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RÉSUMÉ. — Nous proposons ici deux constructions dans l'espace linéaire symplectique qui peuvent être considérées comme les analogues de birapport et dérivée de Schwarz classiques. Le birapport symplectique est l'invariant unique de quatre sous-espaces lagrangiens dans l'espace symplectique linéaire. La dérivée de Schwarz lagrangienne reconstruit le système des équations linéaires de Newton par l'évolution d'un plan lagrangéen. Elle est aussi invariante par rapport aux transformations symplectiques linéaires. On peut comprendre ces objets comme "invariants symplectiques projectifs". Nous considérons les applications de la dérivée de Schwarz lagrangienne à la théorie de Sturm, et démontrons un théorème qui donne une condition de non-oscillation pour le système des équations de Newton.

ABSTRACT. — We present here two constructions in the linear symplectic space which can be considered as analogues of the classical cross-ratio and Schwarzian derivative. The symplectic cross-ratio is the unique invariant of four Lagrangian subspaces in the linear symplectic space. The Lagrange Schwarzian derivative recovers the system of linear Newton equations from the evolution of a Lagrangian plane. It is also invariant under linear symplectic transformations. One can understand these objects as "symplectic projective invariants". We consider applications of the Lagrange Schwarzian derivative to Sturm theory and prove a theorem which gives a nonoscillation condition for the system of Newton equations.

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0. Introduction. Geometrical meaning of the classical Sturm theorems and Schwarzian derivative

Two famous Sturm theorems on zeros of solutions of the linear differential equation (Sturm-Liouville equation)

$$x''(t) = k(t)x(t) \tag{1}$$

have a simple geometrical interpretation ([A1], see also [O1]) illustrated by figures 1 and 2.

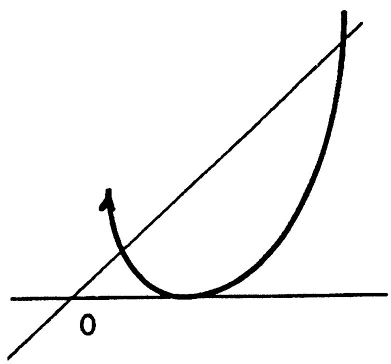


Fig. 1

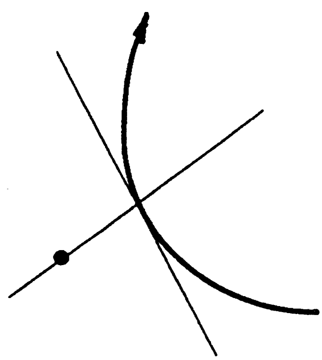


Fig. 2

THEOREM A. — (theorem on zeros) *Given two linearly independent solutions $x_1(t)$ and $x_2(t)$ of the equation (1), between any two zeros of $x_1(t)$ there is at least one zero of $x_2(t)$ (fig. 3).*

THEOREM B.— (disconjugacy theorem) *If the potential $k(t)$ of the equation (1) is non negative for any t, then any solution has at most one zero.*

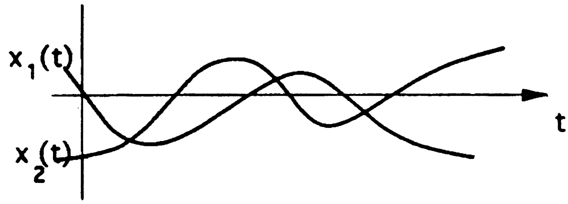


Fig. 3

To prove these theorems consider a point in the plane with coordinates $(x_1(t), x_2(t))$. The "motion" of the point is determined by the "central force" which is proportional to the vector $r(t) = (x_1(t), x_2(t))$ with the coefficient $k(t)$. The kinetic moment (the area of the parallelogram on figure 4) does not depend on t and is not equal to zero. That is, the velocity vector r' is never proportional to r. Consequently, between any two points of intersection of the trajectory with the axis x_1 there is at least one point of intersection with the axis x_2. (In other words, the trajectory on figure 1 can not occur.) This proves theorem A.

Theorem B follows now from a simple remark. In the case when the force acting on the point is repulsive the trajectory is convex. That means that the trajectory lies entirely on one side of any of its tangent line (fig. 2).

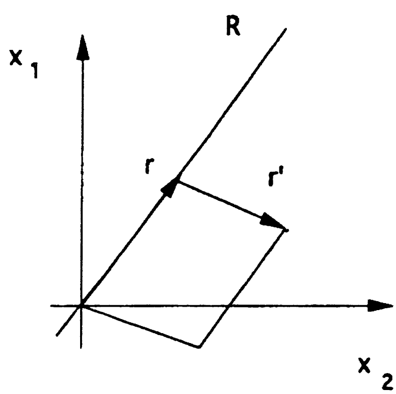


Fig. 4

Note that the kinetic moment (the area of the parallelogram in figure 4) is equal to Wronski determinant

$$W = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$$

The *Schwarzian derivative*

$$S\big(f(x)\big) = \frac{f'''}{f'} - \frac{3(f'')^2}{2(f')^2} \tag{2}$$

(where $f(x)$ is a function such that $f' \neq 0$, $f' = \mathrm{d}f(x)/\mathrm{d}x$) was already known to Lagrange [L]. The Schwarzian derivative bridges geometry and analysis (see [Kl]) and appears unexpectedly in a variety of problems, from differential geometry of curves (where it can be interpreted as curvature [Fl]) to the theory of conformal maps. The Schwarzian derivative owes its universality to two properties:

i) it is *projectively invariant*: $S\big(f(t)\big) = S\big(g(t)\big)$ if and only if

$$\frac{af(t)+b}{cf(t)+d} = g(t)$$

(where a, b, c, d are numbers);

ii) it is the unique (modulo coboundaries up to multiplying) continuous 1-*cocycle* on the group Diff $\mathbb{R}$ of diffeomorphisms of the line with values in the space of quadratic differentials [Fu]. That is, we have the formula

$$S(f \circ g) = S(f)(g)(g')^2 + S(g)$$

(where f, g are diffeomorphisms of $\mathbb{R}$). Thus the map $g \to S(g)dx^2$ to quadratic differentials is a 1-cocycle.

The Schwarzian derivative appears in our context in the "projectivized" picture. Let us consider the evolution of the strainght line R_t (fig. 4) on the plane. The problem is:

how to recover Sturm-Liouville equation from the evolution of a straight line in $\mathbb{R}^2$?

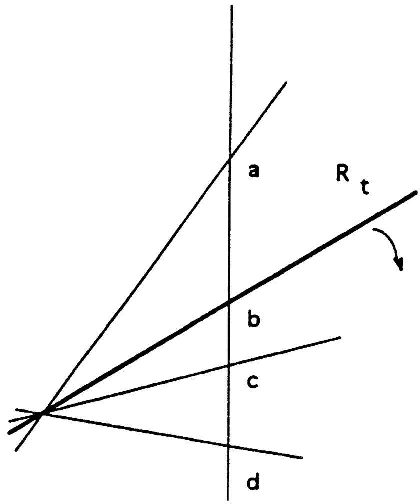


Fig. 5

The answer is given by the following construction. Fix any three distinct straight lines which contain the point 0. Denote by $\phi(t)$ the cross-ratio of these lines and R_t (fig. 5):

$$\phi(t) = \frac{(a-c)(b-d)}{(b-c)(a-d)}\,.$$

The *statement* is

$$k(t) = \frac{1}{2}\, S\big(\phi(t)\big)\,.$$

To prove this formula, remark that the cross-ratio $\phi(t)$ coincides (up to a linear-fractional transformation) with the affine coordinate of R_t (fig. 6). In other words, it equals the quotient of two solutions:

$$\phi(t) = \frac{x_1(t)}{x_2(t)}\,.$$

One can easily verify that $x_2(t) = \mathrm{const}/\sqrt{\phi'(t)}$. Indeed, observe that $x_1(t) = x_2(t)\phi(t)$ and use the fact that Wronski determinant $W\big(x_1(t), x_2(t)\big) = \mathrm{const}$. We obtain immediately

$$k(t) = \frac{x_2''(t)}{x_2(t)} = \frac{1}{2}\, S\big(\phi(t)\big)\,.$$

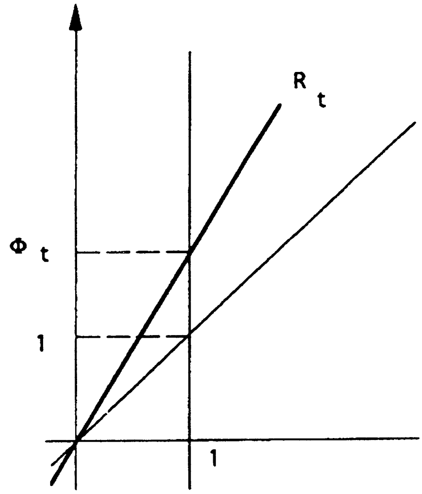


Fig. 6

The main content of this paper

1. Construction of the multidimensional analogue of the Schwarzian derivative that recovers the system of linear Newton equations

$$y''(t) = A(t)y(t)\,, \quad y(t) \in \mathbb{R}^n\,, \quad A^* = A \tag{4}$$

from the evolution of a Lagrangian subspace in a linear symplectic space.

2. Disconjugacy theorem which gives a necessary condition of oscillation (in some sense "the border of oscillation").

The first result was originally published in Russian [O2]. In the present paper we shall give more details and discuss relations of this multidimensional Schwarzian derivative with loop groups.

Remark.— There exist already many different analogues of the Schwarzian derivative which generalise its different properties (see e.g. [Ca], [R], [RS], [T]). The main property of the Lagrange Schwarzian derivative is its projective invariance.

1. Two constructions

"A straight line is just a Lagrangian subspace of the plane" [A2]. The higher-dimensional Sturm theory ([B], [M], [C], [A2]) describes the evolution of a Lagrangian plane in the standard linear symplectic space $(\mathbb{R}^{2n}, \omega)$ under the action of a linear Hamiltonian system (e.g. of the system (4)).

As in one-dimensional case, the higher-dimensional analogue of the Schwarzian derivative corresponds to the "projectivised" picture. In the case of a linear symplectic space it is natural to consider the manifold of all Lagrangian subspaces as an analogue of the projective space. This manifold is called *Lagrange Grassmann manifold* and denoted by Λ_n. The evolution of a Lagrangian space defines a curve in Λ_n.

The Lagrange Schwarzian derivative recovers the system of Newton equations (4) from a curve in Λ_n. To define it we need the notion of the cross-ratio of four Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega)$.

1.1 Cross-ratio in the linear symplectic space

DEFINITION 1.— *Given four Lagrangian subspaces α, β, γ, δ such that α, β, δ are transversal to γ, in the linear symplectic space $(\mathbf{R}^{2n}, \omega)$. Define the cross-ratio as a pair of quadratic forms (up to a linear transformation) in a linear n-dimensional space in the following way.*

First of all, α, γ, δ define a quadratic form in a linear n-dimensional space. Indeed, consider the subspaces γ and δ as "coordinate planes". Let $v \in \delta$ and consider the vector u in α such that its projection on δ along γ equals v. Let w be the projection of u on γ (fig. 7a). Define a quadratic form on δ:

$$\Phi[\alpha, \gamma, \delta](v) = \omega(v, w).$$

The four transveral Lagrangian subspaces α, β, γ, δ of $(\mathbb{R}^{2n}, \omega)$ define a pair of quadratic forms on δ (fig. 7b):

$$\Phi_1 = \Phi[\alpha, \gamma, \delta], \quad \Phi_2 = \Phi[\beta, \gamma, \delta].$$

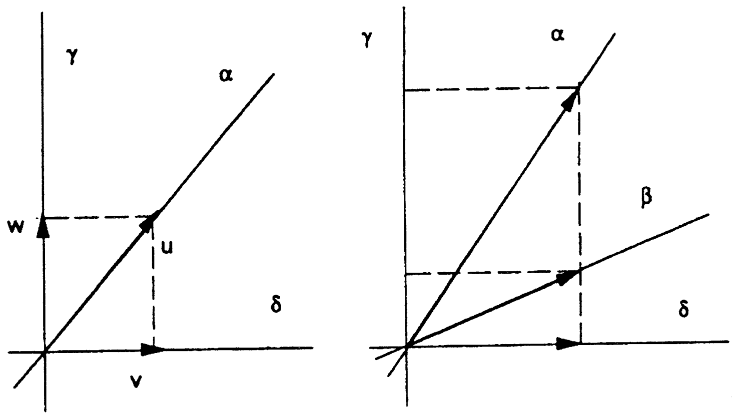


Fig. 7

As a linear space δ is isomorphic $\mathbb{R}^n$. One obtains a pair of quadratic forms (Φ_1, Φ_2) in $\mathbb{R}^n$ defined up to a linear transformation.

We shall call the pair of quadratic forms in $\mathbb{R}^n(\Phi_1, \Phi_2)[\alpha, \beta, \gamma, \delta]$ which is defined up to an equivalence

$$(\Phi_1, \Phi_2) \sim \left(C\Phi_1 C^*, C\Phi_2 C^*\right), \quad C \in \mathrm{GL}_n$$

the cross-ratio in $(\mathbb{R}^{2n}, \omega)$ or the symplectic cross-ratio.

LEMMA 1.— *Given four transverse Lagrangian subspaces α, β, γ, δ*

i) the cross-ratio is the unique invariant of α, β, γ, δ under the action of the group of linear symplectic transformations $\mathrm{Sp}(2n, \mathbb{R})$*;*

ii) the cross-ratio does not depend on the following transpositions of the arguments:

$$(\Phi_1, \Phi_2)[\alpha, \beta, \gamma, \delta] = (\Phi_1, \Phi_2)[\gamma, \delta, \alpha, \beta]$$

$$(\Phi_1, \Phi_2)[\alpha, \beta, \gamma, \delta] = (\Phi_1, \Phi_2)[\beta, \alpha, \delta, \gamma].$$

Proof

i) The first statement is evident, indeed, the unique invariant of three Lagrangian spaces is the corresponding quadratic form (which is defined as we saw up to a linear transformation) [A2]. The invariant of the quadruple $(\alpha, \beta, \gamma, \delta)$ is in fact the invariant of two triples (α, γ, δ) and (β, γ, δ).

ii) The second statement is a simple computation.

Remark 1.— The standard cross-ratio in $\mathbb{R}^2$ is equal to the fraction Φ_1/Φ_2.

Remark 2.— The index of the quadratic form $\Phi[\alpha, \beta, \gamma]$ is called Arnold-Maslov index of the triple (α, β, γ) ([A3], [A2]).

Remark 3.— In the particular case when the form Φ_2 is positive definite the complete list of invariants of four transversal Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega)$ is given by eigenvalues of Φ_1. Indeed, consider an orthonormal basis for the form Φ_2. Denote by F the symmetric matrix which is given by the quadratic form Φ_1 in this basis. This symmetric matrix is defined up to a conjugation by an orthogonal matrix:

$$F \sim O^{-1} F O \,.$$

In general, it is clear that the characteristic polynomial $\det(\Phi_1 - \lambda\Phi_2)$ is invariant.

Suppose that $F = -\mathrm{id}$ (where id is the unit matrix). Then we say that $(\alpha, \beta, \gamma, \delta)$ is a *harmonic division* (*cf.* Sec. 5).

1.2 Definition of the Lagrange Schwarzian derivative

DEFINITION 2.— *Consider a* 1*-parameter smooth family of symmetric positive definite $n \times n$ matrices $M(x)$. A family $B(x)$ is called a square root of $M(x)$ and is denoted by $B(x) = \sqrt{M(x)}$, if*

$$B^* B = M \quad \textit{and} \quad B' B^{-1} \textit{ is symmetric for every } x$$

where $B' = \mathrm{d}B(x)/\mathrm{d}x$, $x \in \mathbb{R}$.

Remark.— We will show in section 4.1 that the square root defined by these two properties is the normal form of all families of matrices $B(x)$ such that $B^* B = M$ under the action of the loop group $C^\infty\big(S^1, \mathrm{O}(n)\big)$.

LEMMA 2.— *Given a family $M(x)$ as above, the square root is defined uniquely up to the equivalence $\sqrt{M(x)} \sim O\sqrt{M(x)}$ where O is an orthogonal matrix that does not depend on x.*

We will prove this lemma in section 4.1.

Consider a 1-parameter family of symmetric matrices $F(x)$ such that $F'(x)$ is positive definite for every x.

DEFINITION 3.— (main definition) *Define the Lagrange Schwarzian derivative by*

$$\mathrm{LS}\big(F(x)\big) = \sqrt{(F')^{-1}}\left(F''' - \frac{3}{2}\,F''(F')^{-1}F''\right)\sqrt{(F')^{-1*}}\,.$$

Remark.— Note that $\mathrm{LS}(F)$ is defined only up to a conjugation by orthogonal matrices. On the other hand it is easy to check that for a fixed matrix C, $\mathrm{LS}\big(C^*F(x)C\big) = \mathrm{LS}\big(F(x)\big)$. Thus we can consider LS as a mapping from the space of 1-parameter families of *symmetric forms* on $\mathbb{R}^n$.

Here are some elementary properties of LS.

PROPOSITION 1

i) $\mathrm{LS}\big(F(x)\big) = \mathrm{LS}\big(G(x)\big)$ *if and only if*

$$G(x) = \big(AF(x) + B\big)\big(CF(x) + D\big)^{-1}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})\,.$$

Thus LS *is invariant under the action of the group* $\mathrm{Sp}(2n, \mathbb{R})$ *locally on the space of quadratic forms on* $\mathbb{R}^n$.

ii) if $g : \mathbb{R} \to \mathbb{R}$ *is a diffeomorphism, then*

$$\mathrm{LS}\,\big(F\big(g(x)\big)\big) = \mathrm{LS}(F)(g)(g')^2 + S(g)\,\mathrm{id}\,.$$

Here $S(g)$ *is the classical Schwarzian derivative and* id *is the unit matrix.*

iii) Given any continuous family of matrices $A(x)$*, the equation* $\mathrm{LS}(F) = A$ *is (locally in* x*) solvable.*

Proof.— see section 4.

2. Main theorem

We formulate here the main results of this paper concerning the relation of the Lagrange Schwarzian derivative with Newton systems. All proofs will be given in section 4.

2.1 Newtonian systems and positive curves on the Lagrange Grassmannian

DEFINITION 4.— ([A2]) *The train of a given point α of the Lagrange Grassmann manifold Λ_n is the set of all Lagrangian planes which are not transversal to α.*

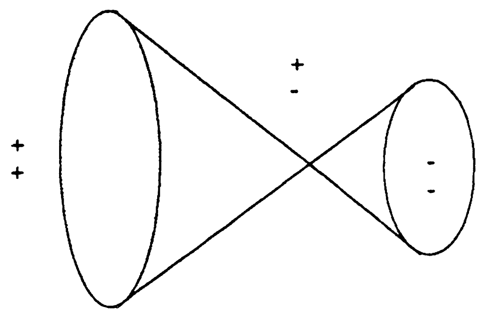


Fig. 8 a)

In the neighborhood of any point α the Lagrange Grassmannian is diffeomorphic to the manifold of quadratic forms in $\mathbb{R}^n$ (fix another Lagrangian subspace β transverse to α, then for any $\gamma \in \Lambda_n$ in a neighborhood of α, this diffeomorphism is given by $\Phi[\gamma, \beta, \alpha]$). The train of α is locally diffeomorphic to the variety of degenerate quadratic forms. Thus the complement of the train of any point of Λ_n is partitioned into $(n+1)$-subsets corresponding to indices of nondegenerate forms (fig. 8 a)).

DEFINITION 5.— *Consider the system of linear Newton equations (4). Define a symplectic structure ("Wronskian") on the $2n$-dimensional space of its solutions by*

$$W(y, z) = \sum_{i=1}^{n} (y_i z_i' - y_i' z_i) \tag{5}$$

where $y = \big(y_1(t), \ldots, y_n(t)\big)$, $z = \big(z_1(t), \ldots, z_n(t)\big) \in \mathbb{R}^n$ are solutions of the system (4).

LEMMA 3.— *$W(y, z)$ does not depend on t and it is a nondegenerate skew-symmetric form in $\mathbb{R}^{2n}$.*

Thus, the space of solutions of the system (4) is identified with the linear symplectic space $(\mathbb{R}^{2n}, \omega)$.

Let $\lambda(s)$ be the n-dimensional space consisting of solutions of the system (4) that vanish if $t = s$.

LEMMA 4.— *$\lambda(s)$ is a Lagrangian subspace of the space of solutions of the system (1).*

Thus, each system (4) defines an evolution of a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega)$. In other words, it defines a curve in the Lagrange Grassmann manifold Λ_n.

DEFINITION 6.— *Fix two transverse Lagrangian subspaces α, β in $(\mathbb{R}^{2n}, \omega)$. The evolution of the Lagrangian space $\lambda(s)$ defines a family of quadratic forms on $\mathbb{R}^n$*

$$\Phi(s) = \Phi\left[\lambda(s), \alpha, \beta\right].$$

(Note that $\Phi(s)$ is defined iff $\lambda(s)$ is transversal to α.) The curve $\lambda(s)$ in λ_n is called positive if the quadratic form $\mathrm{d}\Phi(s)/\mathrm{d}s$ is a positive definite for every s such that $\lambda(s)$ is transversal to α (fig. 8b).

It is easy to check that the definition does not depend on a choice of Lagrangian subspaces α, β.

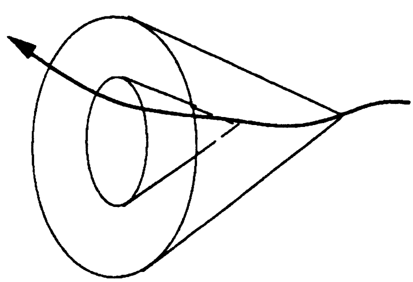


Fig. 8 b)

PROPOSITION 2.— *The curve $\lambda(s)$ in λ_n, corresponding to a system (4), is positive [A2].*

That is, a Newton system (4) canonically defines a positive curve in Λ_n.

2.2 Newton systems and the Lagrange Schwarzian derivative

One can construct the Lagrange Schwarzian derivative similarly to the classical one. Given a positive curve $\lambda(t)$ in Λ_n, fix an arbitrary pair of transversal Lagrangian subspaces α, β in $(\mathbb{R}^{2n}, \omega)$. Fix an arbitrary isomorphism of β with $\mathbb{R}^n$. Then (for values of t such that $\lambda(t)$ transverse to α) the family of quadratic forms

$$\Phi(t) = \Phi\big[\lambda(t), \alpha, \beta\big]$$

defines a family of symmetric matrices.

THEOREM 1.— *Every positive curve $\lambda(t)$ in Λ_n corresponds to a system (4), where $A(t)$ is given by*

$$A = -\frac{1}{2}\,\mathrm{LS}\big(\Phi(t)\big)\,. \tag{6}$$

A does not depend on the choise of α, β.

Note that locally in t there exists a factorisation of the differential operator which defines the system of Newton equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 - A(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + V(t)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - V(t)\right)$$

where $V(t)$ is a family of symmetric matrices. Consequently,

$$A = V' + V^2\,. \tag{7}$$

We denote this expression by $\mu(V)$.

PROPOSITION 3.— *$V(t)$ is given by the formula*

$$V(t) = \left(-\frac{1}{2}\right)\sqrt{(F')^{-1}}\,F''(t)\sqrt{(F')^{-1*}}\,. \tag{8}$$

Remark.— The last formula is an analogue of the logarithmic derivative $\mathrm{d}\big(\log(f')\big)/\mathrm{d}t$ which defines a 1-cocycle on the group $\mathrm{Diff}(S^1)$ with values in the space of 1-forms on S^1. Formula (7) is an analogue of the so-called Miura transformation.

Let us call the right hand side in formula (8) the *Lagrange logarithmic derivative* and denote it $\mathrm{LD}\big(F(t)\big)$.

3. Symplectic Sturm theorems. Disconjugacy theorem

3.1 Sturm theorms in Symplectic space

Let us give here a very brief "esquisse" of Sturm theory in the linear symplectic space (see [A2] for the details).

DEFINITION 7.— *Consider an evolution of a Lagrangian space in the linear symplectic space. Fix an arbitrary Lagrangian subspace. A Lagrangian space is said be "vertical" (in the terminology [A2]) if it is not transversal to the fixed Lagrangian subspace.*

In higher-dimensional Sturm theory instead of zeros of solutions one considers moments of verticality of a Lagrangian plane.

Let us formulate symplectic analogues of the Sturm theorems A and B which also have already become classical ([M], [C], [A2]).

A. Theorem on zeros.— Given the evolutions of two Lagrangian planes $\lambda_1(t)$ and $\lambda_2(t)$ under the action of a Newton system, the difference between the numbers of moments of verticality of $\lambda_1(t)$ and $\lambda_2(t)$ on any segment of time does exceed the number n of degrees of freedom.

DEFINITION 8.— *The system (4) is called disconjugate on the interval I, if the corresponding Lagrange subspace $\lambda(t)$ of the space of solutions fails to be transverse to each fixed Lagrange subspace in at most n values of t. (For example, for Sturm-Liouville equation it means that each solution has at most one zero point on I.)*

B. Disconjugacy theorem.— If the potential energy $A(t)$ is nonnegative definite:

$$A(t) \geq 0\,,$$

then the Newton system is disconjugate on the whole time axis.

Proof.— ([A2]) It is clear that in theorem A instead of evolutions of two Lagrangian planes $\lambda_1(t)$ and $\lambda_2(t)$ one may consider the evolution of one Lagrangian subspace and its moments of nontransversality with two fixed Lagrangian subspaces. The equivalent way to formulate the result is theorem A$'$.

THEOREM A'. — *Given two Lagrangian subspaces α and β in the space of solutions of the system (4), the difference between the numbers of moments of nontransversality of α and β with the Lagrangian plane $\lambda(t)$ on any segment of time does not exceed the number n of degrees of freedom.*

Consider the quadratic form $\Phi(t) = \Phi\big[\lambda(t), \alpha, \beta\big]$. At each moment of nontransversality of $\lambda(t)$ with β the form $\Phi(t)$ is degenerate. The index of the form $\Phi(t)$ increases by the dimension of the kernel of the form $\Phi(t)$ at this moment. Indeed, according to proposition 2, the derivative $\Phi(t)'$ is positive definite. At each moment of nontransversality of $\lambda(t)$ with α, the inertia index of the form $\Phi(t)$ decreases. Consequently, the difference between the numbers of nontransversality of α and β with $\lambda(t)$ does not exceed n. Theorem A is proved. □

Consider the quadratic form $\Phi\big[\alpha, \beta, \lambda(t)\big]$. At each moment of nontransversality of α and β with $\lambda(t)$, the inertia index of this form increases. That is this number is at most n. Theorem B is proved. □

3.2 Nonoscillation property

Let us apply property ii) of the Lagrangian Schwarzian derivative to disconjugacy theorem. Consider the *nonoscillation property* on the whole line:

$$A(t) \geq 0 \quad \text{for each value of } t\,,$$

and apply a diffeomorphism $t \to g(t)$. The potential transforms to

$$g^*A(t) = A\big(g(t)\big)\big(g'(t)\big)^2 + S(g)\ \mathrm{id}\ . \tag{9}$$

The corresponding Newton system is still disconjugate since the action of g on the curve $\lambda(t)$ is a transformation of the parameter:

$$g^*\lambda(t) = \lambda\big(g(t)\big)\,.$$

Whe obtain the theorem 2.

THEOREM 2. — *If the quadratic form*

$$A(t) + S\big(g(t)\big)\ \mathrm{id} \geq 0 \tag{10}$$

(nonnegative definite) then the system (4) is disconjugate on the whole time axis.

Example. — In one dimensional case condition (10) gives a *criteria of nonoscillation* for Sturm-Liouville equation.

COROLLARY

i) If the quadratic form

$$A(t) \geq \frac{1}{(1-t^2)^2}\,\mathrm{id}$$

on the interval $(-1,1)$ *then the Newtonian system is disconjugate on this interval.*

ii) If the quadratic form

$$A(t) \geq \frac{I}{t^2}\,\mathrm{id}$$

on the ray $(0,\infty)$ *then the Newtonian system is disconjugate on it.*

Proof. — Consist in a simple application of the diffeomorphisms $g(t) = (1/\pi)\arctan(t)$ and $g(t) = \mathrm{e}^t$ (which transform the time axis into the interval and into the ray correspondingly).

Remark. — These theorems are multidimensional analogues of Nehari [N] and Kneser theorems correspondingly. The second one is known (see [C]), the first one we failed to find in literature.

4. Relation with loop groups. Proofs

Let G be the group of all smooth functions on S^1 with values in the group of matrices $G = C^\infty(S^1\,,\mathrm{GL}(n))$. Denote by g the corresponding Lie algebra: $g = C^\infty(S^1\,,\mathrm{gl}(n))$.

Notation. — Denote by f the space of functions on S^1 with values in the set of symmetric matrices, and by $\mathcal{F} \subset f$ the subspace of functions with *nondegenerate* derivative.

Consider the following mapping (which we defined in section 1):

$$r : \mathcal{F} \to G$$

defined by $r(F(t)) = \sqrt{(F')^{-1}}$,

$$a : \mathrm{Im}\, r \subset G \to f$$

given by $a(X) = X''X^{-1}$ (note that if $X = \sqrt{M}$ then $X''X^{-1}$ is a symmetric matrix).

Define a mapping $c : G \to g$ by the formula

$$c(G) = G'G^{-1} \tag{11}$$

(as it is well known, the last formula defines a 1-cocycle on G which represents the unique nontrivial class of cohomology $H^1(G, g)$). Recall, that by definition of the square root, the mapping $c \circ r$ has values in the space of symmetric matrices. Thus we have an application

$$c : \operatorname{Im} r \to f\,.$$

All these mappings form the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{F} \arrow[dl, "r"'] \arrow[dd, "LS"] \arrow[dr, "\mathrm{LD}"] & \\
G \supset \sqrt{\mathcal{F}} \arrow[rr, "c"] \arrow[dr, "a"'] & & f \arrow[dl, "\mu"] \\
 & f &
\end{tikzcd}$$

where LS is the Lagrange Schwarzian derivative, LD is the logarithmic derivative, μ is given by formula (7). Consequently,

$$\mathrm{LD} = c \circ r\,, \quad \mathrm{LS} = a \circ r\,.$$

4.1 Proof of lemma 2: the square root and the loop group $C^\infty(S^1, \mathrm{GL}(n))$

Proof.— Let $M(x)$ be a family of nondegenerate symmetric positive definite matrices. Consider a family of matrices $B(x)$ such that $B^*B = M$. Then $B(x)$ is defined up to the equivalence $B \sim Q(x)B$ where $Q(x) \in \mathrm{O}(n)$ for each x. We shall prove that one can find a family $Q(x)$ such that

$$c(QB) = (QB)'(QB)^{-1}$$

is a symmetric matrix.

Indeed, $c(QB) = Qc(B)Q^{-1} + c(Q)$, since c is a 1-cocycle. $c(Q)$ is a family of skew-symmetric matrices, since $Q(x) \in \mathrm{O}(n)$. That is

$$c(QB) - c(QB)^* = Q\big(c(B) - c(B)^*\big)Q^{-1} + 2c(Q)\,.$$

We are looking for $Q(x)$ such that $c(QB) - c(QB)^* = 0$. In other words $c(Q) = (-1/2)Q\big(c(B)^* - c(B)\big)Q^{-1}$. Thus, we get finally the following linear differential equation on $Q(x)$:

$$Q' = Q\big(c(B)^* - c(B)\big)\,.$$

This equation has the unique solution $Q(x) \in \mathrm{O}(n)$ (up to equivalence $Q(x) \sim OQ(x)$ where O is an orthogonal matrix that does not depend on x).

Remark.— Consider the group $C^\infty\big(S^1\,,\,\mathrm{O}(n)\big)$ of all smooth functions on S^1 with values in the orthogonal group. Define an affine action of this group on the Lie algebra g: given a family $Q = Q(x)$ of orthogonal matrices,

$$T_Q m(x) := Qm(x)Q^{-1} + c(Q)\,.$$

(In fact, this action coincides with the coadjoint action of the Kac-Moody algebra, see e.g. [Ki].) Consider also the (left) action of this group on G:

$$R_Q G(x) = QG(x)\,.$$

This action preserves the relation $G(x)^*G(x) = M(x)$. The mapping (11) is equivariant. The normal form of the action T_Q is a family of symmetric matrices (note that the space s is orthogonal to the subalgebra $C^\infty\big(S^1\,,\,\mathrm{so}(n)\big)$ in g). It means that the *square root* $G = \sqrt{M(x)}$ *is the normal form of families of matrices which verify the relation* $G(x)^*G(x) = M(x)$ *under the action of the loop group* $C^\infty\big(S^1\,,\,\mathrm{O}(n)\big)$.

4.2 Proof of theorem 1: solutions of the Newton system

Proof of lemma 3.— Let $y_1(t), \ldots, y_n(t) \in \mathbb{R}^n$ be linearly independent solutions of the system (4). Denote by $Y(t)$ the $n \times n$ matrix such that its column Y_i coincides with the vector y_i. We shall $Y(t)$ the matrix of solutions of the system (4).

Given two matrices of solutions $Y(t)$ and $Z(t)$. Define their "Wronskian" by

$$W(Y,Z) = Y'^*Z - Y^*Z'\,. \tag{5'}$$

It is clear that $W_{ij} = W(y_i, z_j)$. Then

$$\frac{\mathrm{d}W(Y,Z)}{\mathrm{d}t} = Y''^*Z - Y^*Z'' = Y^*AZ - Y^*AZ = 0\,.$$

Thus, W is a bilinear antisymmetric form on the space of solutions of the system (4). One can easily construct Darboux basis on the space of solutions: it is given by the standard initial conditions.

Proof of lemma 4.— Follows immediately from the definition.

Proof of theorem 1.— Consider the curve $\lambda(s)$ in Λ_n by the evolution of a Lagrangian subspace in the space of solutions of the system (4). Fix two arbitrary transversal Lagrangian subspaces ψ and ζ. Let $\big(y_1(t), \dots y_n(t)\,;\, z_1(t), \dots, z_n(t)\big)$ be the Darboux basis in the space of solutions associated with the polarization (ψ, ζ) and $Y(t)$, $Z(t)$ the corresponding matrices of solutions. Then

$$W\big(Y(t)\,,\, Z(t)\big) = \mathrm{id}\,, \quad W\big(Y(t)\,,\, Y(t)\big) = 0\,, \quad W\big(Z(t)\,,\, Z(t)\big) = 0\,. \tag{12}$$

For each value of s, $\lambda(s)$ is a subspace of solutions of the system (4) which are equal to 0 at the moment $t = s$. Let $L_s(t)$ be its matrix of solutions. Then (up to a conjugation)

$$L_s(t) = Y(t)Y(s)^{-1} - Z(t)Z(s)^{-1}\,. \tag{13}$$

Consider the quadratic form $\Phi(s) = \Phi\big[\,\lambda(s), \psi, \zeta\,\big]$ (defined on the subspace ζ, see section 1.1) and denote by $F(s)$ its matrix in the basis $\big(z_1(t), \dots, z_n(t)\big)$.

LEMMA 5

i) The curve $\lambda(s)$ in Λ_n is given by the evolution of a Lagrangian subspace under the action of a Newton system.

ii) Matrices of solutions of this Newton system are given by

$$Y(t) = \sqrt{\big(F'(t)\big)^{-1}}\,, \quad Z(t) = F(t)\sqrt{\big(F'(t)\big)^{-1}}\,. \tag{14}$$

Proof of the lemma.— From formula (13) we get immediately:

$$F(s) = Y(s)^{-1}Z(s)\,.$$

Thas is,

$$Z(s) = Y(s)F(s)\,.$$

Substitute this formula to (12): $W\big(Y(s)\,,\,Y(s)F(s)\big)=\mathrm{id}$. Thus, from the formula ($5'$)

$$Y'^{*}YF-Y^{*}(YF)'=-Y^{*}YF'=\mathrm{id}\;.$$

Whe have the first property of the square root: $Y^{*}Y=(F')^{-1}$. The second property: $Y'Y^{-1}$ *is a symmetric matrix* is equivalent to the fact that ψ is a *Lagrangian* subspace. Indeed, it means that $W\big(Y(s)\,,\,Y(s)\big)=0$. We get $Y'^{*}Y=Y^{*}Y'$. The lemma is proved.

To finish the proof of the theorem, remark that the potential of the Newton system can be calculated from a matrix of its solutions by the formula $A(t)=Y(t)''Y(t)^{-1}$. The matrix of solutions $Y(t)$ is given by formula (14). Thus, we must calculate the derivative of the square root.

The first derivative of the square root.— Let us derivative the formula $Y^{*}Y=-(F')^{-1}$. Then

$$Y^{*\prime}Y+Y^{*}Y'=\big[(F')^{-1}\big]'=-(F')^{-1}F''(F')^{-1}\,.$$

Also (by the second property of the square root, $Y^{*\prime}Y$ is a family of symmetric matrices) $Y^{*\prime}Y+Y^{*}Y'=2Y^{*}Y'$. Thus, finally we get

$$Y'Y^{-1}=\Big(-\frac{1}{2}\Big)\sqrt{(F')^{-1}}\,F''(t)\sqrt{(F')^{-1^{*}}}\tag{8'}$$

(we shall use this formula in the proof of proposition 3).

The second derivative of the square root.— In the same way,

$$\begin{aligned}2(Y^{*\prime}Y'+Y^{*}Y'')&=-\big[(F')^{-1}F''(F')^{-1}\big]'\\&=2(F')^{-1}F''(F')^{-1}F''(F')^{-1}-(F')^{-1}F'''(F')^{-1}\,.\end{aligned}$$

From the previous formula we have $2Y'=-\sqrt{(F')^{-1}}\,F''(t)(F')^{-1}$. So

$$2Y^{*\prime}Y'=\frac{1}{2}\,(F')^{-1}F''(F')^{-1}F''(F')^{-1}$$

and we obtain the formula (6).

This calculation does not depend on the choice of the Darboux basis since for a fixed matrix C,

$$\mathrm{LS}\big(C^{*}F(x)C\big)=\mathrm{LS}\big(F(x)\big)$$

Theorem 1 is proved. □

Proof of proposition 1.— The calculation in the proof of theorem 1 does not depend on the choice of Lagrangian subspaces ψ and ζ because as we saw the curve $\lambda(s)$ defines the system (4) uniquely. Thus the Lagrange Schwarzian derivative is a projective invariant, and i) is proved. The property ii) is evident. The third one follows from the fact that any Newton system defines the evolution of a Lagrangian subspace.

Proof of proposition 2.— Becomes evident now because of the formula $Y^*Y = \left(F'\right)^{-1}$.

Proof of proposition 3.— Consider the linear differential equation $Y' = V(t)Y$. If $Y = Y(t)$ is solution of the system (4) then $V(t)$ satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 - A(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + V(t)\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - V(t)\right) .$$

The formula (8) follows now from (8′).

Remark.— Another version of the Schwarzian derivative was constructed in [T]. This symplectic Schwarzian defines a 1-cocycle on the group of symplectic diffeomorphisms on a manifold with values in some space of vector fields.

We ask here an analogous question: *Is Lagrange Schwarzian derivative a 1-cocycle* in any sense?

5. Symplectic projective geometry: does it exist?

There is at least one reason to suppose that such a theory exists: the construction of Tits building on the linear symplectic group (see [Br]).

We present here several small remarks and definitions which probably could be related to this theory (certainly, in the case of positive answer to the question). It is a result of discussions of B. Khesin, S. Tabachnikov and the author.

Consider the following configuration in $(\mathbb{R}^{2n}, \omega)$. Let α, β be two Lagrangian subspaces. Fix k points $a_1, \ldots, a_k$ in α and k points $b_1, \ldots, b_k$ in β. Consider $n+1$-dimensional affine spaces

$$L_{IJ} = \left(a_{i_1}, a_{i_2}, \ldots, a_{i_m}, b_{j_1}, b_{j_2}, \ldots, b_{j_{n-m+2}}\right) .$$

where $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_{n-m}\}$.

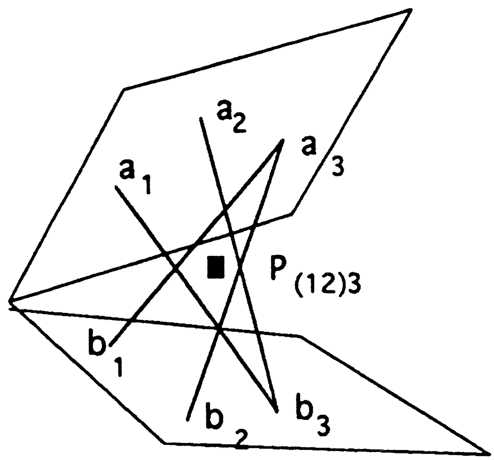


Fig. 9

Consider the points of intersections $p_{IJ} = L_{IJ} \cap L_{JI}$ (fig. 9). The question is : *whether these points belong to the same affine Lagrangian space ?* (L_{IJ} are not necessarily Lagrangian.)

PROPOSITON 4

i) If the pairs of points (a_i, b_i) are given by intersections of α and β with k affine Lagrange spaces parallel to the same Lagrangian subspace γ (fig. 10) then the points p_{IJ} belong to a Lagrangian subspace δ such that four Lagrangian subspaces $(\alpha, \beta, \gamma, \delta)$ are harmonic.

ii) Let $a_1, \ldots, a_{n+2} \in \alpha$ and $b_1, \ldots, b_{n+2} \in \beta$ and there exists $k \in \{1, \ldots, n+2\}$ such that for any i affine Lagrangian subspaces

$$(a_1, \ldots, \widehat{a_i}, \widehat{a_k}, \ldots, a_{n+2}, b_k) \quad \textit{and} \quad (b_1, \ldots, \widehat{b_i}, \widehat{b_k}, \ldots, b_{n+2}, a_k)$$

are parallel Lagrangian subspaces then the affine spaces

$$(b_1, \ldots, \widehat{b_j}, \widehat{b_k}, \ldots, b_{n+2}, a_j) \quad \textit{and} \quad (a_1, \ldots, \widehat{a_j}, \widehat{a_k}, \ldots, a_{n+2}, b_j)$$

are parallel and Lagrangian (fig. 11).

iii) If $k = n$ and the vectors $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ are proportional to a Darboux basis in $(\mathbb{R}^{2n}, \omega)$ then p_{IJ} belong to an affine Lagrangian space.

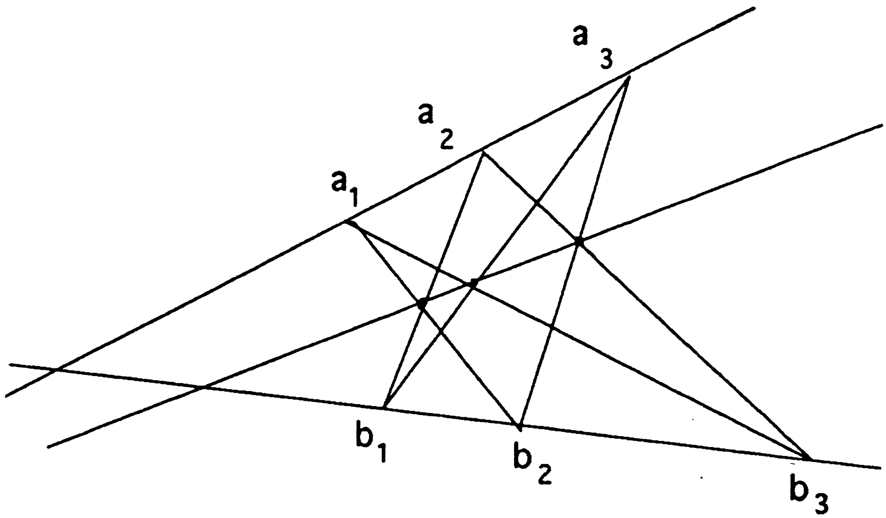
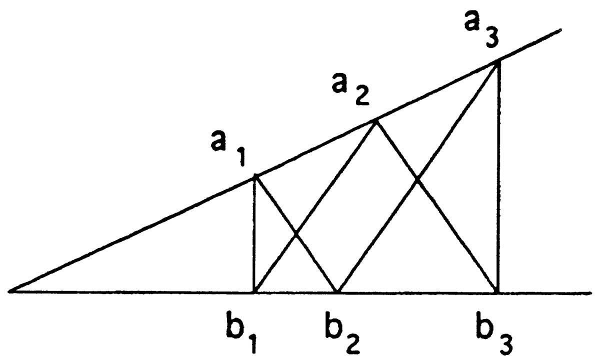


Fig. 10



$$\left.\begin{array}{l} a_1b_2 \parallel a_2b_3 \\ a_2b_1 \parallel a_3b_2 \end{array}\right\} \quad \Rightarrow \quad a_1b_1 \parallel a_3b_3 .$$

Fig. 11

Remark.— i) is due to B. A. Khesin, ii) is due to S. L. Tabachnikov.

However, we do not know of any general configuration theorem in $(\mathbb{R}^{2n}, \omega)$.

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