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On a theorem of Enriques – Swinnerton-Dyer(*)

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RÉSUMÉ. — On propose ici une nouvelle démonstration de l'énoncé classique suivant : chaque surface sur le corps k qui, sur la clôture algébrique de k devient isomorphe à un plan projectif avec quatre points en position générale éclatés, a un point rationnel. Nous retrouvons toutes ces surfaces comme les "quotients" d'une variété de Grassmann $G(3,5)$ par rapport à l'action de tores maximaux du groupe linéaire $GL(5)$.

ABSTRACT. — We propose a new proof of the following classical statement: every surface over a field k, which over an algebraic closure of k becomes isomorphic to the projective plane with four points in general position blown-up, has a rational point. In fact all such surfaces can be obtained as "quotients" of a Grassmannian variety $G(3,5)$ by the action of maximal tori of the general linear group $GL(5)$.

1. Introduction

Let k be a perfect field. The aim of this note is to give a new proof of the following statement formulated by Enriques [3] in 1897 and proved by Swinnerton-Dyer [11] in 1970.

THEOREM . — *Any del Pezzo k-surface of degree 5 has a k-point.*

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This statement is usually used to prove the k-rationality of such a surface. The proof of [11] is indirect, so it appears that the present proof, which is conceptually very simple, is of some interest(2).

Recall that a del Pezzo surface of degree 5 is defined as a k-form of the projective plane $\mathbb{P}^2$ with 4 points in general position blown-up (we shall call this a split del Pezzo surface of degree 5; "general position" simply means that no three points are collinear). In fact, we prove that for any del Pezzo k-surface X of degree 5 there exists a maximal k-torus $T \subset GL(5)$, such that X is isomorphic to the orbit space of T on the set of semistable points of the natural action of T on the Grassmannian $G(3,5)$. If k is infinite we have plenty of semistable k-points since semistable points form a dense open subset of $G(m,n)$. For arbitrary k the existence of a k-point on X follows from a simple statement known as the lemma of Lang – Nishimura ([6], [9]).

The action of the group of diagonal matrices of $GL(n)$ on $G(m,n)$ is briefly discussed in Section 2. In Section 3 we show that for $m = 3$ and $n = 5$ the corresponding space of (semi)stable orbits is no other than $\mathbb{P}^2$ with four points blown-up, a fact probably well known to experts (*cf.* [2]). Note that the automorphism group of the split del Pezzo surface of degree 5 is precisely the Weyl group $W(A_4)$, isomorphic to the group of permutations on five elements. We prove the main theorem in Section 4 (Theorem 4.4) by combining these geometric facts with a formal argument in Galois cohomology.

2. Torus action on Grassmannians : the split case

Les $V = k \oplus \cdots \oplus k$, $\dim(V) = n$, be the vector space with a fixed decomposition into the direct sum of one-dimensional subspaces. Let $SL(n)$ be the group of linear transformations of V with determinant 1, and $D \subset SL(n)$ be the subgroup of diagonal matrices. Consider the Grassmannian $G(m,n)$ of m-dimensional subspaces of V with the natural (right) action of $SL(n)$. The restriction of this action to D was studied in a series of papers by I. M. Gelfand and his colleagues (see, for example, [4]). Let us recall some of their constructions. In what follows $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

(2) After this paper has been sent to a journal, the author became aware that N. I. Shepherd-Barron has recently obtained another simple proof of this theorem ("The rationality of quintic del Pezzo surfaces – A short proof." Bull. London Math. Soc. 24 (1992), pp. 249-250).

Let $I_n := \{1, 2, \ldots, n\}$. Choose $e_i \in V$ to be a vector whose i-th coordinate is non-zero, and all the other coordinates are zeros. For $I \subseteq I_n$ define $V_I \subseteq V$ as the subspace generated by e_i, $i \in I$. Let f be a function from the subsets of I_n to non-negative integers. Define a constructible algebraic set $U_f \subset G(m,n)$ whose points are the subspaces $S \subset V$ such that $\dim(S \cap V_I) = f(I)$ for all $I \subseteq I_n$. We have a decomposition $G(m,n) = \bigcup_f U_f$. Obviously, U_f are D-invariant. The unique dense open set $U_0 = U_{f_0}$ parametrizes the subspaces S in general position with respect to all V_I. It is given by

$$f_0(I) = \max\{0\,,\, m + \#I - n\}\,.$$

It is often simpler to work not with f but with another function defined by:

$$r(I) = m - f(I_n \setminus I)\,.$$

Let $S \subset V$ be the subspace corresponding to a point of $G(m,n)$.

Choose a basis in S, and decompose it with respect to the coordinate system $V = k \oplus \cdots \oplus k$. Let M be the resulting matrix. One checks that for a subset $I \subseteq I_n$ the value $r(I)$ is the rank of the submatrix of M of size $(m \times \#I)$ consisting of the columns with numbers in I (see, e.g. [4, (1.1)]). In particular, the function $r_0(I) = m - f_0(I_n \setminus I) = \min\{\#I\,,\, m\}$ describes the matrices whose every m columns are linearly independent.

We are interested in "the quotient" of $G(m,n)$ by D. For this reason we consider stable and semistable points of $G(m,n)$ with respect to the ample sheaf $\mathcal{O}(1)$ corresponding to the Plücker embedding. (This makes sense because $SL(n)$ acts linearly on V, thus $\mathcal{O}(1)$ has an $SL(n)$-linearization, see [8, Chap. 4, § 4].)

LEMMA 2.1.— *The set $G(m,n)^{\mathrm{s}}$ (resp. $G(m,n)^{\mathrm{ss}}$) of stable (resp. semistable) points of $G(m,n)$ with respect to D and $\mathcal{O}(1)$ is the union of U_f for f satisfying $f(I) < (m/n)\,\#I$ (resp. $f(I) \leq (m/n)\,\#I$) for all $I \subset I_n$.*

Proof.— This follows from the proof of [8, Prop. 4.3]. □

If m and n are coprime then $(m/n)\,\#I$ is never an integer for $\#I < n$, and the lemma implies that $G(m,n)^{\mathrm{s}} = G(m,n)^{\mathrm{ss}}$.

The condition of stability can be reformulated as follows:

$$r(I) > (m/n)\,\#I \quad \text{for all nonempty } I \subseteq I_n\,. \tag{1}$$

This implies that M does not contain a zero column.

By geometric invariant theory [8, (1.10)] there exists a quasiprojective scheme Y, and a morphism $\phi : G(m,n)^{\mathrm{ss}} \to Y$ satisfying $\phi(xt) = \phi(x)$, $t \in D$, which is the universal categorical quotient [8, Def. 0.5]. According to the remark following the proof of [8, (1.11)], Y is proper over k. Moreover, there is an open set $Y' \subseteq Y$ such that $\phi^{-1}(Y') = G(m,n)^{\mathrm{s}}$, and $\phi : G(m,n)^{\mathrm{s}} \to Y'$ is the universal geometric quotient [8, Def. 0.6]. Y' has the property that every fibre $\phi^{-1}(y)$, $y \in Y'$, is an orbit of D. Note that up to an isomorphism, Y and Y' do not depend on the choice of a decomposition $V = k \oplus \cdots \oplus k$, or, equivalently, on the choice of a split maximal torus $D \subset SL(n)$.

LEMMA 2.2.— *Let $\varepsilon \in GL(n)$ be a diagonal matrix $[\varepsilon_i \delta_{ij}]$, $\varepsilon_i \in k^*$.*

Define the decomposition $I_n = \bigcup_{r=1}^{p} J_r$ such that $\varepsilon_i = \varepsilon_j$ if and only if $\{i,j\} \subseteq J_r$ for some r. A subspace $S \subset V$ is ε-invariant if and only if

$$S = \bigoplus_{r=1}^{p} (S \cap V_{J_r})\,.$$

Let $i : SL(n) \to PGL(n)$ be the canonical isogeny such that $\mathrm{Ker}(i)$ is the center of $SL(n)$. Let $T := i(D)$.

COROLLARY 2.3.— *Let $x \in U_f \subset G(m,n)$. Then the stabilizer of x in T is trivial if and only if there does not exist a decomposition*

$$I_n = \bigcup_{r=1}^{p} J_r\,,\ p \geq 2\,, \quad \textit{such that} \ \sum_{r=1}^{p} f(J_r) = m\,.$$

In particular, this is true for the points of $G(m,n)^{\mathrm{s}}$.

PROPOSITION 2.4.— *The restriction of ϕ to $G(m,n)^{\mathrm{s}} \to Y'$ endows $G(m,n)^{\mathrm{s}}$ with the structure of a Y'-torsor under T. In particular, Y' is smooth.*

Proof.— If $\#I = m$ the condition $r(I) = m$ defines an invariant dense open set $Z_I \subset G(m,n)$ (given by the non-vanishing of the corresponding determinant, or in other words, the corresponding Plücker coordinate). These form an open covering of $G(m,n)$. Let us construct a family of invariant open subsets of Z_I such that each of them is a trivial torsor under T.

In fact, we shall use the constructions of chapter 3 of the book [8]. Assume $I = \{1, \ldots, m\}$. Define an R-partition of $\{1, \ldots, m\}$ as an ordered set of subsets $S_1, \ldots, S_{n-m}$ which cover $\{1, \ldots, m\}$, and such that ([8, Def. 3.3]):

$$\#\big(S_i \cap (S_{i-1} \cup \cdots \cup S_1)\big) = 1 \quad \text{for } i = 2, \ldots, n-m\,.$$

To each R-covering we associate an open set $Z_R \subseteq Z_I$ defined as the intersection of Z_I with all Z_J's such that

$$J = I \cup \{m + j\} \setminus \{i\}\,, \quad \text{where } i \in S_j\,.$$

One then checks similarly to [*loc. cit.*] that

$$Z_R \cong T \times \mathbf{A}^{(m-1)(n-m-1)}\,.$$

It is not hard to verify that the union of Z_R's for all possible permutations of I_n coincides with the subset of $G(m, n)$ consisting of points satisfying the condition of corollary 2.3 (*cf.* [8, Prop. 3.3]). These two facts imply the proposition. □

COROLLARY 2.5. — *Let m and n be coprime. Then*

$$G(m, n)^{\mathrm{s}} = G(m, n)^{\mathrm{ss}}\,,$$

and $Y = Y'$ is a smooth projective variety.

Remark 2.6. — Let N be the normalizer of D in $SL(n)$, then the Weyl group $W = W(A_{n-1}) := N/D$ of the root system A_{n-1} is the symmetric group Σ_n permuting the components of the decomposition $V = k \oplus \cdots \oplus k$. It acts on D, and thus on T. Clearly $G(m, n)^{\mathrm{s}}$ and $G(m, n)^{\mathrm{ss}}$ are invariant under N, thus W acts by automorphisms on Y and Y'.

The following trivial remark will be important in what follows. The group Σ_n of permutations of the components of the decomposition $V = k \oplus \cdots \oplus k$ is naturally a subgroup of $GL(n)$. This makes it possible to identify W with a subgroup of $GL(n)$. As such, it naturally acts on $G(m, n)$. This action preserves $G(m, n)^{\mathrm{s}}$ and $G(m, n)^{\mathrm{ss}}$, and the corresponding morphisms to Y and Y' are W-equivariant.

3. Del Pezzo surfaces of degree 5: the split case

DEFINITION 3.1. — *A split del Pezzo surface of degree* 5 *is defined as the blowing-up of* $\mathbb{P}^2$ *in points* $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ *and* $(1:1:1)$.

Note that we could as well define a split del Pezzo surfaces of degree 5 as the blowing-up of four points in $\mathbb{P}^2$, no three of them collinear. Indeed, $PGL(3)$ acts transitively on such quadruples. By the universal property of blowing-up [5, II.7.15], there is a unique isomorphism of the corresponding blowings-up extending this action.

PROPOSITION 3.2. — *Let* $(m,n) = (3,5)$, *then* $Y = Y'$ *is a split del Pezzo surface of degree* 5.

Proof. — The stability condition (1) implies that every two columns are not proportional. Let $I \subset I_5$, $\#I = 3$. The condition that the columns of M with numbers in I are linearly independent defines a dense open set $Z_I^{\mathrm{s}} = Z_I \cap G(3,5)^{\mathrm{s}}$. It is D-invariant, so its image $\phi(Z_I^{\mathrm{s}})$ is also open. Define a dense open set $Z \subset G(3,5)^{\mathrm{s}}$ as the intersection of the Z_I^{s}'s for all possible three-element subsets of $\{1, 2, 3, 4\}$. Now let $S \subset V$ be the subspace corresponding to a point of Z. From the way we defined Z it follows that:

- every three out of the first four columns of M are linearly independent;
- no two columns are proportional.

Changing the basis, and multiplying the columns of M by non-zero numbers (this is the action of D), we can arrange that M is of the following form:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & x \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 1 & 1 & z \end{bmatrix}$$

Here x, y, z are uniquely determined up to multiplication by a common non-zero constant. Conversely, taking any point

$$(x:y:z) \in \mathbb{P}^2 \setminus \{(1:0:0), (0:1:0), (0:0:1), (1:1:1)\}$$

one checks immediately that the corresponding matrix M satisfies the stability condition (1), and so the space generated by its rows defines a point in $G(3,5)^{\mathrm{s}}$. Thus the map which sends S to $(x:y:z) \in \mathbb{P}^2$ is an isomorphism of $\phi(Z) \subset Y$ onto $\mathbb{P}^2 \setminus \{(1:0:0), (0:1:0), (0:0:1), (1:1:1)\}$. By Corollary 2.5, Y is a smooth projective surface, and this isomorphism extends to a birational morphism $\sigma : Y \to \mathbb{P}^2$ (Zariski's Main Theorem [5, V.5.2]).

Let us denote $L_I = Y \setminus \phi(Z_I^s)$, $I \subset I_5$, $\#I = 3$. We now prove that:

(a) $L_I \cap L_J = \emptyset$ if and only if $\#(I \cup J) = 4$;

(b) every L_I is isomorphic to $\mathbb{P}^1$.

It follows from (a) and (b) that $Y \setminus \phi(Z)$ is the disjoint union of four smooth proper curves of genus 0. Thus σ^{-1} is the blowing-up of the above four points in $\mathbb{P}^2$ (*cf.* [5, V.5.4]), and the proposition will be proved.

Note that the stability condition (1) has it that $r(K) = 3$ for any 4-element subset $K \subset I_5$. To prove (a) one checks that $\#(I \cup J) = 4$ and $r(I) = r(J) = r(I \cap J) = 2$ automatically imply that $r(I \cup J) = 2$, which is not possible.

In order to prove (b) we can assume by symmetry that $I = \{3, 4, 5\}$. Then $L_{\{3,4,5\}}$ is covered by the following open sets:

$$A = L_{\{3,4,5\}} \setminus \left(L_{\{1,2,3\}} \cup L_{\{1,2,4\}}\right),$$
$$B = L_{\{3,4,5\}} \setminus \left(L_{\{1,2,4\}} \cup L_{\{1,2,5\}}\right),$$
$$C = L_{\{3,4,5\}} \setminus \left(L_{\{1,2,3\}} \cup L_{\{1,2,5\}}\right).$$

Choose a point in $\phi^{-1}(A)$, and a basis in the corresponding vector space S. Let M be the matrix obtained by decomposing this basis with respect to the standard basis of $V = k \oplus \cdots \oplus k$. We have $r(\{1, 2, 3\}) = 3$, $r(\{1, 2, 4\}) = 3$. It follows from (a) that $r(\{1, 3, 4\}) = 3$, $r(\{2, 3, 4\}) = 3$. This means that every three out of the first four columns of M are linearly independent. On the other hand, the last three columns are linearly dependent. Now changing the basis, and multiplying the columns of M by non-zero numbers, we can arrange that M is of the following form:

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & x \end{bmatrix}$$

Here $x \in k$ is uniquely defined, and any $x \neq 1$ would do. This proves that A is isomorphic to $\mathbf{P}^1$ minus two points. We leave to the reader the routine verification that A, B, C glue together to produce $\mathbf{P}^1$. This completes the proof of the proposition. □

From now on we fix the notation Y for the split del Pezzo surface of degree 5. Recall that Y contains precisely 10 exceptional curves of the first kind (see, e.g., [7, Chap. 4]).

COROLLARY 3.3.— *(of the proof) The genus zero curves L_I are exceptional curves of the first kind on Y. There are 10 of these, therefore every exceptional curve of the first kind on Y coincides with L_I for some $I \subset I_5$, $\#I = 3$.*

Proof.— The curves L_I for $I \subset \{1, 2, 3, 4\}$ can be smoothly blown down as it follows from the proof of Proposition 3.2. By symmetry, the same is true for any L_I. □

The following statement seems to be well known to experts (*cf.* [2, VII]).

PROPOSITION 3.4.— *The natural map*

$$\nu : \mathrm{Aut}(Y) \to \mathrm{Aut}\big(\mathrm{Pic}(Y)\big)$$

is an isomorphism onto the group of automorphisms of $\mathrm{Pic}(Y)$ *leaving invariant the canonical class* $K_Y \in \mathrm{Pic}(Y)$ *and the scalar product* $(\cdot\,,\,\cdot)$ *given by the intersection index. The group* $\nu\big(\mathrm{Aut}(Y)\big)$ *is isomorphic to the Weyl group* $W = W(A_4)$*, implying* $\mathrm{Aut}(Y) \cong W$.

Proof.— We know from Remark 2.6 that W acts on Y. We prove that $\mathrm{Ker}(\nu) = 1$, $\mathrm{Im}(\nu) \cong W$. Indeed, let $\alpha \in \mathrm{Ker}(\nu)$, then α fixes the classes $[L_I] \in \mathrm{Pic}(Y)$ of exceptional curves of the first kind. Since L_I is the only curve in its class of linear equivalence, L_I is α-invariant. By the proof of Proposition 3.2, the complement in Y to the union of L_I, for $I \subset \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{P}^2 \setminus \big\{(1:0:0)\,,\,(0:1:0)\,,\,(0:0:1)\,,\,(1:1:1)\big\}$. Thus α defines a birational automorphism of $\mathbb{P}^2$, which is in fact biregular by Zariski's Main Theorem. It follows that α comes from an element of $PGL(3)$ fixing the four points as above. Thus α must be the identity map. Next we consider $\nu\big(\mathrm{Aut}(Y)\big)$. This group fixes the canonical class $K_Y \in \mathrm{Pic}(Y)$. On the other hand, the scalar product $(\cdot\,,\,\cdot)$ given by the intersection index, is also $\nu\big(\mathrm{Aut}(Y)\big)$-invariant. The restriction of $(\cdot\,,\,\cdot)$ to the orthogonal complement $K_Y^{\perp}$ is negative definite, and the elements with norm -2 form a root system A_4 [7, IV]. By [7, IV.1] the subgroup of $\mathrm{Aut}\big(\mathrm{Pic}(Y)\big)$ leaving invariant K_Y and $(\cdot\,,\,\cdot)$ is the Weyl group $W = W(A_4)$. Thus $\nu\big(\mathrm{Aut}(Y)\big) \subseteq W$. By Remark 2.6, $\nu\big(\mathrm{Aut}(Y)\big)$ contains $\nu(W) \cong W$, implying that $\mathrm{Aut}(Y) \cong W$. □

4. Del Pezzo surfaces of degree 5 and Galois cohomology

Let us recall some standard facts on forms and Galois cohomology [10, 1.5; 2.1; 3.1]. Let X be a variety over k. We denote by $\bar{k}$ the algebraic closure of k, $\overline{X} := X \times_k \bar{k}$, and $\Gamma := \mathrm{Gal}(\bar{k}/k)$ is the Galois group. The group $\mathrm{Aut}(\overline{X})$ of $\bar{k}$-automorphisms of $\overline{X}$ is equiped with a continuous invariant action of Γ:

$$a \to {}^s a = (1 \otimes s) a (1 \otimes s^{-1}), \quad s \in \Gamma .$$

In what follows this action comes from an action of a finite factor of Γ, so we shall make this assumption from now on.

If $k \subseteq K \subseteq \bar{k}$, then $\mathrm{Aut}(X \times_k K)$ is the set of fixed elements of $\mathrm{Aut}(\overline{X})$ with respect to the Galois group $\mathrm{Gal}(\bar{k}/k)$. If K/k is a Galois extension, a 1-cocycle $a \in Z^1(K/k, \mathrm{Aut}(X \times_k K))$ is a continuous map

$$a : \mathrm{Gal}(K/k) \to \mathrm{Aut}(X \times_k K)$$

such that $a_{st} = a_s \cdot {}^s a_t$. The cocycles a and a' are cohomologous if there exists $b \in \mathrm{Aut}(X \times_k K)$ such that $a'_s = b^{-1} \cdot a_s \cdot {}^s b$. This is an equivalence relation, and the pointed set of orbits is $H^1(K/k, \mathrm{Aut}(X \times_k K))$ (the neutral element comes from the zero cocycle).

A k-variety Z is a K/k-form of X if $Z \times_k K$ is isomorphic to $X \times_k K$. Let $E(K/k, X)$ be the pointed set of such forms considered up to an isomorphism, with the isomorphism class of X as the neutral element. Let K/k be a finite Galois extension. Then there is a canonical injection of pointed sets

$$\theta : E(K/k, X) \to H^1(K/k, \mathrm{Aut}(X \times_k K)) .$$

Let $Z \in E(K/k, X)$, then a 1-cocycle $a \in \theta(Z)$ can be chosen in the following way. Fix an isomorphism

$$\rho : Z \times_k K \xrightarrow{\sim} X \times_k K ,$$

and take $a = (a_s)$ to be the function $\mathrm{Gal}(K/k) \to \mathrm{Aut}(X \times_k K)$ such that the natural action of $\mathrm{Gal}(K/k)$ on $Z \times_k K$ (via the second factor) translates as its twisted action on $X \times_k K$:

$$\rho(1 \otimes s)\, \rho^{-1}(x) = a_s (1 \otimes s) x , \quad s \in \mathrm{Gal}(K/k) ,\ x \in X \times_k K .$$

The cohomology class of a does not depend on ρ.

If X is a quasiprojective k-variety, and K/k is a finite Galois extension, then θ is bijective [10, III.1.3]. In fact, the corresponding form is the quotient scheme $(X \times_k K)/\operatorname{Gal}(K/k)$ with respect to the twisted action of $\operatorname{Gal}(K/k)$.

PROPOSITION 4.1. — *Let X be a quasiprojective k-variety. Assume that $\operatorname{Aut}(X) = \operatorname{Aut}(\overline{X})$, and that this group is finite. Let $\operatorname{Inn}(\operatorname{Aut}(X))$ be the group of inner automorphisms of $\operatorname{Aut}(X)$, and let*

$$\operatorname{Hom}(\Gamma\,,\ \operatorname{Aut}(X))/\operatorname{Inn}(\operatorname{Aut}(X))$$

be the set of orbits of $\operatorname{Inn}(\operatorname{Aut}(X))$ on $\operatorname{Hom}(\Gamma\,,\ \operatorname{Aut}(X))$ with respect to the natural action. Then there is a canonical bijection of pointed sets

$$\theta : E(\overline{k}/k\,,\ X) \xrightarrow{\sim} \operatorname{Hom}(\Gamma\,,\ \operatorname{Aut}(X))/\operatorname{Inn}(\operatorname{Aut}(X))\,.$$

Proof. — Since $\operatorname{Aut}(X) = \operatorname{Aut}(\overline{X})$, this group has a trivial action of Γ. Thus 1-cocycles are no other that homomorphisms, and the equivalence relation of 1-cocycles is just the conjugation. A homomorphism $\Gamma \to \operatorname{Aut}(\overline{X})$ has a finite image, thus the corresponding form can be recovered as a quotient scheme, and so θ is bijective. □

DEFINITION 4.2. — *A del Pezzo surface of degree 5 is defined as a $\overline{k}/k$-form of the split del Pezzo surface of degree 5.*

COROLLARY 4.3. — *There is a natural bijection between the following pointed sets:*

(i) the set of isomorphism classes of del Pezzo k-surfaces of degree 5 with the class of the split surface as the neutral element;

(ii) the pointed set $H^1(\Gamma, W)$;

(iii) the pointed set of orbits $\operatorname{Hom}(\Gamma, W)/\operatorname{Inn}(W)$ with the trivial homomorphism as the neutral element.

Proof. — By Proposition 3.4 we have $\operatorname{Aut}(Y) \cong W$, but we also have $\operatorname{Aut}(\overline{Y}) \cong W$ by the same result, so we are in the situation of Proposition 4.1. □

THEOREM 4.4.— *Any del Pezzo k-surface of degree 5 has a k-point.*

Proof.— Let us consider a twisted version of the whole set-up of Section 2. Let us identify W with the group Σ_5 of permutational matrices in $GL(5)$. Fix a homomorphism $h : \Gamma \to W \cong \Sigma_5$. Define the following action of Γ on $V \otimes \bar{k} = \bar{k} \otimes \cdots \otimes \bar{k}$:

$$s(v) = h(s)(1 \otimes s)v\,, \quad s \in \Gamma\,,\ v \in V \otimes \bar{k}\,. \tag{2}$$

This obviously induces an action of Γ on $G(3,5) \times_k \bar{k}$, and thus on $G(3,5)^{\mathrm{s}} \times_k \bar{k}$. By the general theory, we can consider the corresponding $\bar{k}/k$-forms ${}_hG(3,5)$ and ${}_hG(3,5)^{\mathrm{s}}$.

The map $\phi : G(3,5)^{\mathrm{s}} \to Y$ gives rise to ${}_h\phi : {}_hG(3,5)^{\mathrm{s}} \to {}_hY$ (recall that W normalizes the torus D, and hence ϕ is W-equivariant). Clearly ${}_hY$ is a form of Y. Since Σ_5 normalizes the diagonal torus of $GL(5)$, we get from (2) that the corresponding twisted action of Γ on $\overline{Y}$ is given by

$$s(x) = h(s)(1 \otimes s)x\,, \quad s \in \Gamma\,,\ x \in \overline{Y}\,.$$

Thus ${}_hY$ is a del Pezzo surface of degree 5 whose cohomology class is represented by $h \in \mathrm{Hom}(\Gamma, W)$. It follows from Corollary 4.3 that we obtain all del Pezzo surfaces of degree 5 in this way.

Now let us go back to ${}_hG(3,5)$. This is a homogeneous space of $GL(5)$ twisted by a cocycle $h : \Gamma \to W$. Due to the fact that $W \cong \Sigma_5$ naturally lies in $GL(5)$, the cocycle h lifts to a cocycle with coefficients in $GL(5)$. Any such is a coboundary by Hilbert's Theorem 90. It follows that ${}_hG(3,5)$ is isomorphic to $G(3,5)$.

If k is infinite, then k-points are Zariski dense on $G(3,5)$, and so there is a k-point on ${}_hG(3,5)^{\mathrm{s}}$, and hence on ${}_hY$. Following [11] we may end the proof in the finite field case by refering to a general theorem of Weil [12] that a smooth projective rational surface defined over a finite field k always has a k-point (see also [7, 23.1]). However, a simple general argument is available, which I owe to J.-L. Colliot-Thélène:

LEMMA (Lang [6], Nishimura [9]).— *If $f : X \to Z$ is a rational map of integral k-varieties, where Z is proper and X has a smooth k-point, then Z has a k-point.*

Applying this with $X = G(3,5)$ and $Z = {}_hY$ we prove the theorem. □

One can interprete ${}_hG(3,5)^s$ as an "almost universal" torsor on ${}_hY$: it is a torsor under the algebraic k-torus dual to the Γ-module $K_{\overline{Y}}^{\perp}$. (Recall that a universal torsor is a torsor under the dual torus of the whole Picard group $\mathrm{Pic}(\overline{Y})$, see the details in [1].) Thus it is not surprising that in our proof k-points are first traced on ${}_hG(3,5)^s$: this agrees with the philosophy of the descent theory [1] that the universal torsors over a rational variety in a certain sense "untwist" its arithmetic.

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