ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

ENRIQUE FERNÁNDEZ-CARA
FRANCISCO GUILLÉN

# Some new existence results for the variable density Navier-Stokes equations

*Annales de la faculté des sciences de Toulouse 6e série*, tome 2, nº 2 (1993), p. 185-204

<http://www.numdam.org/item?id=AFST_1993_6_2_2_185_0>

Annales de la Faculté des Sciences de Toulouse Vol. II, n° 2, 1993

# Some new existence results for the variable density Navier-Stokes equations[*]

ENRIQUE FERNÁNDEZ-CARA and FRANCISCO GUILLÉN[1]

RÉSUMÉ. — Dans cet article, on établit de nouveaux résultats d'existence pour un système d'équations aux dérivées partielles qui décrit le comportement d'un fluide visqueux, incompressible et non homogène. D'abord, on considère le cas avec des conditions aux limites de type Dirichlet non homogènes. Dans un deuxième résultat, on impose des conditions aux limites s'annulant mais, en même temps, on suppose que la viscosité n'est pas constante. Plus précisément, on suppose qu'elle est une fonction continue de la densité. Dans les démonstrations, les arguments sont similaires mais, dans le deuxième cas, on a besoin d'une analyse détaillée du comportement de la densité.

ABSTRACT. — In this paper, we prove new existence results for a nonlinear partial differential system modelling the behavior of a viscous, nonhomogeneous and incompressible flow. First, we consider the case of general (nonzero) Dirichlet boundary conditions. In a second result, we assume that the boundary data vanish but viscosity is nonconstant — more precisely, a continuous function of the density. In the proofs, the arguments are similar, although a more detailed analysis of the behavior of the density is needed in the second case.

## Contents



(*) Reçu le 17 novembre 1992

(1) Departamento de Análisis Matemático, Universidad de Sevilla, C/Tarfia s/n, 41012 Sevilla (Spain)

## Notation

$\mathbf{uv}$ is the tensor product of $\mathbf{u}$ and $\mathbf{v}$; $\mathbf{uv}$ is a $N \times N$ matrix whose $(i,j)$-th component is $u_i v_j$

$\nabla \mathbf{u}$ is the gradient of $\mathbf{u}$; $\nabla \mathbf{u}$ is a $N \times N$ matrix whose $(i,j)$-th component is $\partial u_i / \partial x_j$

$\nabla \cdot \mathbf{u} = \sum_i \frac{\partial u_i}{\partial x_i}$ is the divergence of $\mathbf{u}$

$\nabla \times \mathbf{u} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$ is the curl of $\mathbf{u}$

$(\mathbf{u} \cdot \nabla)\mathbf{v}$ is the vector whose $i$-th component is $\sum_j u_j (\partial v_i / \partial x_j)$

$\nabla \cdot (\mathbf{uv}) = (\nabla \cdot \mathbf{u})\mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{v}$

$C(\mathbb{R}_+)$ is the space of continuous real functions defined on $\mathbb{R}_+$

$C(0,T;X)$ is the Banach space of continuous functions $f : [0,T] \to X$ (here $X$ is a Banach space).

## 1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set whose boundary $\partial\Omega \in C^2$ and let $T$ be a positive real number. It is well known that the motion of a viscous, nonhomogeneous and incompressible fluid in $\Omega$ during the time interval $[0,T]$ is described by the solution of the variable density Navier-Stokes equations:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{uu}) - \nabla \cdot \big(\mu(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})\big) + \nabla p = \rho \mathbf{f}, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.3}$$

It will be assumed that these are satisfied in the open set $Q = \Omega \times (0,T)$. We complete the problem by adding to (1.1)-(1.3) the nonhomogeneous boundary condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Sigma \equiv \partial\Omega \times (0,T) \tag{1.4}$$

and the initial conditions

$$\rho\big|_{t=0} = \rho_0 , \tag{1.5}$$

$$(\rho \mathbf{u})\big|_{t=0} = \rho_0 \mathbf{u}_0 . \tag{1.6}$$

Here, the unknowns are $\rho$, $\mathbf{u} = (u_1, u_2, u_3)$ and $p$. They respectively give the mass density, the velocity field and the pressure distribution of the fluid as functions of position $\mathbf{x}$ and time $t$. The functions $\mathbf{f}$, $\mathbf{g}$, $\rho_0$, $\mathbf{u}_0$ and $\mu$ are the data of the problem ($\mu$ is the dynamic viscosity of the fluid; of course, $\mu > 0$ and $\rho_0 \geq 0$ in $\Omega$).

It is not difficult to rewrite the conservation laws of linear momentum and mass, (1.1) and (1.2) respectively, in a "nonconservative form":

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \nabla\cdot\big(\mu(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})\big) + \nabla p = \rho\mathbf{f} , \tag{1.1'}$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho = 0 . \tag{1.2'}$$

Obviously, when $\rho \equiv$ Const., (1.1)-(1.6) reduces to the well known incompressible Navier-Stokes problem.

When $\mathbf{g} \equiv 0$ and $\mu$ is constant, several results have already been obtained for (1.1)-(1.6). For a bounded $\Omega$, the existence of a weak solution has been established by S. A. Antonzev and A. V. Kazhikhov ([1], [7]) and J.-L. Lions [12], under the assumption

$$\rho_0 \geq \alpha > 0 \quad \text{in } \Omega .$$

This has been generalized by J. Simon [17] to the more general case in which one merely assumes $\rho_0 \geq 0$. O. A. Ladyzhenskaya and V. A. Solonnikov [10] and H. Okamoto [14] proved the existence and uniqueness — in certain "regular" spaces — of a local solution, *i.e.* a solution in $\Omega \times [0, T_*)$ for some $T_* \leq T$. In [18], Simon found a global "nonsmooth" solution for which the initial condition (1.6) is satisfied in a weak sense — he also proved that, under some regularity for the data, this solution is more regular in a short interval of time — and, recently, it was shown in [3] that global existence remains essentially true in an unbounded three-dimensional $\Omega$ (for unbounded domains, M. Padula had previously obtained in [15]-[16] results that are similar to those in [10]; see also [6]).

In this paper, our main aim is to prove the existence of a global solution of (1.1)-(1.6) (as in [18] and [3]) first for nonvanishing g and then for nonconstant $\mu$. Results of this kind have already been obtained:

- when the fluid is homogeneous, *i.e.* it is governed by the Navier-Stokes equations (see e.g. O. A. Ladyzhenskaya [9] and J.-L. Lions [11]; see also R. Témam [19]);
- also, when $\mu$ is a constant and the fluid is compressible and satisfies some additional properties (for barotropic flows, see Fiszon and Zajacszkowski [4] and Lukaszewicz [13]; for other flows, see Valli and Zajacszkowski [20]).

This paper is organized as follows. In section 2, we first present an argument by means of which it is possible to reduce (1.1)-(1.6) to an equivalent similar problem for which boundary values are zero (here, for simplicity, we assume that g does not depend on $t$). Then, we state our first main result, theorem 1, which gives the existence of a global weak solution for constant $\mu$. In section 3, we present the proof of theorem 1. It relies on the definition of a family of approximations to (1.1)-(1.6) that can be obtained by solving certain "semi-discretized" problems (more precisely, some nonlinear problems in which (1.1) but not (1.2) has been discretized). The fact that $g \neq 0$ makes more difficult to obtain uniform *a priori* estimates and to solve (1.2), (1.5) by the method of characteristics. In section 4, we consider the case in which g depends on $t$. Finally in section 5, we state and prove theorem 3, an existence result for (1.1)-(1.6) with $g = 0$ and nonconstant $\mu$. It will be seen that the proof is similar to the one in section 3 but, now, some recent results on the transport problem satisfied by $\rho$ — mainly due to R. Di Perna and P. L. Lions, see [2] — are needed.

## 2. The first main result (theorem 1)

As announced, we will first consider the case in which $g \neq 0$ but $\mu$ is a positive constant. Let us introduce our assumptions on $\Omega$:

$\Omega$ is a bounded open connected set, $\partial\Omega \in C^2$;
either $\Omega$ is simply connected (and then $\Gamma \equiv \partial\Omega$), or it is not and, in this case, $\partial\Omega \equiv \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_p$, with $\Gamma_0$ being the "outer" boundary and $\Gamma_1, \ldots, \Gamma_p$ being the "inner" boundaries.

In this section, we assume that g does not depend on $t$ (a more general situation will be considered in section 4). It will be imposed that

$$\mathbf{g} \in H^{3/2}(\partial\Omega)^3 \tag{2.1}$$

and, also, that the compatibility conditions

$$\int_{\Gamma_i} \mathbf{g}\cdot\mathbf{n}\,d\sigma = 0, \quad \forall\, i : 0 \le i \le p$$

or (2.2)

$$\int_{\Gamma} \mathbf{g}\cdot\mathbf{n}\,d\sigma = 0 \quad (\text{if } \Omega \text{ is simply connected})$$

are satisfied. For a given g in these conditions, let us introduce the couple $(\mathbf{a}, q)$, the unique solution to the Stokes problem

$$\begin{cases} -\Delta\mathbf{a} + \nabla q = 0\,, \quad \nabla\cdot\mathbf{a} = 0 & \text{in } \Omega, \\ \mathbf{a} = \mathbf{g} & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Due to (2.1)-(2.2) and the regularity of $\partial\Omega$, we know that

$$\mathbf{a} \in H^2(\Omega)^3\,, \quad \nabla q \in L^2(\Omega)^3 \quad \text{and} \quad \|\mathbf{a}\|_{H^2} + \|\nabla q\|_{L^2} \le C\|\mathbf{g}\|_{H^{3/2}(\partial\Omega)^3} \tag{2.4}$$

for some constant $C$ (*cf.* [9, p. 78]). Also, from known results (for instance, see Corollary 3.3, p. 47 in [5]), we have

$$\exists\, \mathbf{b} \in H^2(\Omega)^3 \quad \text{such that} \quad \mathbf{a} = \nabla\times\mathbf{b} \text{ and } \nabla\cdot\mathbf{b} = 0\,. \tag{2.5}$$

Our aim is to solve the nonlinear system (1.1)-(1.6). Notice that, after the introduction of the new variable $\mathbf{u}^* = \mathbf{u} - \mathbf{a}$ (which in the sequel will be denoted again by $\mathbf{u}$), (1.1)-(1.6) also reads

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot\big(\rho(\mathbf{u}+\mathbf{a})\mathbf{u}\big) + (\rho\cdot\nabla)\mathbf{a} - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{F} + \mu\Delta\mathbf{a} \text{ in } Q\,, \tag{2.6}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\big(\rho(\mathbf{u}+\mathbf{a})\big) = 0 \quad \text{in } Q\,, \tag{2.7}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } Q\,, \tag{2.8}$$

$$\mathbf{u} = 0 \quad \text{on } \Sigma\,, \tag{2.9}$$

$$\rho\big|_{t=0} = \rho_0 \quad \text{in } \Omega\,, \tag{2.10}$$

$$(\rho \mathbf{u})\big|_{t=0} = \rho_0 \overline{\mathbf{u}}_0 \quad \text{in } \Omega\,, \tag{2.11}$$

where we have set $\mathbf{F} = \mathbf{f} - (\mathbf{a}\cdot\nabla)\mathbf{a}$ and $\overline{\mathbf{u}}_0 = \mathbf{u}_0 - \mathbf{a}$ (recall that $\mu$ is a positive constant). As usual, let us set

$$\mathcal{V} = \left\{\mathbf{v}\,;\, \mathbf{v} \in C_0^\infty(\Omega)^3\,,\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\right\}$$

and let $H$ and $V$ be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and $H_0^1(\Omega)^3$ respectively. It is well known that

$$H = \left\{\mathbf{v}\,;\, \mathbf{v} \in L^2(\Omega)^3\,,\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\,,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\right\}$$

and

$$V = \left\{\mathbf{v}\,;\, \mathbf{v} \in H_0^1(\Omega)^3\,,\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\right\}.$$

On the other hand, recall that, when $X$ is a Banach space, $0 < s \le 1$ and $1 \le q \le +\infty$, the Nikolskii space $N^{s,q}(0,T\,;\,X)$ is given as follows [8]:

$$N^{s,q}(0,T\,;\,X) = \Big\{f\,;\, f \in L^q(0,T\,;\,X)\,,\\ \sup_{0<h<T} h^{-s}\big\|\tau_h f - f\big\|_{L^q(0,T-h\,;\,X)} < +\infty\Big\}.$$

Here, $\tau_h f(t) = f(t+h)$ for $t$ a.e. in $[\,0\,,\,T-h\,]$. This is a Banach space for the norm $\|\cdot\|_{N^{s,q}(0,T\,;\,X)}$, where

$$\|f\|_{N^{s,q}(0,T\,;\,X)} = \|f\|_{L^q(0,T\,;\,X)} + \sup_{0<h<T} h^{-s}\big\|\tau_h f - f\big\|_{L^q(0,T-h\,;\,X)}\,.$$

In particular, $N^{s,\infty}(0,T\,;\,X)$ is a space of $X$-valued functions $f = [\,0\,,\,T\,] \to X$ which are bounded and Hölder-continuous. Our first main result is the following theorem.

THEOREM 1.— *Assume* $\overline{\mathbf{u}}_0 \in H$, $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \ge 0$ *a.e. and* $\mathbf{f} \in L^1\big(0,T\,;\,L^2(\Omega)^3\big)$. *Then, problem (2.6)-(2.11) possesses at least one weak solution. More precisely, there exist functions*

$$\mathbf{u} \in L^2(0,T\,;\,V)\,,\quad p \in W^{-1,\infty}\big(0,T\,;\,L^2(\Omega)\big) \text{ and } \rho \in L^\infty\big(0,T\,;\,L^\infty(\Omega)\big)$$

*such that*

$$\rho\mathbf{u} \in L^\infty\big(0,T\,;\, L^2(\Omega)^3\big) \cap N^{1/4,2}\big(0,T\,;\, W^{-1,3}(\Omega)^3\big)\,,$$

$$\inf_\Omega \rho_0 \le \rho(\mathbf{x},t) \le \sup_\Omega \rho_0 \quad \textit{a.e. in } Q$$

*and satisfy*

*(2.6) as an equality in* $W^{-1,\infty}\big(0,T\,;\, H^{-1}(\Omega)^3\big)$,

*(2.7) as an equality in* $L^\infty\big(0,T\,;\, H^{-1}(\Omega)\big)$ *and* $L^2\big(0,T\,;\, W^{-1,6}(\Omega)\big)$,

*(2.10) as an equality in* $H^{-1}(\Omega)$ *(for instance) and*

*(2.11) in the following weak sense:*

$$\int_\Omega \rho\mathbf{u}\cdot\mathbf{v}\,\mathbf{dx} \in C\big([\,0\,,\,T\,]\big)$$

*and*

$$\left(\int_\Omega \rho\mathbf{u}\cdot\mathbf{v}\,\mathbf{dx}\right)(0) = \int_\Omega \rho_0\overline{\mathbf{u}}_0\cdot\mathbf{v}\,\mathbf{dx}\,, \quad \forall\,\mathbf{v}\in V\,.$$

*Moreover,* $\big\|\rho^{1/2}\mathbf{u}\big\|_{L^2} \in C\big([\,0\,,\,T\,]\big)$ *and* $\big\|\rho^{1/2}\mathbf{u}\big\|_{L^2}(0) = \big\|\rho^{1/2}\overline{\mathbf{u}}_0\big\|_{L^2}$.

Notice that $\overline{\mathbf{u}}_0 \in H$ if and only if $\nabla\cdot\mathbf{u}_0 = 0$ in $\Omega$ and $\mathbf{u}\cdot\mathbf{n} = \mathbf{g}\cdot\mathbf{n}$ on $\partial\Omega$. In addition, if $\rho_0 \ge \alpha > 0$, it is not difficult to prove that

$$\mathbf{u} \in L^\infty(0,T\,;\,H) \cap N^{1/4,2}\big(0,T\,;\, L^2(\Omega)^3\big)\,.$$

## 3. Proof of theorem 1

For the proof, we will introduce a semi-Galerkin approach, we will try to find appropriate *a priori* estimates and, then, we will take limits on a sequence. There are some differences with the standard argument which is used in the case of homogeneous boundary conditions ([17], [3]):

a) to be able to handle adequate semi-Galerkin approximations, we have to use basis functions of a particular kind; among other things, these have to be smooth and compactly supported in $\Omega$;

b) to be able to solve the transport equations satisfied by $\rho$, $\mathbf{a}$ has to be regularized appropriately by introducing certain approximations which vanish on $\partial\Omega$;

c) there are some new terms in the equations that have to be bounded and for which convergence properties have to be derived.

### 3.1 Regularization of F

Since $\mathbf{F} = \mathbf{f} - (\mathbf{a}\cdot\nabla)\mathbf{a} \in L^1\big(0,T\,;\,L^2(\Omega)^3\big)$, we can introduce a regularizing sequence $\{\mathbf{F}^m\}$ for $\mathbf{F}$, with

$$\mathbf{F}^m \in C\big([\,0\,,\,T\,]\,;\,L^2(\Omega)^3\big) \quad \text{and} \quad \mathbf{F}^m \to \mathbf{F} \text{ in } L^1\big(0,T\,;\,L^2(\Omega)^3\big)\,. \tag{3.1}$$

Obviously, it can be assumed that, for some function $K \in L^1(0,T)$, on has:

$$\big\|\mathbf{F}^m\big\|_{L^2} \leq K \quad \text{a.e. in } (0,T)\,. \tag{3.2}$$

### 3.2 Regularization of a

Let us also introduce a sequence $\{\mathbf{a}^m\}$ of regular and divergence-free functions which vanish on $\partial\Omega$ and converge towards $\mathbf{a}$. Such a sequence exists; indeed, it suffices to set

$$\mathbf{a}^m = \nabla\times\mathbf{b}^m \quad \text{with } \mathbf{b}^m = \zeta_{1/2m} * (X_{\Omega_{1/m}}\mathbf{b})\,, \tag{3.3}$$

where $\{\zeta_h\}$ is a usual regularizing sequence and $\mathbf{b}$ is given by (2.5). Here, $f * g$ denotes the convolution of $f$ and $g$ and $X_{\Omega_{1/m}}$ is the cut-off function which is equal to 1 on

$$\Omega_{1/m} = \big\{\mathbf{x}\,;\,\mathbf{x}\in\Omega\,,\ \operatorname{dist}(\mathbf{x},\partial\Omega) > 1/m\big\}$$

and vanishes elsewhere. Obviously,

$$\mathbf{a}^m \in C^\infty(\overline{\Omega})\,, \quad \nabla\cdot\mathbf{a}^m = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{a}^m = 0 \text{ on } \partial\Omega\,. \tag{3.4}$$

Also, $\mathbf{a}^m \to \mathbf{a}$ in $H^1_{\mathrm{loc}}(\Omega)^3$. More precisely, if $n \in \mathbb{N}$ is given and large enough, from the continuity properties of the convolution product, one deduces the following:

$$\mathbf{b}^m \to \mathbf{b} \text{ in } H^2(\Omega_{2/n})^3 \quad \text{and} \quad \big\|\mathbf{b}^m\big\|_{H^2(\Omega_{2/n})^3} \leq \big\|\mathbf{b}\big\|_{H^2(\Omega)^3}\,, \ \forall\, m \geq n\,.$$

This gives:

$$
\begin{aligned}
&\mathbf{a}^m \to \mathbf{a} \text{ in } H^1(\Omega_{2/n})^3\,, \quad \forall\, n \\
&\|\mathbf{a}^m\|_{H^1(\Omega_{2/n})^3} \le C\|\mathbf{b}^m\|_{H^2(\Omega_{2/n})^3} \\
&\qquad\qquad \le C\|\mathbf{b}\|_{H^2(\Omega)^3}\,, \quad \forall\, n\,,\ \forall\, m \ge n\,.
\end{aligned} \tag{3.5}
$$

In particular, we see that $\|\mathbf{a}^m\|_{H^1(\Omega_{2/m})^3}$ is bounded independently from $m$.

### 3.3 The choice of a special basis of $V$

Let us consider a sequence $\{\mathbf{v}^m\}$ in $V$ with the following properties:

$$
\begin{cases}
\mathbf{v}^m \in V \cap C^1(\overline{\Omega}) \text{ has compact support,} \\
\{\mathbf{v}^m\} \text{ is an orthonormal sequence of } L^2(\Omega)^3\,,
\end{cases} \tag{3.6a}
$$

$$
\forall\, \mathbf{v} \in V\,,\ \exists\, \{\mathbf{w}^k\} \text{ with } \mathbf{w}^k \in \langle \mathbf{v}^1, \ldots, \mathbf{v}^{m_k}\rangle \text{ and } \mathbf{w}^k \to \mathbf{v} \text{ in } V\,. \tag{3.6b}
$$

Such a sequence can be obtained as follows. Since $V$ is a separable Hilbert space, there exists a dense set $\{\mathbf{u}^m\,;\, m \ge 1\}$. For each $m$, one can find a sequence $\{\mathbf{w}_j^m\}_j$ in $\mathcal{V}$ such that

$$
\mathbf{w}_j^m \to \mathbf{u}^m \quad \text{in } V\,.
$$

After a renumerotation of the set $\{\mathbf{w}_j^m\,;\, j,\, m \ge 1\}$, making use of the usual orthonormalization technique with respect to the scalar product in $L^2$, one easily obtains a sequence $\{\mathbf{v}^m\}$ satisfying (3.6a) and (3.6b).

Since each $\mathbf{v}^m$ is compactly supported in $\Omega$, there exists a sequence $\{m'(m)\}$ in $\mathbb{N}$, strictly increasing with respect to $m$ and such that $m'(m) \ge m$ and $\operatorname{supp} \mathbf{v}^m \subset \Omega_{2/m'(m)}$. Hence, introducing the finite dimensional spaces $V^m = \langle \mathbf{v}^1, \ldots, \mathbf{v}^m\rangle$, it is clear that

$$
\operatorname{supp} V^m \subset \Omega_{2/m'(m)} \tag{3.7}
$$

and "$V^m \to V$", *i.e.*

$$
\forall\, \mathbf{v} \in V\,,\ \exists\, \mathbf{z}^1, \mathbf{z}^2, \ldots \in V \text{ such that } \mathbf{z}^m \in V^m \text{ and } \mathbf{z}^m \to \mathbf{v} \text{ in } V\,. \tag{3.8}
$$

For simplicity, in the sequel, when no confusion is possible, we use the symbol $m'$ to denote $m'(m)$.

### 3.4 The definition and existence of approximate solutions

For given $m$, it will be said that the couple $(\rho^m, \mathbf{u}^m)$ is a $m$-th approximate solution if $\rho^m \in C^1(\overline{Q})$, $\mathbf{u}^m \in C^1([0,T]; V^m)$ and

$$\int_\Omega \rho^m \left[\frac{\partial \mathbf{u}^m}{\partial t} + ((\mathbf{u}^m + \mathbf{a}^{m'}) \cdot \nabla)\mathbf{u}^m + (\mathbf{u}^m \cdot \nabla)\mathbf{a} - \mathbf{F}^m\right] \cdot \mathbf{v} + \\ + \mu \int_\Omega (\nabla \mathbf{u}^m + \nabla \mathbf{a}) : \nabla \mathbf{v} = 0\,, \ \forall\, \mathbf{v} \in V^m\,, \tag{3.9}$$

$$\frac{\partial \rho^m}{\partial t} + (\mathbf{u}^m + \mathbf{a}^{m'}) \cdot \nabla \rho^m = 0 \quad \text{in } Q\,, \tag{3.10}$$

$$\rho^m\big|_{t=0} = \rho_0^m \quad \text{and} \quad \mathbf{u}^m\big|_{t=0} = \mathbf{u}_0^m \quad \text{in } \Omega\,. \tag{3.11}$$

Here, we assume that

$$\begin{aligned} &\rho_0^m \in C^1(\overline{\Omega})\,,\ \rho_0^m \to \rho_0 \quad \text{weakly-}* \text{ in } L^\infty(\Omega), \\ &\rho_0^m \to \rho_0 \text{ strongly in } L^q(\Omega)\,, \quad \forall\, q < \infty\,, \\ &\frac{1}{m} + \inf_\Omega \rho_0 \le \rho_0^m \le \frac{1}{m} + \sup_\Omega \rho_0 \quad \text{in } \Omega\,, \\ &\mathbf{u}_0^m \in V^m\,,\ \mathbf{u}_0^m \to \overline{\mathbf{u}}_0 \quad \text{in } H\,. \end{aligned} \tag{3.12}$$

From the definition of $\mathbf{a}^{m'}$, one sees that (3.10) is a transport equation for $\rho^m$, where the velocity field, given by $\mathbf{u}^m + \mathbf{a}^{m'}$, vanishes on $\Sigma$. Consequently, (3.10) can be solved by the method of characteristics. Also, observe that in all terms in (3.9), we are in fact integrating in $\Omega_{2/m'(m)}$. Finally, notice that an equivalent conservative form of (3.9)-(3.11) is given by the system

$$\int_\Omega \left[\frac{\partial \rho^m \mathbf{u}^m}{\partial t} + \nabla \cdot (\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})\mathbf{u}^m) + (\rho^m \mathbf{u}^m \cdot \nabla)\mathbf{a} - \rho^m \mathbf{F}^m\right] \cdot \mathbf{v} + \\ + \mu \int_\Omega (\nabla \mathbf{u}^m + \nabla \mathbf{a}) : \nabla \mathbf{v} = 0\,, \ \forall\, \mathbf{v} \in V^m\,, \tag{3.13}$$

$$\frac{\partial \rho^m}{\partial t} + \nabla \cdot (\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})) = 0 \quad \text{in } Q\,, \tag{3.14}$$

together with (3.11).

Recall that, in the case of zero boundary conditions considered in [17], the existence of a $m$-th approximate solution can be demonstrated arguing as follows:

a) rewrite (3.9)-(3.11) as a fixed point equation and introduce the linearized approximate solutions,

b) derive approximate estimates for these linearized solutions and

c) apply Schauder's Theorem to deduce that (3.9)-(3.11) possesses at least one solution.

Here, we can repeat this argument to prove that a solution exists. Only the obtention of uniform estimates (the second step above) is different, due to the new terms in (3.9)-(3.10). However, since this difficulty also arises when one is trying to find *a priori* estimates for the $m$-th approximate solutions, it will be postponed to the next paragraph.

### 3.5 "A priori" estimates for the approximate solutions

Let us first estimate $\rho^m$. Obviously, $\rho^m$ is a constant on each characteristic line. Thus, we see from (3.12) and (3.14) that

$$\frac{1}{m} + \inf_\Omega \rho_0 \le \rho_0^m \le \frac{1}{m} + \sup_\Omega \rho_0 \le 1 + \sup_\Omega \rho_0 \equiv M \, ;$$

therefore,

$$\rho^m \quad \text{is uniformly bounded in } L^\infty(Q) \, . \tag{3.15}$$

On the other hand, from (3.13) and (3.14) one sees that, for $t$ a.e. in $[\,0\,,\,T\,]$,

$$\int_\Omega \left[ \frac{\partial}{\partial t}\Big(\rho^m \frac{|\mathbf{u}^m|^2}{2}\Big) + \nabla\Big(\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})\frac{|\mathbf{u}^m|^2}{2}\Big) + \mu|\nabla \mathbf{u}^m|^2 \right] =$$

$$= \int_\Omega \left[ \rho^m \mathbf{F}^m \cdot \mathbf{u}^m - \mu \nabla \mathbf{a} : \nabla \mathbf{u}^m - (\rho^m \mathbf{u}^m \cdot \nabla)\mathbf{a} \cdot \mathbf{u}^m \right] .$$

Observe that

$$\int_\Omega \rho^m \mathbf{F}^m \cdot \mathbf{u}^m \le M^{1/2} K(t) \left( \int_\Omega \rho^m |\mathbf{u}^m|^2 \right)^{1/2} ,$$

$$-\mu \int_\Omega \nabla \mathbf{a} : \nabla \mathbf{u}^m \le \left\| \nabla \mathbf{a} \right\|_{L^2}^2 + \frac{\mu}{4} \int_\Omega |\nabla \mathbf{u}^m|^2$$

and

$$-\int_\Omega (\rho^m \mathbf{u}^m \cdot \nabla)\mathbf{a} \cdot \mathbf{u}^m \le \frac{C}{\mu} \left\| \nabla \mathbf{a} \right\|_{L^3}^2 \int_\Omega \rho^m |\mathbf{u}^m|^2 + \frac{\mu}{4} \int_\Omega |\nabla \mathbf{u}^m|^2 .$$

Thus, one also has the following energy inequality:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^m|\mathbf{u}^m|^2 + \mu\int_\Omega |\nabla\mathbf{u}^m|^2 \le \\
&\qquad \le M^{1/2}K(t)\left(\int_\Omega \rho^m|\mathbf{u}^m|^2\right)^{1/2} + C_1\int_\Omega \rho^m|\mathbf{u}^m|^2 + C_2 \qquad (3.16)\\
&\qquad \le \left(\frac{1}{2}K(t) + C_1\right)\int_\Omega \rho^m|\mathbf{u}^m|^2 + \frac{M}{2}K(t) + C_2
\end{aligned}$$

with $C_i$ only depending on $\mu$ and $\mathbf{g}$ for $i = 1, 2$. From Gronwall's Lemma and the fact that the function $\rho_0|\mathbf{u}_0^m|^2$ is uniformly bounded in $L^1(\Omega)$, we deduce that $(\rho^m)^{1/2}\mathbf{u}^m$ is uniformly bounded in $L^\infty(0,T;\,L^2(\Omega)^3)$. From (3.6), we also derive that

$$\mathbf{u}^m \quad \text{is uniformly bounded in } L^2(0,T;\,V)\,. \qquad (3.17)$$

Hence, it is found that

$$\begin{cases} (\rho^m)^{1/2}\mathbf{u}^m \text{ and } \rho^m\mathbf{u}^m \text{ are uniformly bounded in} \\ \qquad L^\infty(0,T;\,L^2(\Omega)^3) \cap L^2(0,T;\,L^6(\Omega)^3) \end{cases} \qquad (3.18)$$

and, from interpolation theory, we easily obtain:

$$\rho^m\mathbf{u}^m\mathbf{u}^m \text{ is uniformly bounded in } L^{4/3}(0,T;\,L^2(\Omega)^9)\,. \qquad (3.19)$$

Our goal is to take limits in the conservative form of the problem (3.9)-(3.11). Consequently, we also need uniform bounds for $\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})$ and $\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})\mathbf{u}^m$. Recall that

$$\mathbf{a}^{m'(m)} \text{ is uniformly bounded in } H^1_{\mathrm{loc}}(\Omega)\,; \qquad (3.20)$$

accordingly, it is not difficult to deduce that

$$\begin{cases} \rho^m(\mathbf{u}^m + \mathbf{a}^{m'}) \text{ is uniformly bounded in} \\ \qquad L^\infty(0,T;\,L^2_{\mathrm{loc}}(\Omega)^3) \cap L^2(0,T;\,L^6_{\mathrm{loc}}(\Omega)^3) \end{cases} \qquad (3.21)$$

and

$$\rho^m(\mathbf{u}^m + \mathbf{a}^{m'})\mathbf{u}^m \text{ is uniformly bounded in } L^{3/4}(0,T;\,L^2_{\mathrm{loc}}(\Omega)^9)\,. \qquad (3.22)$$

Of course, global estimates of the same kind hold true in each $\Omega_{2/n}$.

### 3.6 "A priori" estimates for the time derivatives

From (3.14) and (3.21), one has

$$\begin{cases} \dfrac{\partial \rho^m}{\partial t} \text{ is uniformly bounded in} \\ \qquad L^\infty(0,T\,;\, H^{-1}_{\text{loc}}(\Omega)) \cap L^2(0,T\,;\, W^{-1,6}_{\text{loc}}(\Omega))\,. \end{cases} \tag{3.23}$$

Let us see that

$$\left| \frac{\partial}{\partial t} \int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v} \right| \le (G + \psi_m) \|\nabla \mathbf{v}\|_{L^2}\,, \quad \forall\, \mathbf{v} \in V^m\,, \tag{3.24}$$

where $G \in L^1(0,T)$ is a fixed function and $\{\psi_m\}$ is bounded in the space $L^{4/3}(0,T)$. Indeed, from (3.13), one finds:

$$\begin{aligned} \left| \frac{\partial}{\partial t} \int_\Omega \rho^m \mathbf{u}^m \cdot \mathbf{v} \right| \le & \Big( \|\rho^m (\mathbf{u}^m + \mathbf{a}^{m'}) \mathbf{u}^m \|_{L^2(\Omega_{2/m'})} + \\ & + \mu \|\nabla(\mathbf{u}^m + \mathbf{a})\|_{L^2} + CM \|\mathbf{F}^m\|_{L^2} \Big) \|\nabla \mathbf{v}\|_{L^2} + \\ & + \left| \int_\Omega (\rho^m \mathbf{u}^m \cdot \nabla)\mathbf{a} \cdot \mathbf{v} \right|, \end{aligned}$$

whence we obtain (3.24) (it suffices to use (3.2), (3.17) and (3.22) to bound the first term in the right and, on the other hand, to apply Hölder's inequality and Sobolev's embedding Theorem to estimate the second one).

It is now possible to obtain new estimates for $\rho^m \mathbf{u}^m$ which involve Nikolskii spaces. The argument is similar to those in [17] and [3] and consists of four steps.

*First step.*— Taking into account (3.24), we see as in [17] and [3] that a constant $C > 0$ exists with the following property:

$$\begin{aligned} I_1 \equiv & \int_0^{T-h} \left( \int_\Omega \left[ \rho^m \mathbf{u}^m (t+h) - \rho^m \mathbf{u}^m (t) \right] \cdot \left[ \mathbf{u}^m (t+h) - \mathbf{u}^m (t) \right] \right) \, dt \\ & \le C h^{1/2} \end{aligned} \tag{3.25}$$

for all $h$ with $0 < h < T$.

*Second step.*— There exists a new constant $C > 0$ such that

$$I_2 \equiv \int_0^{T-h} \left( \int_\Omega \left[\rho^m(t+h) - \rho^m(t)\right] \mathbf{u}^m(t) \cdot \left[\mathbf{u}^m(t+h) - \mathbf{u}^m(t)\right] \right) dt \\ \leq C h^{1/2} \tag{3.26}$$

for all $h$ in $(0, T)$. Indeed, if $w \in W_0^{1,3/2}(\Omega)$ and $\operatorname{supp}(\omega) \subset \Omega_{2/m'}$, it is then clear that

$$\int_\Omega \left[\rho^m(t+h) - \rho^m(t)\right] w = \int_t^{t+h} \left( \int_{\Omega_{2/m'}} \rho^m(\mathbf{u}^m + \mathbf{a}^{m'})(s) \cdot \nabla w \right) ds\,.$$

Hence, using (3.21) and taking $w = \mathbf{u}^m(t) \cdot \left(\mathbf{u}^m(t+h) - \mathbf{u}^m(t)\right)$, we easily arrive at (3.26).

*Third step.*— From the value of $I_1 - I_2$, we deduce the existence of $C > 0$ such that

$$\left\| \tau_h \rho^m (\tau_h \mathbf{u}^m - \mathbf{u}^m) \right\|_{L^2(0,T-h\,;\,L^2(\Omega)^3)} \leq C h^{1/4}\,. \tag{3.27}$$

*Fourth step.*— As in [3], from (3.14) and the estimates (3.21), one has

$$\left\| \tau_h \rho^m - \rho^m \right\|_{L^\infty(0,T-h\,;\,W^{-1,6}(\Omega_{2/m'}))} \leq \\ \leq C \max_{0 \leq t \leq T} \left\{ \int_t^{t+h} \left\| \rho^m(\mathbf{u}^m + \mathbf{a}^{m'})(s) \right\|_{L^6(\Omega_{2/m'})^3} ds \right\} \\ \leq C h^{1/2}\,.$$

Since $\mathbf{u}^m(t) \in V^m$ and this space is continuously embedded in $H_0^1(\Omega_{2/m'})^3$, (3.17) and the fact that the mapping

$$(v, \lambda) \in H_0^1 \times W^{-1,6} \to v\lambda \in W^{-1,3}$$

is continuous yields:

$$\left\| (\tau_h \rho^m - \rho^m) \mathbf{u}^m \right\|_{L^2(0,T-h\,;\,W^{-1,3}(\Omega_{2/m'})^3)} \leq C h^{1/2}\,.$$

This, together with (3.27) leads to the desired property:

$$\rho^m \mathbf{u}^m \text{ is uniformly bounded in } N^{1/4,2}\left(0, T;\, W_{\mathrm{loc}}^{-1,3}(\Omega)^3\right)\,. \tag{3.28}$$

We can now continue exactly as in the proof of Theorem 1 in [3]. Since the argument is rather standard, we omit the details (see also [6]).

## 4. The case in which the boundary data depend on $t$

In this section, we will generalize the previous result (theorem 1) to the case in which $\mathbf{g} = \mathbf{g}(\mathbf{x}, t)$. We will consider again the formulation (2.6)-(2.11), but now with different functions $\mathbf{F}$ and $\overline{\mathbf{u}}_0$.

THEOREM 2.— *Assume that* $\Omega$, $\rho_0$, $\mathbf{u}_0$ *and* $\mathbf{f}$ *are as in theorem 1. Also, assume there exist functions* $\mathbf{a}$ *and* $\mathbf{b}$ *such that*

$$\mathbf{a} \in L^2\big(0,T\,;\, H^2(\Omega)^3\big)\,, \quad \frac{\partial \mathbf{a}}{\partial t} \in L^1\big(0,T\,;\, L^2(\Omega)^3\big)\,, \quad \mathbf{a}\big|_\Sigma = \mathbf{g}\,, \tag{4.1}$$

$$\mathbf{b} \in L^2\big(0,T\,;\, H^2(\Omega)^3\big)\,, \quad \mathbf{a} = \nabla \times \mathbf{b} \ \textit{in } Q \tag{4.2}$$

*and set*

$$\mathbf{F} = \mathbf{f} - \frac{\partial \mathbf{a}}{\partial t} - (\mathbf{a}\cdot\nabla)\mathbf{a} \quad \textit{and} \quad \overline{\mathbf{u}}_0 = \mathbf{u}_0 - \mathbf{a}(0)\,.$$

*Then, problem (2.6)-(2.11) possesses at least one weak solution.*

*Sketch of the proof.*— It is very similar to the proof of theorem 1. The unique differences are due to the fact that now $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}^m$ and $\mathbf{b}^m$ depend on $t$, which lead to some changes.

Notice that $\mathbf{F} \in L^1\big(0,T\,;\, L^2(\Omega)^3\big)$ and $\mathbf{a}(0)$ makes sense in $L^2(\Omega)^3$. If we extend $\mathbf{b}$ by zero outside $Q$ and $\mathbf{a}^m$ is given by (3.3), we have

$$\mathbf{a}^m \in C^\infty(\overline{Q})\,, \quad \nabla\cdot\mathbf{a}^m = 0 \text{ in } Q \quad \text{and} \quad \mathbf{a}^m = 0 \text{ on } \Sigma\,. \tag{4.3}$$

Furthermore, from (4.2) one sees that

$$\begin{cases} \mathbf{a}^m \to \mathbf{a} \text{ in } L^2\big(0,T\,;\, H^1_{\mathrm{loc}}(\Omega)^3\big) \\ \big\|\mathbf{a}^m\big\|_{L^2(0,T\,;\,H^1(\Omega_{2/m})^3)} \text{ is uniformly bounded}\,. \end{cases} \tag{4.4}$$

Thus, with the same choice for the basis functions $\mathbf{v}^m$ and the integers $m'(m)$, arguing as in section 3, one deduces the existence of a weak solution to (2.6)-(2.11) (now, one has to replace in the definition of the $m$-th approximate solution $\mathbf{a}$ by an adequate regular apprimation; see [6] for more details). □

To end this section, we present a construction of the functions $\mathbf{a}$ and $\mathbf{b}$.

PROPOSITION .— *Assume $\Omega$ is as in theorem 1 and 2. Also, assume that*

$$\mathbf{g} \in L^2(0,T;\, H^{3/2}(\partial\Omega)^3)\,, \quad \frac{\partial \mathbf{g}}{\partial t} \in L^1(0,T;\, L^2(\partial\Omega)^3) \tag{4.5}$$

*and (2.2) holds for $t$ a.e. in $(0,T)$. Then, there exist functions* $\mathbf{a}$ *and* $\mathbf{b}$ *satisfying (4.1) and (4.2).*

*Proof.*— Let us introduce the linear operator $A : H^{1/2}(\partial\Omega)^3 \to H^1(\Omega)^3$, given as follows:

$$A\mathbf{h} = \mathbf{u} \quad \Leftrightarrow \quad \begin{cases} \mathbf{u} \in H^1(\Omega)^3\,, \quad \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\,, \quad \mathbf{u}\big|_{\partial\Omega} = \mathbf{h}\,, \\ (\nabla\mathbf{u}, \nabla\mathbf{w})_{L^2} = 0\,,\ \forall\, \mathbf{w} \in V\,. \end{cases} \tag{4.6}$$

Let us also introduce $B : L^2(\Omega)^3 \to L^2(\partial\Omega)^3$, with

$$B\mathbf{v} = \left(-\frac{\partial \mathbf{z}}{\partial \mathbf{n}} + q\mathbf{n}\right)\Big|_{\partial\Omega}\,, \quad \forall\, \mathbf{v} \in L^2(\Omega)^3\,, \tag{4.7}$$

where $(\mathbf{z}, q)$ is the unique strong solution to the Stokes problem

$$\begin{cases} \mathbf{z} \in H^2(\Omega)^3 \cap V\,, \quad q \in H^1(\Omega)\,, \quad \displaystyle\int_\Omega q(\mathbf{x})\,d\mathbf{x} = 0\,, \\ -\Delta\mathbf{z} + \nabla q = \mathbf{v}\,, \quad \nabla\cdot\mathbf{z} = 0 \text{ in } \Omega\,, \quad \mathbf{z}\big|_{\partial\Omega} = 0\,. \end{cases} \tag{4.8}$$

From the existence and regularity results for this Stokes problem, it is clear that

$$A \in \mathcal{L}(H^{1/2}(\partial\Omega)^3;\, H^1(\Omega)^3)\,, \quad A \in \mathcal{L}(H^{3/2}(\partial\Omega)^3;\, H^2(\Omega)^3)\,,$$

$B$ coincides with the restriction to $L^2(\Omega)^3$ of the adjoint operator $A^*$ and

$$B \in \mathcal{L}(L^2(\Omega)^3;\, L^2(\partial\Omega)^3)\,.$$

Furthermore, from the definitions of $A$ and $B$, one has

$$(A\mathbf{h}\,,\, \mathbf{v})_{L^2(\Omega)^3} = (\mathbf{h}\,,\, B\mathbf{v})_{L^2(\partial\Omega)^3}\,, \quad \forall\, \mathbf{v} \in L^2(\Omega)^3\,,\ \forall\, \mathbf{h} \in H^{1/2}(\partial\Omega)^3\,. \tag{4.9}$$

Let us set $\mathbf{a}(t) = A\mathbf{g}(t)$ for $t$ a.e. in $(0,T)$. Obviously,

$$\mathbf{a} \in L^2(0,T;\, H^2(\Omega)^3) \quad \text{and} \quad \mathbf{a}\big|_\Sigma = \mathbf{g}\,;$$

moreover, it is not difficult to derive that

$$\frac{\partial \mathbf{a}}{\partial t} \in L^1(0,T;\, L^2(\Omega)^3)$$

from (4.9). In other words, **a** satisfies (4.1). From (4.5), one also deduces that, for $t$ a.e. in $(0,T)$

$$\exists\ \mathbf{b}(t) \in H^2(\Omega)^2 \text{ such that } \mathbf{a}(t) = \nabla \times \mathbf{b}(t) \quad \text{and} \quad \nabla \cdot \mathbf{b}(t) = 0 \text{ in } \Omega$$

(see e.g. [5, p. 47]). If each $\mathbf{b}(t)$ is found as in [5], then it is easy to check that

$$\|\mathbf{b}(t)\|_{H^2} \le C \|\mathbf{a}(t)\|_{H^1}$$

and, consequently, $\mathbf{b} \in L^2(0,T;\, H^2(\Omega)^3)$, i.e. **b** satisfied (4.2). □

Notice that, in fact, the first assumption in (4.5) can be weakened. It suffices to impose

$$\mathbf{g} \in L^2(0,T;\, W^{2/3,3}(\Omega)^3)$$

(see [6] for more details).

## 5. The second main result (theorem 3) and its proof

This section is devoted to establish a new existence result for (1.1)-(1.6). Here, it will be assumed that $\Omega \subset \mathbb{R}^3$ is a general bounded open set with $\partial\Omega \in W^{1,\infty}$. We also assume that $\mathbf{g} \equiv 0$ but $\mu$ is not a constant but continuous function of the density.

THEOREM 3.— *Assume that $\rho_0$, $\mathbf{u}_0$ and $\mathbf{f}$ are as in theorem 1. Also, assume that $\Omega$ is bounded, $\mathbf{g} = 0$ and*

$$\mu \in C(\mathbf{R}_+), \quad \mu(s) \ge \beta > 0, \ \forall\, s \ge 0. \tag{5.1}$$

*Then, problem (1.1)-(1.6) possesses at least one weak solution $(\rho, \mathbf{u}, p)$. The functions $\rho$, $\mathbf{u}$ and $p$ satisfy the same properties as in theorem 1 (with obvious changes).*

*Proof*.— It is similar to the proof of theorem 1, even easier in some aspects. There are only two serious differences (here, we conserve the notation of section 3).

*a)* The demonstration of the inequalities

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^m|\mathbf{u}^m|^2 + \beta\|\nabla\mathbf{u}^m\|_{L^2}^2 \leq \int_\Omega \rho^m\mathbf{f}^m\cdot\mathbf{u}^m\,, \tag{5.2}$$

which replace (3.16) in this case. Of course, $(\rho^m, \mathbf{u}^m)$ is here and in the sequel a $m$-th approximate solution to (1.1)-(1.6).

*b)* The proof that

$$\int_\Omega \mu(\rho^m)(\nabla\mathbf{u}^m + {}^t\nabla\mathbf{u}^m) : \nabla\mathbf{v} \to \int_\Omega \mu(\rho)(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}) : \nabla\mathbf{v} \tag{5.3}$$

in $\mathcal{D}'(0,T)$ for every $\mathbf{v} \in \bigcup_{n\geq 1} V^n$.

The energy inequalities (5.2) can be derived as follows. First, it is clear from the context that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^m|\mathbf{u}^m|^2 + \int_\Omega \mu(\rho^m)(\nabla\mathbf{u}^m + {}^t\nabla\mathbf{u}^m) : \nabla\mathbf{u}^m \leq \int_\Omega \rho^m\mathbf{f}^m\cdot\mathbf{u}^m\,. \tag{5.4}$$

We notice that, for some $C$,

$$0 < \beta \leq \mu(\rho^m) \leq C \quad \text{in } Q\,. \tag{5.5}$$

In particular, this gives

$$\int_\Omega \mu(\rho^m)(\nabla\mathbf{u}^m + {}^t\nabla\mathbf{u}^m) : \nabla\mathbf{u}^m \geq \beta\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \int_\Omega \sum_{i,j}\frac{\partial u^{m^i}}{\partial x_j}\,\frac{\partial u^{m^j}}{\partial x_i}\right). \tag{5.6}$$

But, since $\nabla\cdot\mathbf{u} \equiv 0$ and $\mathbf{u}\big|_\Sigma \equiv 0$, the last integral vanishes. From (5.4) and (5.6), we obtain (5.2).

In order to prove (5.3), we begin by selecting a subsequence (again indexed with $m$) which is weakly convergent in $L^2$:

$$\mu(\rho^m)(\nabla\mathbf{u}^m + {}^t\nabla\mathbf{u}^m) \to \chi \quad \text{weakly in } L^2(Q)^9 \tag{5.7}$$

(this is possible, since $\mu(\rho^m)(\nabla\mathbf{u}^m + {}^t\nabla\mathbf{u}^m)$ is uniformly bounded in this space). Obviously, if

$$\chi = \mu(\rho)(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})\,,$$

then (5.3) holds. Hence, it suffices to check that $\mu(\rho^m)$ converges strongly in $L^2$ towards $\mu(\rho)$.

In principle, from (3.15) we can only assert weak convergence for $\rho^m$:

$$\rho^m \rightarrow \rho \quad \text{weakly in } L^p(Q)\,,\ \forall\, p \in [\,1\,,\,\infty)\,. \tag{5.8}$$

On the other hand, we know that $\rho$ is a weak solution of the transport problem

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 \text{ in } Q\,, \quad \rho\big|_{t=0} = \rho_0 \text{ in } \Omega\,.$$

Thus, from the results in [2], we also deduce the following for every $p < \infty$:

$$\rho \in C\big(0, T\,;\, L^p(\Omega)\big) \quad \text{and} \quad \big\|\rho(t)\big\|_{L^p} \equiv \big\|\rho_0\big\|_{L^p}\,. \tag{5.9}$$

Consequently, taking into account the choice of $\rho_0^m$ and the fact that $\big\|\rho^m(t)\big\|_{L^p} \equiv \big\|\rho_0^m\big\|_{L^p}$, one has:

$$\big\|\rho^m\big\|_{L^p(Q)} \rightarrow \big\|\rho\big\|_{L^p(Q)}\,.$$

This, together with (5.8), yields strong convergence in $L^p$; for $p = 2$, we obtain:

$$\rho^m \rightarrow \rho \quad \text{strongly in } L^2(Q)\,. \tag{5.10}$$

Using (5.5), (5.10) and the fact that $\mu$ is continuous, it is now immediate that $\mu(\rho^m)$ converges strongly towards $\mu(\rho)$. □

A natural generalization of both theorems 1 and 3 concerns (1.1)-(1.6) with, at the same time, nonvanishing g and nonconstant $\mu$. In this case, $\rho$ is the solution of a transport problem where the velocity field is not zero on $\partial\Omega$. This leads to the fact that the identity (5.9) does not hold necessarily. However, (5.9) was needed in the proof of theorem 3 to obtain strong convergence for the sequence $\{\rho^m\}$ which, in turn, has been crucial to establish the convergence of $\big\{\mu(\rho^m)\big\}$ towards $\mu(\rho)$. Hence, it is not *a priori* clear whether similar arguments work in this case.

## References

[1] Antonzev (S. A.) and Kazhikhov (A. V.) .— *Mathematical study of flows of nonhomogeneous fluids*, Lectures at the University of Nobosibirsk (U.S.S.R.), 1973 (in russian).

[2] Di Perna (R.) and Lions (P.-L.) .— *Ordinary differential equations transport theory and Sobolev spaces*, Invent. Math. **98** (1989), pp. 511-547.

[3] FERNÁNDEZ-CARA (E.) and GUILLÉN (F.) .— *The existence of nonhomogeneous, viscous and incompressible flow in unbounded domains*, Comm. P.D.E. **17** (7 & 8) (1992), pp. 1253-1265.

[4] FISZON (W.) and ZAJACZKOWSKI (W. M.) .— *Existence and uniqueness of solutions of the initial boundary value problem for the flow of a barotropic viscous fluid global in time*, Arch. Mech. **35** (1983), pp. 517-532.

[5] GIRAULT (V.) and RAVIART (P. A) .— *Finite element methods for Navier-Stokes equations*, Springer- Verlag, Berlin (1986).

[6] GUILLÉN (F.) .— *Thesis*, University of Sevilla (1992).

[7] KAZHIKHOV (A. V.) .— *Resolution of boundary value problems for nonhomogeneous viscous fluids*, Dokl. Akad. Nauk. **216** (1974), pp. 1008-1010.

[8] KUFNER (A.) *and al.* .— *Function Spaces*, Noordhoff International Publishing Leyden (1977).

[9] LADYZHENSKAYA (O. A.) .— *The mathematical theory of viscous incompressible flow*, Gordon & Breach, New York (1969).

[10] LADYZHENSKAYA (O. A.) and SOLONNIKOV (V. A.) .— *Unique solvablity of an initial and boundary value problem for viscous incompressible nonhomogeneous fluids*, J. Soviet. Math. **9** (1978), pp. 697-749.

[11] LIONS (J.-L.) .— *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthiers-Villars, Paris (1969).

[12] LIONS (J.-L.) .— *On some problems connected with Navier-Stokes equations*, In "Nonlinear Evolution Equations", M.C. Crandall, ed. Academic Press, New York (1978).

[13] LUKASZEWICZ (G.) .— *An existence theorem for compressible viscous and heat conducting fluids*, Math. Meth. Appl. Sci. **6** (1984), pp. 234-247.

[14] OKAMOTO (H.) .— *On the equation of nonstationary stratified fluid motion: uniqueness and existence of solutions*, J. Fac. Sci. Univ. Tokyo, Sect. 1A Math. **30** (1984), pp. 615-643.

[15] PADULA (M.) .— *An existence theorem for nonhomogeneous incompressible fluids*, Rend. Circ. Mat. Palermo (4), **31** ( 1982), pp. 119-124.

[16] PADULA (M.) .— *On the existence and uniqueness of nonhomogeneous motions in exterior domains*, Math. Z. **203** (1990), pp. 581-604.

[17] SIMON (J.) .— *Existencia de solución del problema de Navier-Stokes con densidad variable*, Lectures in the University of Sevilla (1989).

[18] SIMON (J.) .— *Nonhomogeneous viscous incompressible fluids: existence of velocity, density and pressure*, SIAM J. Math. Anal. **21**, n° 5 (1990), pp. 1093-1117.

[19] TÉMAM (R.) .— *Navier-Stokes evaluations. Theory and numerical analysis*, North-Holland, Amsterdam (1977).

[20] VALLI (A.) and ZAJACSZKOWSKI (W. M.) .— *Navier-Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case*, Comm. Math. Phys. **103** (1986), pp. 259-296.