the indeterminacy in the definition of torsion the pair (W, X) should be endowed with homological orientation and Euler structure as defined in [Tu1] and [Tu2]. Another interesting idea would be to replace in the definition of TQFT's the projective modules by locally trivial vector bundles over certain topological spaces. We will not pursue these ideas here.

Remark 2.— As an exercise the reader may prove the following two assertions. Let $(\mathcal{T}, \tau)$ be an anomaly-free TQFT based on a cobordism theory $(\mathfrak{B}, \mathfrak{A})$.

- Let X be an $\mathfrak{A}$-space and g be an $\mathfrak{A}$-homeomorphism $X \to X$. Gluing the top base of the cylinder $X \times [\,0\,,\,1\,]$ to its bottom base along g we get a $\mathfrak{B}$-structure (with void boundary) in the mapping torus M_g of g. Then $\tau(M_g) = \mathrm{tr}(g_\sharp)$ (this generalizes Theorem 3.1.3).
- Let (M, X, Y) be a $(\mathfrak{B}, \mathfrak{A})$-cobordism. Under the identification

$$\mathcal{T}(-X) = \big(\mathcal{T}(X)\big)^*, \quad \mathcal{T}(-Y) = \big(\mathcal{T}(Y)\big)^*$$

induced by d_X, d_Y we have

$$\tau(M, -Y, -X) = \tau\big(M, X, Y\big)^*.$$

4. Proof of theorems

4.1. Lemma

LEMMA .— *Let P and Q be modules over a commutative ring with unit K. Let*

$$b : K \to Q \otimes_K P, \quad d : P \otimes_K Q \to K$$

be K-linear homomorphisms satisfying the identities

$$(\mathrm{id}_Q \otimes d)(b \otimes \mathrm{id}_Q) = k\,\mathrm{id}_Q \quad \textit{and} \quad (d \otimes \mathrm{id}_P)(\mathrm{id}_P \otimes b) = k'\,\mathrm{id}_P$$

where k and k' are invertible elements of K. Then $k = k'$ and both b and d are non-degenerate.

Proof.— Denote the homomorphisms $Q \to P^*$, $P \to Q^*$ adjoint to d and the homomorphisms $P^* \to Q$, $Q^* \to P$ adjoint to b by f_1, f_2, f_3, f_4 respectively. We will show that these four homomorphisms are isomorphisms. The first identity between b and d implies that $f_3 f_1 = k\,\mathrm{id}_Q$. Indeed, if $b(1) = \sum_i q_i \otimes p_i$ with $q_i \in Q$, $p_i \in P$ then this identity indicates that for any $q \in Q$

$$\sum_i d(p_i, q) q_i = kq\,.$$

The homomorphism f_1 transforms q into the linear functional $p \mapsto d(p, q)$ and the homomorphism f_3 transforms this functional $f_1(q)$ into

$$\sum_i f_1(q)(p_i) q_i = \sum_i d(p_i, q) q_i = kq\,.$$

A similar argument deduces from the same identity that $f_2 f_4 = k\,\mathrm{id}_{Q^*}$. The second identity between b and d similarly implies that $f_1 f_3 = k'\,\mathrm{id}_{P^*}$ and $f_4 f_2 = k'\,\mathrm{id}_P$. Therefore the homomorphisms f_1, f_2, f_3, f_4 are isomorphisms. The equalities

$$k f_3 = (f_3 f_1) f_3 = f_3 (f_1 f_3) = k' f_3$$

imply that $k = k'$.

4.2. Proof of Theorem 3.1.1

Let $(\mathcal{T},\tau)$ be a TQFT. Let X be an $\mathfrak{A}$-space. Set $P=\mathcal{T}(X)$ and $Q=\mathcal{T}(-X)$. Denote by J the cylinder $X\times[0,1]$ with the $\mathfrak{B}$-structure given by the axiom (2.1c) so that $\partial J=(-X)\amalg X$. This $\mathfrak{B}$-space gives rise to two cobordisms: $(J,\emptyset,\partial J)$ and $(J,-\partial J,\emptyset)$. Denote the corresponding operators $K\to\mathcal{T}(\partial J)=Q\otimes_K P$ and $P\otimes_K Q=\mathcal{T}(-\partial J)\to K$ by $b=b_X$ and $d=d_X$ respectively. We will prove that the operators $\{d_X\}_X$ satisfy the self-duality axiom (1.3a).

It follows from the definition of d_X and the axioms that this pairing is natural with respect to $\mathfrak{A}$-homeomorphisms and multiplicative with respect to disjoint union. Let us verify that $d_{-X}=d_X\,\mathrm{Perm}_{Q,P}$ where $\mathrm{Perm}_{Q,P}$ is the flip $Q\otimes P\to P\otimes Q$. Consider the homeomorphism $g:X\times[0,1]\to X\times[0,1]$ defined by the formula $g(x\times t)=x\times(1-t)$ where $x\in X$, $t\in[0,1]$. The axiom (2.1c) implies that g inverts the $\mathfrak{B}$-structure in the cylinder $J=X\times[0,1]$ so that g may be regarded as a $\mathfrak{B}$-homeomorphism $X\times[0,1]\to(-X)\times[0,1]$. Therefore g yields a homeomorphism of cobordisms

$$\big(X\times[0,1],\,X\times 0\cup(-X)\times 1,\,\emptyset\big)\longrightarrow\big((-X)\times[0,1],\,(-X)\times 0\cup X\times 1,\,\emptyset\big).$$

The homomorphism

$$P\otimes Q=\mathcal{T}\big(X\times 0\cup(-X)\times 1\big)\longrightarrow\mathcal{T}\big((-X)\times 0\cup X\times 1\big)=Q\otimes P$$

induced by g is just the flip $\mathrm{Perm}_{P,Q}$. Therefore the naturality of τ with respect to $\mathfrak{B}$-homeomorphisms of cobordisms implies that $d_X=d_{-X}\,\mathrm{Perm}_{P,Q}$. A similar argument shows that $b_{-X}=\mathrm{Perm}_{Q,P}b_X$.

It remains to prove non-degeneracy of d_X. With this view we prove that d_X and b_X satisfy conditions of Lemma 4.1. Let us take four disjoint copies J_1, J_2, J_3, J_4 of the cylinder $J=X\times[0,1]$. Clearly,

$$\partial J_i=(-X_i^-)\amalg X_i^+$$

where X_i^-, X_i^+ are copies of X with $i=1,2,3,4$. Consider the cobordisms $(J_1\amalg J_2,\,-\partial J_1\amalg X_2^-,\,X_2^+)$ and $(J_3\amalg J_4,\,X_3^-,\,X_3^+\amalg\partial J_4)$. The operators τ corresponding to these two cobordisms may be identified with $d\otimes\mathrm{id}_P$ and $\mathrm{id}_P\otimes b$ respectively. Gluing these two cobordisms along the identification

$$X_3^+\amalg\partial J_4=X\amalg(-X)\amalg X=-\partial J_1\amalg X_2^-$$

we get a cobordisms, say, M between X_3^- and X_2^+. Applying the axiom (2.2c) we conclude that

$$\tau(M) = k(d \otimes \mathrm{id}_P)(\mathrm{id}_P \otimes b)$$

for certain invertible $k \in K$. It follows from the axiom (2.1d) that M is $\mathfrak{B}$-homeomorphic to the cylinder J via a homeomorphism extending the identifications of their bases $X_3^- = X$, $X_2^+ = X$.

Naturality of τ with respect to $\mathfrak{B}$-homeomorphisms and axiom (2.2d) imply that $\tau(M) = \tau(J)$. Therefore $(d_X \otimes \mathrm{id}_P)(\mathrm{id}_P \otimes b_X) = k^{-1}\,\mathrm{id}_P$. Replacing in this formula X with $-X$ and P with Q, and using the expressions for d_{-X}, b_{-X} established above we get an equality equivalent to the formula $(\mathrm{id}_Q \otimes d)(b \otimes \mathrm{id}_Q) = k'\,\mathrm{id}_Q$ with invertible $k' \in K$. Now, Lemma 4.1 implies that the pairing $d_X : \mathcal{T}(X) \otimes \mathcal{T}(-X) \to K$ is non-degenerate. This completes the proof of Theorem 3.1.1.

4.3. Lemma

LEMMA .— *If $(\mathcal{T}, \tau)$ is anomaly-free then for any $\mathfrak{B}$-space M we have*

$$\tau(M\,,\, -\partial M\,,\, \emptyset) = d_{\partial M}\big(\tau(M) \otimes \mathrm{id}_{\mathcal{T}(-\partial M)}\big) : \mathcal{T}(-\partial M) \to K\,.$$

Proof.— Set $X = \partial M$. Let X_i^-, X_i^+ be copies of X and J_i be the cylinder $X_i \times [\,0\,,1\,]$ with $i = 1, 2$. Consider the cobordisms

$$(M \amalg -J_1\,,\, -X_1^-\,,\, X \amalg -X_1^+) \quad \text{and} \quad (J_2\,,\, X_2^- \amalg -X_2^+\,,\, \emptyset)\,.$$

The operators τ corresponding to these two cobordisms may be identified with $\tau(M) \otimes \mathrm{id}_{\mathcal{T}(-X)}$ and d_X respectively. Gluing these two cobordisms along $X \amalg -X_1^+ = X_2^- \amalg -X_2^+$ we get a cobordism, say, M' between $-X_1^-$ and $\emptyset$. It follows from the axiom (2.2c) (with $k = 1$) that

$$\tau(M'\,,\, -X_1^-\,,\, \emptyset) = d_X\big(\tau(M) \otimes \mathrm{id}_{\mathcal{T}(-X)}\big)\,.$$

Consider the cobordism $(M'', -X, \emptyset)$ obtained by gluing the cylinder $(-X) \times [\,0\,,1\,]$ to $(M, -X, \emptyset)$ along $(-X) \times 1 = -X$. It is easy to deduce from the axiom (2.1d) that the cobordisms $(M'', -X, \emptyset)$ and $(M'\,,\, -X_1^-\,,\, \emptyset)$ are $\mathfrak{B}$-homeomorphic via a homeomorphism extending the identity $-X_1^- = -X$. Therefore

$$\begin{aligned}\tau(M, -X, \emptyset) = \tau(M'', -X, \emptyset) &= \tau(M'\,,\, -X_1^-\,,\, \emptyset)\\ &= d_X\big(\tau(M) \otimes \mathrm{id}_{\mathcal{T}(-X)}\big)\,.\end{aligned}$$

Here the first equality follows from the axioms (2.2c), (2.2d) and the second equality follows from the naturality of τ.

4.4. Proof of Theorem 3.1.2

We have

$$\begin{aligned}\tau(M,\emptyset,\emptyset) &= \tau\big(M_2\,,\,-\partial(M_2)\,,\,\emptyset\big)\circ g_\sharp\circ\tau\big(M_1\,,\,\emptyset\,,\,\partial(M_1)\big)\\ &= d_{\partial(M_2)}\big(\tau(M_2)\otimes g_\sharp\big(\tau(M_1)\big)\big)\\ &= d_{-\partial(M_1)}\Big(\overline{g}_\sharp^{-1}\big(\tau(M_2)\big)\otimes\tau(M_1)\Big)\\ &= d_{\partial(M_1)}\Big(\tau(M_1)\otimes\overline{g}_\sharp^{-1}\big(\tau(M_2)\big)\Big)\,.\end{aligned}$$

Here the first equality follows from the axiom (2.2c) (with $k = 1$), the second equality follows from Lemma 4.3, the third equality follows from the naturality of d_X, and the last equality follows from the symmetry of d_X.

4.5. Lemma

LEMMA .— *If under the conditions of Lemma 4.1 the modules P, Q are projective then* $\mathrm{Dim}(P) = \mathrm{Dim}(Q) = k^{-1}(d\circ\mathrm{Perm}_{P,Q}\circ b)(1)\in K$.

Proof.— If we identify P with Q^* via the isomorphism $P\to Q^*$ induced by d then d is identified with the evaluation pairing $d_Q : Q^*\otimes Q\to K$ whereas b is identified with a homomorphism $K\to Q\otimes Q^*$ which satisfies the identities $(\mathrm{id}_Q\otimes d)(b\otimes\mathrm{id}_Q) = k\,\mathrm{id}_Q$ and $(d\otimes\mathrm{id}_{Q^*})(\mathrm{id}_{Q^*}\otimes b) = k\,\mathrm{id}_{Q^*}$. Such a homomorphism $K\to Q\otimes Q^*$ is uniquely determined by the evaluation pairing d and equals kb_Q where $b_Q = d_Q^* : K\to Q\otimes Q^*$. It follows directly from the definition of $\mathrm{Dim}(Q)$ that

$$\mathrm{Dim}(Q) = d_Q\,\mathrm{Perm}_{Q,P}b_Q(1) = k^{-1}(d\circ\mathrm{Perm}_{P,Q}\circ b)(1)\,.$$

The equality $\mathrm{Dim}(P) = \mathrm{Dim}(Q)$ follows from the duality of these two modules.

4.6. Proof of Theorem 3.1.3

Let d_X and b_X be the linear operators defined in Section 4.2. The proof of Theorem 3.1.1 shows that these operators satisfy the equalities of Lemma 4.1. Since $(\mathcal{T},\tau)$ is anomaly-free we may assume that $k = k' = 1$. The previous lemma shows that

$$\mathrm{Dim}\big(\mathcal{T}(X)\big) = \big(d_X\,\mathrm{Perm}_{\mathcal{T}(-X),\mathcal{T}(X)}b_X\big)(1) = d_{-X}b_X(1)\,.$$

It follows from the definition of d_X, b_X and the axiom (2.2c) that $d_{-X}b_X = \tau(Z)$ where Z is the $\mathfrak{B}$-space obtained by gluing $X\times[0,1]$ to $(-X)\times[0,1]$ along the identity homeomorphism of the boundaries. Gluing first along $X\times 0=(-X)\times 0$ we get a $\mathfrak{B}$-space which $\mathfrak{B}$-homeomorphic to $X\times[0,1]$ because of (2.1c) and (2.1d). Therefore Z is $\mathfrak{B}$-homeomorphic to $X\times S^1$. This implies Theorem 3.1.3.

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