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RÉSUMÉ. — Nous définissons la croissance de l'élargissement transverse des feuilletages qui peuvent être considérés comme une croissance du cardinal maximal des ensembles séparés d'un pseudo-groupe d'holonomie. Nous prouvons que la croissance de l'élargissement transverse est un invariant topologique des feuilletages, et nous calculons la croissance de l'élargissement transverse de quelques feuilletages typiques de codimension 1.

ABSTRACT. — We define the expansion growth of foliations which is, roughly speaking, growth of the maximum cardinality of separating sets with respect to a holonomy pseudogroup. We prove that the expansion growth is topological invariant for foliations and we compute the expansion growth of several typical foliations of codimension 1.

0. Introduction

The entropy of foliations is defined by Ghys, Langevin and Walczak [GLW]. Their definition was done by generalizing Bowen's definition of the entropy of dynamical systems [B]. Let $\mathcal{F}$ be a codimension q foliation of class C^0 on a compact manifold M. When we fix a finite foliation cover $\mathcal{U}$ of $(M, \mathcal{F})$, we obtain the holonomy pseudogroup $\mathcal{H}$ of local homeomorphisms of $\mathbb{R}^q$ induced by $\mathcal{U}$. Then we can define an integer $s_n(\varepsilon)$ ($n \in \mathbb{N}$, $\varepsilon > 0$) to be the maximum cardinality of (n, ε)-separating sets with respect to the holonomy pseudogroup $\mathcal{H}$. $s_n(\varepsilon)$ is monotone increasing on n and monotone decreasing on ε. The entropy $h(\mathcal{F}, \mathcal{U})$ of the foliation $\mathcal{F}$ is defined by the following formula:

$$h(\mathcal{F}, \mathcal{U}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon)\,.$$

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When we fix a sufficiently small positive real number ε, we notice that the monotone increasing map $s_n(\varepsilon)$ with respect to n represents the expansion of the foliation. For example, if $\mathcal{F}$ is a Reeb component then the growth of $s_n(\varepsilon)$ with respect to n is the linear growth. If $\mathcal{F}$ is a linear foliation on $\mathbb{T}^m$ then the growth of $s_n(\varepsilon)$ is the constant growth.

In this paper, we consider the growth type of $s_n(\varepsilon)$ defined in the growth set which is an extension of the usual growth set (*cf.* [HH2]) and we prove that the growth of $s_n(\varepsilon)$ depends only on $(M, \mathcal{F})$. Therefore it becomes a topological invariant for foliations. Moreover we define an integer $r_n(\varepsilon)$ to be the minimum cardinality of (n, ε)-spanning sets with respect to the holonomy pseudogroup $\mathcal{H}$ and we show that the growth of $s_n(\varepsilon)$ is equal to the growth of $r_n(\varepsilon)$. We call it the expansion growth of $(M, \mathcal{F})$. The expansion growth of foliations produces a number of numerical topological invariants for foliations. We also show that many of them are non-trivial invariants.

In section 1, we define the growth set which is an extansion of the usual growth set. The expansion growth of foliations is defined as an element of this growth set. In section 2, we define the expansion growth of a foliation and we describe several important properties of the expansion growth. In section 3, we compute the expansion growth of several typical codimension 1 foliations. We also construct foliations which have various expansion growths.

1. Growth

In this section, we define the growth of an increasing sequence of incresing functions.

Let $\mathcal{I}$ be the set of nonnegative increasing functions on $\mathbb{N}$:

$$\mathcal{I} = \left\{ g : \mathbb{N} \to [0\,,\,\infty)\,;\; g(n) \le g(n+1) \text{ for all } n \in \mathbb{N} \right\}.$$

Let $\widetilde{\mathcal{I}}$ be the set of increasing sequences of $\mathcal{I}$.

$$\widetilde{\mathcal{I}} = \left\{ (g_j)_{j \in \mathbb{N}} \subset \mathcal{I}\,;\; g_j(n) \le g_{j+1}(n) \text{ for all } j \in \mathbb{N} \text{ and } n \in \mathbb{N} \right\}.$$

We regard $\mathcal{I}$ as a subset of $\widetilde{\mathcal{I}}$ by the map

$$\mathcal{I} \ni g \mapsto (g,\, g,\, g,\, \ldots) \in \widetilde{\mathcal{I}}\,.$$

We define the growth type of an element of $\widetilde{\mathcal{I}}$. We define a preorder $\preceq$ in $\widetilde{\mathcal{I}}$ as follows. For $(g_j)_{j\in\mathbb{N}}, (h_k)_{k\in\mathbb{N}} \in \widetilde{\mathcal{I}}$,

$$(g_j)_{j\in\mathbb{N}} \preceq (h_k)_{k\in\mathbb{N}} \iff \exists\, B \in \mathbb{N}\,,\ \forall\, j \in \mathbb{N}\,,\ \exists\, k \in \mathbb{N}\,,\ \exists\, A > 0$$
$$\text{such that } g_j(n) \le A h_k(Bn) \text{ for any } n \in \mathbb{N}\,.$$

In the definition, we note that $B \in \mathbb{N}$ is independent of $j \in \mathbb{N}$ and $A > 0$ may depends on $j \in \mathbb{N}$. The preorder $\preceq$ induces an equivalence relation $\simeq$ in $\widetilde{\mathcal{I}}$.

$$(g_j)_{j\in\mathbb{N}} \simeq (h_k)_{k\in\mathbb{N}} \iff (g_j)_{j\in\mathbb{N}} \preceq (h_k)_{k\in\mathbb{N}} \text{ and } (g_j)_{j\in\mathbb{N}} \succeq (h_k)_{k\in\mathbb{N}}\,.$$

We define $\widetilde{\mathcal{E}}$ to be the set of equivalence classes in $\widetilde{\mathcal{I}}$:

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{I}}/\simeq\ .$$

$\widetilde{\mathcal{E}}$ is the set of all growth types of increasing sequences of increasing functions and has the partial order $\le$ induced by the preorder $\preceq$. The equivalence class of $(g_j)_{j\in\mathbb{N}} \in \widetilde{\mathcal{I}}$ is written by $[g_j]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}$ and is called the growth type of $(g_j)_{j\in\mathbb{N}}$. The equivalence class of $g \in \mathcal{I} \subset \widetilde{\mathcal{I}}$ is simply written by $[g]$. Let $\mathcal{E}$ be the set of such growth types:

$$\mathcal{E} = \big\{[g]; g \in \mathcal{I}\big\} \subset \widetilde{\mathcal{E}}\,.$$

$\mathcal{E}$ is equal to the partial ordered set of all growths of monotone increasing functions in the usual sense (*cf.* [HH2]) and $\widetilde{\mathcal{E}}$ can be considered as an extension of it.

For example, the following relation is easily seen

$$[0] \lneq [1] \lneq [n] \lneq [n^2] \lneq \cdots \lneq [1,\, n,\, n^2,\, \ldots]$$
$$\lneq [2^n] = [3^n] \lneq [1,\, 2^n,\, 3^n,\, \ldots]\,.$$

Here $[0]$ (resp. $[1]$) is the growth of the constant function whose value is 0 (resp. 1). $[e^n] \in \mathcal{E}$ is said to have exactly exponential growth. For $k \in \mathbb{N} \cup \{0\}$, $[n^k] \in \mathcal{E}$ is said to have exactly polynomial growth of degree k. $[g_j]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}$ is said to be quasi-exponential if

$$\lim_{j\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log g_j(n) > 0\,.$$

Next we define the finite sum and the finite product of elements of $\widetilde{\mathcal{E}}$. For $[g_j]_{j\in \mathbb{N}}, [h_k]_{k\in \mathbb{N}} \in \widetilde{\mathcal{E}}$,

$$[g_j]_{j\in \mathbb{N}} + [h_k]_{k\in \mathbb{N}} = [g_j + h_j]_{j\in \mathbb{N}},$$
$$[g_j]_{j\in \mathbb{N}} \cdot [h_k]_{k\in \mathbb{N}} = [g_j \cdot h_j]_{j\in \mathbb{N}}.$$

For example, the following relations are easily seen.

$$\xi \cdot [0] = [0], \quad \xi + [0] = \xi \cdot [1] = \xi \text{ for } \xi \in \widetilde{\mathcal{E}}.$$

$$\xi + [1] = \xi \text{ for } \xi \neq [0] \in \widetilde{\mathcal{E}}.$$

$$\xi_1 \leq \zeta_1, \ \xi_2 \leq \zeta_2 \quad \Longrightarrow \quad \xi_1 + \zeta_1 \leq \xi_2 + \zeta_2, \ \xi_1 \cdot \zeta_1 \leq \xi_2 \cdot \zeta_2.$$

2. Expansion growth of foliations

In this section, we define the expansion growth of a foliation on a compact manifold and describe several properties of it. Let $\mathcal{F}$ be a C^0 codimension q foliation on a compact $(p+q)$-dimensional manifold M. We assume that $\partial M = \emptyset$, however the same discussion is applicable in the case where $\partial M \neq \emptyset$ provided $\mathcal{F}$ is tangent or transverse to ∂M.

$\mathcal{A} = (\mathcal{U}^s, \mathcal{U}, \mathcal{U}^b)$ is called a triple of foliation covers of $(M, \mathcal{F})$ if it satisfies the following conditions.

(1) $\mathcal{U}^b, \mathcal{U}$ and $\mathcal{U}^s$ are open coverings of M.

(2) $\mathcal{U}^b = \{(U_i^b, \varphi_i^b)\}_{i=1}^A$, $\mathcal{U} = \{(U_i, \varphi_i)\}_{i=1}^A$ and $\mathcal{U}^s = \{(U_i^s, \varphi_i^s)\}_{i=1}^A$.

(3) $U_i^b \supset U_i \supset U_i^s$.

(4) $\varphi_i = \varphi_i^b|_{U_i}$ and $\varphi_i^s = \varphi_i^b|_{U_i^s}$.

(5) $\varphi_i^b(U_i^b) = B_3^p(o) \times B_3^q(o_i)$, $\varphi_i(U_i) = B_2^p(o) \times B_2^q(o_i)$ and $\varphi_i^s(U_i^s) = B_1^p(o) \times B_1^q(o_i)$ where $o = (0, \ldots, 0) \in \mathbb{R}^p$, $o_i = (7i, \ldots, 7i) \in \mathbb{R}^q$, and $B_k^p(z) = \{x \in \mathbb{R}^p; |x - z| < k\} \subset \mathbb{R}^p$.

(6) If $U_i^b \cap U_{i'}^b \neq \emptyset$, then there exists a local homeomorphism $\phi_{ii'}^b$ of $\mathbb{R}^q$ such that $\Phi_i^b = \phi_{ii'}^b \circ \Phi_{i'}^b$ on $U_i^b \cap U_{i'}^b$, where $\Phi_i^b = \mathrm{pr} \circ \varphi_i^b$ and $\mathrm{pr} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ is the projection to the second factor.

We put $B_i = B_2^q(o_i) \subset \mathbb{R}^q$. We call the set $(\varphi_i)^{-1}(B_2^p(o) \times \{z\})$ $(z \in B_i)$ a plaque of $\mathcal{U}$ (or a plaque of U_i). Let $\mathcal{P}^{\mathcal{U}}$ be the set of all plaques of $\mathcal{U}$.

Fix a metric d on M. We define the diameter of $\mathcal{U}$ to be the maximum of the diameters of $\overline{U}_i$ ($i = 1, \ldots, A$):

$$\operatorname{diam}(\mathcal{U}) = \max_{i=1,\ldots,A} \operatorname{diam}(\overline{U}_i).$$

We define width$(\mathcal{U}^s, \mathcal{U})$ and width$(\mathcal{U}, \mathcal{U}^b)$ as follows:

$$\operatorname{width}(\mathcal{U}^s,\mathcal{U}) = \min_{i=1,\ldots,A} d\big(\overline{U}_i^s\,,\ M - U_i\big)\,,$$

$$\operatorname{width}(\mathcal{U},\mathcal{U}^b) = \min_{i=1,\ldots,A} d\big(\overline{U}_i\,,\ M - U_i^b\big)\,.$$

Here for K, $K' \subseteq M$,

$$d(K, K') = \inf_{\substack{x\in K \\ x'\in K'}} d(x, x')\,.$$

Let Leb$(\mathcal{U})$ denote the Lebesgue number of $\mathcal{U}$.

Let x be a point of M and let n be a natural number. Let $P_1, \ldots, P_n$ be plaques of $\mathcal{U}$. $(P_1, \ldots, P_n)$ is said to be an n-chain of $\mathcal{P}^{\mathcal{U}}$ at x if x is a point of P_1 and $P_l \cap P_{l+1} \neq \emptyset$. Let $U_{i_1}, \ldots, U_{i_n}$ be foliation neighborhoods of $\mathcal{U}$. $(U_{i_1}, \ldots, U_{i_n})$ is said to be an n-chain of $\mathcal{U}$ at x if there exists an n-chain $(P_{1,x}, \ldots, P_{n,x})$ of $\mathcal{P}^{\mathcal{U}}$ at x (uniquely determined) such that $P_{l,x}$ is a plaque of U_{i_l}. When an n-chain $(U_{i_1}, \ldots, U_{i_n})$ of $\mathcal{U}$ at x is given, the n-chain of $\mathcal{P}^{\mathcal{U}}$ at x with respect to $(U_{i_1}, \ldots, U_{i_n})$ is often written by $(P_{1,x}, \ldots, P_{n,x})$. For $x_1, \ldots, x_N \in M$, Let $\mathcal{C}_n^{\mathcal{U}}(x_1)$ be the set of all n-chains of $\mathcal{U}$ at x_1 and put

$$\mathcal{C}_n^{\mathcal{U}}(x_1, \ldots, x_N) = \mathcal{C}_n^{\mathcal{U}}(x_1) \cap \cdots \cap \mathcal{C}_n^{\mathcal{U}}(x_N)\,.$$

For $K \subseteq M$, let $\mathcal{P}_n^{\mathcal{U}}(K) \subseteq \mathcal{P}^{\mathcal{U}}$ be the set of P_n such that there exists $x \in K$ and an n-chain $(P_1, \ldots, P_n)$ of $\mathcal{P}^{\mathcal{U}}$ at x.

For plaques P and P' of U_i, put

$$w(P, P') = \max\left\{ \max_{x\in\overline{P}} d(x, \overline{P}'^b)\,,\ \max_{x'\in\overline{P}'} d(x', \overline{P}^b) \right\}\,.$$

Here P^b (resp. P'^b) is the plaque of U_i^b containing P (resp. P'). For x, $y \in M$, we define

$$d_n^{\mathcal{A}}(x, y) = \max\left\{ d(x, y)\,,\ \max_{(U_{i_1},\ldots,U_{i_n})\in\mathcal{C}_n^{\mathcal{U}}(x,y)} w(P_{n,x}\,,\ P_{n,y}) \right\}\,.$$

Here $P_{n,x}$ is the plaque of U_{i_n} where $(P_{1,x}, \ldots, P_{n,x})$ is an n-chain of $\mathcal{P}^{\mathcal{U}}$ at x uniquely determined by $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x)$. If $\mathcal{C}_n^{\mathcal{U}}(x, y) = \emptyset$, then $d_n^{\mathcal{A}}(x, y) = d(x, y)$. We remark that usually $d_n^{\mathcal{A}}$ is not continuous. However, we can easily show the following lemma.

LEMMA 2.1.— *$d_n^{\mathcal{A}}$ is lower semi-continuous.*

Let n be a natural number and let ε be a positive number. For x, $y \in M$, x and y are said to be $(n, \varepsilon, \mathcal{A})$-separated if $d_n^{\mathcal{A}}(x, y) \geq \varepsilon$. Otherwise x and y are said to be $(n, \varepsilon, \mathcal{A})$-close. Let K be a subset of M. $S \subseteq M$ is said to be an $(n, \varepsilon, \mathcal{A}, K)$-separating set if S is a subset of K and for any x, $y \in S$, x and y are $(n, \varepsilon, \mathcal{A})$-separated. $R \subseteq M$ is said to be an $(n, \varepsilon, \mathcal{A}, K)$-spanning set if for any $x \in K$, there exists $y \in R$ such that x and y are $(n, \varepsilon, \mathcal{A})$-close. Since M is compact, S is a finite set (*cf.* lemma 2.14 and theorem 2.16). Put

$$s_n^{\mathcal{A}}(\varepsilon, K) = \max\{\#S\,;\, S \text{ is an } (n, \varepsilon, \mathcal{A}, K)\text{-separating set}\},$$
$$r_n^{\mathcal{A}}(\varepsilon, K) = \max\{\#R\,;\, R \text{ is an } (n, \varepsilon, \mathcal{A}, K)\text{-spanning set}\}.$$

PROPOSITION 2.2

$$0 \leq r_n^{\mathcal{A}}(\varepsilon, K) \leq s_n^{\mathcal{A}}(\varepsilon, K) < \infty.$$

$r_n^{\mathcal{A}}(\varepsilon, K)$ and $s_n^{\mathcal{A}}(\varepsilon, K)$ are monotone increasing on $n \in \mathbb{N}$ and monotone decreasing on $\varepsilon > 0$. If $K \subseteq K'$ then $r_n^{\mathcal{A}}(\varepsilon, K) \leq r_n^{\mathcal{A}}(\varepsilon, K')$ and $s_n^{\mathcal{A}}(\varepsilon, K) \leq s_n^{\mathcal{A}}(\varepsilon, K')$.

Proof.— We only prove that $r_n^{\mathcal{A}}(\varepsilon, K) \leq s_n^{\mathcal{A}}(\varepsilon, K)$.

Let $S \subseteq K$ be an $(n, \varepsilon, \mathcal{A}, K)$-separating set with the maximum cardinality. If there exists

$$z \in K - \bigcup_{y \in S} \{x\,;\, d_n^{\mathcal{A}}(x, y) < \varepsilon\},$$

then $S \cup \{z\}$ is also an $(n, \varepsilon, \mathcal{A}, K)$-separating set, which contradicts the assumptions. So

$$K \subseteq \bigcup_{y \in S} \{x\,;\, d_n^{\mathcal{A}}(x, y) < \varepsilon\}.$$

S is an $(n, \varepsilon, \mathcal{A}, K)$-spanning set. Hence

$$r_n^{\mathcal{A}}(\varepsilon, K) \leq \#S = s_n^{\mathcal{A}}(\varepsilon, K). \square$$

Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a monotone decreasing sequence of positive numbers which converges to 0. By the above proposition, $(s_n^{\mathcal{A}}(\varepsilon_j, K))_{j\in\mathbb{N}}$, $(r_n^{\mathcal{A}}(\varepsilon_j, K))_{j\in\mathbb{N}}$ are elements of $\widetilde{\mathcal{I}}$.

The following theorem is the main result of this paper.

THEOREM 2.3.— *Let $\mathcal{F}$ be a codimension q foliation of classe C^0 on a compact $(p+q)$-dimensional manifold M and let K be a subset of M. Let d be a metric on M, let $\mathcal{A}$ be a triple of foliation covers of $(M, \mathcal{F})$, and let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a monotone decreasing sequence of positive numbers which converges to 0. Then,*

$$[s_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} = [r_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}$$

and this growth type is independent of the choice of d, $\mathcal{A}$ and $(\varepsilon_j)_{j\in\mathbb{N}}$.

Before proving the above theorem, we prove two lemmas.

Let ε be a positive number and let $\mathcal{A} = (\mathcal{U}^s, \mathcal{U}, \mathcal{U}^b)$ be a triple of foliation covers of $(M, \mathcal{F})$. ε is said to be small for $\mathcal{A}$ if

$$0 < \varepsilon < \min\{\mathrm{Leb}(\mathcal{U}^s),\ \mathrm{width}(\mathcal{U}^s, \mathcal{U}),\ \mathrm{width}(\mathcal{U}, \mathcal{U}^b)\}.$$

Let $\mathcal{B} = (\mathcal{V}^s, \mathcal{V}, \mathcal{V}^b)$ be another triple of foliation covers of $(M, \mathcal{F})$, where $\mathcal{V} = \{(V_j, \psi_j)\}_{j=1}^{B}$. A plaque of $\mathcal{V}$ is indicated by a letter Q. $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ if

$$\mathrm{diam}(\mathcal{V}^b) < \min\{\mathrm{Leb}(\mathcal{U}),\ \mathrm{width}(\mathcal{U}, \mathcal{U}^b)\}.$$

LEMMA 2.4.— *Suppose that* $\mathrm{diam}(\mathcal{V}^b) < \mathrm{Leb}(\mathcal{U})$ *and*

$$0 < \varepsilon < \mathrm{width}(\mathcal{V}, \mathcal{V}^b).$$

Then for all $n \in \mathbb{N}$,

$$s_n^{\mathcal{B}}(2\varepsilon, K) \le r_n^{\mathcal{A}}(\varepsilon, K).$$

Hence if $\mathcal{B}$ is a refinement of $\mathcal{A}$ and ε is small for $\mathcal{B}$, then the inequality holds.

Proof.— Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $S \subseteq K$ be an $(n, 2\varepsilon, \mathcal{B}, K)$-separating set with the maximum cardinality and let $R \subseteq M$ be an $(n, \varepsilon, \mathcal{A}, K)$-spanning set with the minimum cardinality. We can choose a map κ from S

to R such that $d_n^{\mathcal{A}}(x, \kappa(x)) < \varepsilon$ for any $x \in S$. We show that κ is injective. Suppose that there exists $x \neq y \in S$ such that $z = \kappa(x) = \kappa(y) \in R$.

By $d(x,z) \leq d_n^{\mathcal{A}}(x,z) < \varepsilon$ and $d(z,y) \leq d_n^{\mathcal{A}}(z,y) < \varepsilon$, we have $d(x,z) < 2\varepsilon$. By $d_n^{\mathcal{B}}(x,y) \geq 2\varepsilon$, there exists an n-chain $(V_{j_1}, \ldots, V_{j_n}) \in \mathcal{C}_n^{\mathcal{V}}(x,y)$ such that $w(Q_{n,x}, Q_{n,y}) \geq 2\varepsilon$. (Here the n-chain $(Q_{1,x}, \ldots, Q_{n,x})$ at x is determined by $(V_{j_1}, \ldots, V_{j_n}) \in \mathcal{C}_n^{\mathcal{V}}(x)$.) Moreover we may assume that $x_n \in \overline{Q}_{n,x}$ satisfies $w(Q_{n,x}, Q_{n,y}) = d(x_n, \overline{Q}^b_{n,y}) \geq 2\varepsilon$ where $Q^b_{n,y}$ is the plaque of $V^b_{j_n}$ containing $Q_{n,y}$.

Put $x_0 = x \in Q_{1,x}$, $z_0 = z$ and $y_0 = y$. By $\operatorname{diam}(\mathcal{V}^b) < \operatorname{Leb}(\mathcal{U})$, for each $m = 1, \ldots, n$, there exists $U_{i_m} \in \mathcal{U}$ containing $\overline{V}^b_{j_m}$. Obviously, $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x,y)$. For each $m = 1, \ldots, n-1$, take $x_m \in \overline{Q}_{m,x} \cap \overline{Q}_{m+1,x}$.

By induction, we show that for $k = 1, \ldots, n$,

$$\begin{cases} (U_{i_1}, \ldots, U_{i_k}) \in \mathcal{C}_n^{\mathcal{U}}(x,z,y) \\ \text{there exists } z_k \in P_{k,z} \text{ such that } d(x_k, z_k) < \varepsilon\,. \end{cases}$$

Suppose that for $k = m-1$, the above statement is true. By

$$d(x_{m-1}, z_{m-1}) < \varepsilon < \operatorname{width}(\mathcal{V}, \mathcal{V}^b)$$

and $x_{m-1} \in \overline{V}_{j_m}$, we have $z_{m-1} \in \overline{V}^b_{j_m} \subset U_{i_m}$. So $(U_{i_1}, \ldots, U_{i_m}) \in \mathcal{C}_m^{\mathcal{U}}(x,z,y)$. By $d_n^{\mathcal{A}}(x,z) < \varepsilon$ and $x_m \in \overline{Q}_{m,x} \subset P_{m,x}$, we have

$$d(x_m, \overline{P}^b_{m,z}) \leq d_n^{\mathcal{A}}(x,z) < \varepsilon < \operatorname{width}(\mathcal{V}, \mathcal{V}^b)\,.$$

Since x_m is a point of $\overline{V}_{j_m}$, there exists $z_m \in \overline{P}^b_{m,z} \cap V^b_{j_m} \subset P_{m,z}$ such that $d(x_m, z_m) = d(x_m, \overline{P}^b_{m,z}) < \varepsilon$. Therefore for $k = m$, the above statement is true.

Finally we obtain

$$\begin{cases} (U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x,z,y) \\ z_n \in P_{n,z} \text{ such that } d(x_n, z_n) < \varepsilon\,. \end{cases}$$

On the other hand, by $d_n^{\mathcal{A}}(z,y) < \varepsilon$ and $z_n \in P_{n,z}$ we have $d(z_n, \overline{P}^b_{n,y}) \leq d_n^{\mathcal{A}}(z,y) < \varepsilon$. By $d(x_n, z_n) < \varepsilon$, we have $d(x_n, \overline{P}^b_{n,y}) < 2\varepsilon$. By $\overline{Q}^b_{n,y} \subset \overline{P}^b_{n,y}$, we deduce the contradiction $d(x_n, \overline{Q}^b_{n,y}) \geq 2\varepsilon$. □

LEMMA 2.5.— *Suppose that* $\mathrm{diam}(\mathcal{V}) < \mathrm{width}(\mathcal{U}, \mathcal{U}^b)$ *and*

$$0 < \varepsilon < \min\{\mathrm{Leb}(\mathcal{V}^s), \mathrm{width}(\mathcal{V}^s, \mathcal{V})\}.$$

Then for all $n \in \mathbb{N}$,

$$s_n^{\mathcal{A}}(\varepsilon, K) \leq r_{Bn}^{\mathcal{B}}\left(\frac{\varepsilon}{2}, K\right).$$

Hence if $\mathcal{B}$ is a refinement of $\mathcal{A}$ and ε is small for $\mathcal{B}$, then the inequality holds.

Proof.— Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Let $S \subseteq K$ be an $(n, \varepsilon, \mathcal{A}, K)$-separating set with the maximum cardinality and let $R \subseteq M$ be a $\big(Bn, (\varepsilon/2), \mathcal{B}, K\big)$-spanning set with the minimum cardinality. We can choose a map κ from S to R such that $d_{Bn}^{\mathcal{B}}\big(x, \kappa(x)\big) < \varepsilon/2$ for any $x \in S$. We will show that κ is injective. Suppose that there exists $x \neq y \in S$ such that $z = \kappa(x) = \kappa(y) \in R$.

By $d(x,z) \leq d_{Bn}^{\mathcal{B}}(x,z) < \varepsilon/2$ and $d(z,y) \leq d_{Bn}^{\mathcal{B}}(z,y) < \varepsilon/2$, we have $d(x,y) < \varepsilon$. By $d_n^{\mathcal{A}}(x,y) \geq \varepsilon$, there exists $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x,y)$ such that $w(P_{n,x}, P_{n,y}) \geq \varepsilon$. Moreover we may assume that $x_{Bn} \in \overline{P}_{n,x}$ satisfies $w(P_{n,x}, P_{n,y}) = d(x_{Bn}, \overline{P}_{n,y}^{\,b}) \geq \varepsilon$.

Put $x_0 = x \in P_{1,x}$, $z_0 = z$ and $y_0 = y$. For each $l = 1, \ldots, n-1$, take $x'_{Bl} \in \overline{P}_{l,x} \cap \overline{P}_{l+1,x}$ and for each $l = 1, \ldots, n$, take a path γ_l from $x'_{B(l-1)}$ to x'_{Bl} contained in $\overline{P}_{l,x}$.

Consider x_0, z_0, y_0 and $\gamma_1 \subset U_{l_1}^b$.

By induction, we show that for $k = 1, \ldots, B$, there exists

$$\begin{cases} (V_{j_1}, \ldots, V_{j_k}) \in \mathcal{C}_k^{\mathcal{V}}(x,z,y), \\ x_k \in \overline{Q}_{k,x}^{\,s} \cap \gamma_1, \\ z_k \in Q_{k,z} \text{ such that } d(x_k, z_k) < \dfrac{\varepsilon}{2}, \\ y_k \in Q_{k,y} \text{ such that } d(z_k, y_k) < \dfrac{\varepsilon}{2}. \end{cases}$$

Suppose that for $k = l-1$, the above statement is true. By

$$\mathrm{diam}\big(\{x_{l-1}, z_{l-1}, y_{l-1}\}\big) < \varepsilon < \mathrm{Leb}(\mathcal{V}^s),$$

there exists $V_{j_l}^s \in \mathcal{V}^s$ containing $x_{l-1}, z_{l-1}, y_{l-1}$. So $(V_{j_1}, \ldots V_{j_l}) \in \mathcal{C}_l^{\mathcal{V}}(x,z,y)$. Let x_l be the maximal point of $\gamma_1 \cap \overline{Q}_{l,x}^{\,s}$ with respect to the orientation of γ_1. By $d^{\mathcal{B}}_{B_n}(x,z) < \varepsilon/2$, we have $d(x_l, \overrightarrow{Q}^b_{l,z}) < \varepsilon/2 <$ width$(\mathcal{V}^s, \mathcal{V})$. Since x_l is a point of $\overline{V}^s_{j_l}$, there exists $z_l \in \overrightarrow{Q}^b_{l,z} \cap V_{j_l} \subset Q_{l,z}$ such that $d(x_l, z_l) = d(x_l, \overrightarrow{Q}^b_{l,z}) < \varepsilon/2$. Moreover by $d^{\mathcal{B}}_{B_n}(z,y) < \varepsilon/2$, we have $d(z_l, \overrightarrow{Q}^b_{l,y}) < \varepsilon/2$. There exists $y_l \in \overrightarrow{Q}^b_{l,y}$ such that $d(z_l, y_l) = d(z_l, \overrightarrow{Q}^b_{l,y}) < \varepsilon/2$. So $d(x_l, y_l) < \varepsilon <$ width$(\mathcal{V}^s, \mathcal{V})$. Again since x_l is a point of $\overline{V}^s_{j_l}$, we have $y_l \in Q_{l,y} \subset V_{j_l}$. (If $x_l \neq x'_B$, then we remark that x_l is a point of $\partial Q^s_{l,x}$ and the subset of γ_1 from x_l to x_B do not intersect with $V^s_{j_1}, \ldots, V^s_{j_l}$.) Therefore for $k = l$, the above statement is true.

Since the number of foliation neighborhoods of $\mathcal{V}^s$ which intersects γ_1 is at most B, x_B coincides with x'_B.

We obtain that

$$\begin{cases} (V_{j_1}, \ldots, V_{j_B}) \in \mathcal{C}^{\mathcal{V}}_B(x,z,y), \\ x_B \in \overline{Q}^s_{B,x}, \\ z_B \in Q_{B,z} \text{ such that } d(x_B, z_B) < \dfrac{\varepsilon}{2}, \\ y_B \in Q_{B,y} \text{ such that } d(z_B, y_B) < \dfrac{\varepsilon}{2}. \end{cases}$$

Moreover since $V_{j_1}, \ldots V_{j_B}$ intersect $\gamma_1 \subset \overline{U}_1$, by

$$\operatorname{diam}(\mathcal{V}) < \operatorname{width}(\mathcal{U}, \mathcal{U}^b),$$

we have $V_{j_1}, \ldots V_{j_B} \subset U^b_1$.

We apply the same argument for x_B, z_B, y_B and $\gamma_2 \subset U^b_{i_2}$. Moreover we can continue for $\gamma_3, \ldots, \gamma_n$.

Finally we obtain that

$$\begin{cases} (V_{j_1}, \ldots, V_{j_{Bn}}) \in \mathcal{C}^{\mathcal{V}}_{Bn}(x,z,y), \\ x_{Bn} \in \overline{Q}^s_{Bn,x}, \\ z_{Bn} \in Q_{Bn,z} \text{ such that } d(x_{Bn}, z_{Bn}) < \dfrac{\varepsilon}{2}, \\ y_{Bn} \in Q_{Bn,y} \text{ such that } d(z_{Bn}, y_{Bn}) < \dfrac{\varepsilon}{2}, \\ V_{B(i-1)+1}, \ldots, V_{Bi} \subset U^b_i \text{ for } i = 1, \ldots, n. \end{cases}$$

By $y_{Bn} \in Q_{Bn,y} \subset P^b_{n,y}$ and $d(x_{Bn}, y_{Bn}) < \varepsilon$, we deduce the contradiction $d(x_{Bn}, \overline{P}^b_{n,y}) \geq \varepsilon$. □

Proof of theorem 2.3

First by the definition of the growth type and proposition 2.2, it is easy to see that $[s_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}}$, $[r_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}$ are independent of the choice of $(\varepsilon_j)_{j\in\mathbb{N}}$.

Next we show that $[s_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} = [r_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}}$ and this growth is indepenent of the choice of $\mathcal{A}$. Let $\mathcal{A}'$ be another triple of foliation covers of $(M, \mathcal{F})$. Then we can take a triple of foliation covers $\mathcal{B}$ such that $\mathcal{B}$ is a refinement of $\mathcal{A}$ and $\mathcal{A}'$. So we may compare $\mathcal{A}$ with $\mathcal{B}$.

Let ε be small for $\mathcal{B}$. By proposition 2.2, lemma 2.4 and lemma 2.5, we have

$$\begin{aligned} r_{Bn}^{\mathcal{B}}\left(\frac{\varepsilon}{2}, K\right) &\geq s_n^{\mathcal{A}}(\varepsilon, K) \geq r_n^{\mathcal{A}}(\varepsilon, K) \\ &\geq s_n^{\mathcal{B}}(2\varepsilon, K) \geq r_n^{\mathcal{B}}(2\varepsilon, K) \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

Therefore

$$\begin{aligned} [r_{Bn}^{\mathcal{B}}(\varepsilon_j, K)]_{j\in\mathbb{N}} &\geq [s_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} \geq [r_n^{\mathcal{A}}(\varepsilon_j, K)]_{j\in\mathbb{N}} \\ &\geq [s_n^{\mathcal{B}}(\varepsilon_j, K)]_{j\in\mathbb{N}} \geq [r_n^{\mathcal{B}}(\varepsilon_j, K)]_{j\in\mathbb{N}}. \end{aligned}$$

By $[r_{Bn}^{\mathcal{B}}(\varepsilon_j, K)]_{j\in\mathbb{N}} = [r_n^{\mathcal{B}}(\varepsilon_j, K)]_{j\in\mathbb{N}}$, the above inequalities are equalities.

Finally we show that this growth is independent of the choice of d. Let d' be another metric on M. Then by the compactness of M, d and d' are uniformly equivalent. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for x, $y \in M$ if $d(x, y) < \delta$ then $d'(x, y) < \varepsilon$ and if $d'(x, y) < \delta$ then $d(x, y) < \varepsilon$. Let R be an $(n, \delta, \mathcal{A}, K)$-spanning set with respect to d with the minimum cardinality.

$$r_n^{\mathcal{A},d}(\delta, K) = \#R.$$

Here $r_n^{\mathcal{A}}(\delta, K)$ with respect to d is written by $r_n^{\mathcal{A},d}(\delta, K)$. For each $x \in K$, there exists $y \in R$ such that $d_n^{\mathcal{A}}(x, y) < \delta$. Then by the definition of $d_n^{\mathcal{A}}$, we obtain ${d'}_n^{\mathcal{A}}(x, y) < \varepsilon$. Hence R is an $(n, \varepsilon, \mathcal{A}, K)$-spanning set with respect to d'

$$r_n^{\mathcal{A},d'}(\varepsilon, K) \leq \#R = r_n^{\mathcal{A},d}(\delta, K).$$

So we have

$$[r_n^{\mathcal{A},d'}(\varepsilon_j, K)]_{j\in\mathbb{N}} \leq [r_n^{\mathcal{A},d}(\varepsilon_j, K)]_{j\in\mathbb{N}}.$$

The converse inequality is similarly shown. □

DEFINITION 2.6.— *By theorem 2.3,*

$$\left[s_n^{\mathcal{A}}(\varepsilon_j, K)\right]_{j\in\mathbb{N}} = \left[r_n^{\mathcal{A}}(\varepsilon_j, K)\right]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}$$

depends only on $(M,\mathcal{F})$ and $K \subseteq M$ and it is a topological invariant for foliations on compact manifolds. This growth is written by

$$\eta(K,\mathcal{F}) \text{ (or simply } \eta(K)) \in \widetilde{\mathcal{E}}$$

and we call it the expansion growth of $(M,\mathcal{F})$ on K. For a leaf L, $\eta(L)$ is called the expansion growth of the leaf L. $\eta(L)$ depends only on the inclusion from L to M.

Remark 2.7

(1) $\eta(K) = [0]$ if and only if $K = \emptyset$.

(2) If L is a compact leaf then $\eta(L) = [1]$.

(3) If $K \subseteq K'$, then $\eta(K) \leq \eta(K')$.

(4) $\eta(K,\mathcal{F})$ is determined by K and $\mathcal{F}|\mathrm{sat}^{\mathcal{F}}(K)$. Here $\mathrm{sat}^{\mathcal{F}}(K)$ is the $\mathcal{F}$-saturation of K.

Now we obtain many numerical topological invariants for foliations on compact manifolds. Corollary 2.8 is deduced from theorem 2.3.

COROLLARY 2.8.— *For $l \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, the following number $\eta_m^l(K,\mathcal{F})$ is a numerical topological invariant for foliations on compact manifolds:*

$$\begin{aligned}\eta_m^l(K,\mathcal{F}) &= \lim_{j\to\infty} \limsup_{n\to\infty} \frac{1}{(\log)^m(n)} (\log)^l\left(s_n^{\mathcal{A}}(\varepsilon_j, K)\right) \\ &= \lim_{j\to\infty} \limsup_{n\to\infty} \frac{1}{(\log)^m(n)} (\log)^l\left(r_n^{\mathcal{A}}(\varepsilon_j, K)\right)\end{aligned}$$

where $(\log)^l(k) = \log\circ\cdots\circ\log(k)$ (l times). *Here if $\eta(K,\mathcal{F}) = [0]$ or $[1]$ then put $\eta_m^l(K,\mathcal{F}) = 0$.*

We will see that $\eta_l^l(K,\mathcal{F})$ and $\eta_l^{l+1}(K,\mathcal{F})$ ($l \in \mathbb{N}$) are non-trivial by theorem 3.7.

Next we describe several properties of $\eta(K,\mathcal{F})$.

PROPOSITION 2.9

$$\eta(\overline{K}) = \eta(K)\,.$$

Proof.— Obviously,

$$\eta(\overline{K}) \geq \eta(K) .$$

We show the converse inequality Fix a metric d and a triple of foliation covers $\mathcal{A}$. For all $\varepsilon > 0$, we can take a positive number δ such that $0 < \delta < \varepsilon$. Let R be an $(n, \delta, \mathcal{A}, K)$-spanning set with the minimum cardinality.

$$r_n^{\mathcal{A}}(\delta, K) = \# R .$$

$$K \subseteq \bigcup_{y \in R} \{x ; d_n^{\mathcal{A}}(x, y) < \delta\} \subseteq \bigcup_{y \in R} \{x ; d_n^{\mathcal{A}}(x, y) \leq \delta\} .$$

Since $\bigcup_{y \in R} \{x ; d_n^{\mathcal{A}}(x, y) \leq \delta\}$ is a closed set by lemma 2.1, we have

$$\overline{K} \subseteq \bigcup_{y \in R} \{x ; d_n^{\mathcal{A}}(x, y) \leq \delta\} \subseteq \bigcup_{y \in R} \{x ; d_n^{\mathcal{A}}(x, y) < \varepsilon\} .$$

So R is an $(n, \varepsilon, \mathcal{A}, \overline{K})$-spanning set.

$$r_n^{\mathcal{A}}(\varepsilon, \overline{K}) \leq \# R = r_n^{\mathcal{A}}(\delta, K) .$$

Therefore

$$\eta(\overline{K}) = \left[r_n^{\mathcal{A}}(\varepsilon_j, \overline{K})\right]_{j \in \mathbb{N}} \leq \left[r_n^{\mathcal{A}}(\varepsilon_j, K)\right]_{j \in \mathbb{N}} = \eta(K) . \square$$

PROPOSITION 2.10.— *Let K, K' be subsets of M. Then*

$$\eta(K \cup K') = \eta(K) + \eta(K') .$$

Proof.— Let n be a natural number, let ε be a positive real number, and let $\mathcal{A}$ be a triple of foliation covers. By $s_n^{\mathcal{A}}(\varepsilon, K \cup K') \geq s_n^{\mathcal{A}}(\varepsilon, K)$, $s_n^{\mathcal{A}}(\varepsilon, K')$, we have

$$2 s_n^{\mathcal{A}}(\varepsilon, K \cup K') \geq s_n^{\mathcal{A}}(\varepsilon, K) + s_n^{\mathcal{A}}(\varepsilon, K') .$$

So

$$\eta(K \cup K') \geq \eta(K) + \eta(K') .$$

We show the converse inequality. Let S be an $(n, \varepsilon, \mathcal{A}, K \cup K')$-separating set with the maximum cardinality.

$$s_n^{\mathcal{A}}(\varepsilon,\, K \cup K') = \# S\,.$$

Then $S \cap K$ is an $(n, \varepsilon, \mathcal{A}, K)$-separating set and $S \cap K'$ is an $(n, \varepsilon, \mathcal{A}, K')$-separating set. So

$$\begin{aligned} s_n^{\mathcal{A}}(\varepsilon, K) + s_n^{\mathcal{A}}(\varepsilon, K') &\geq \#(S \cap K) + \#(S \cap K') \\ &\geq \# S = s_n^{\mathcal{A}}(\varepsilon,\, K \cup K')\,. \end{aligned}$$

Hence

$$\eta(K) + \eta(K') \geq \eta(K \cup K')\,. \square$$

PROPOSITION 2.11.— *Let $\mathcal{A} = (\mathcal{U}^s, \mathcal{U}, \mathcal{U}^b)$ be a triple of foliation covers. For $K \subseteq M$, we have*

$$\eta(K) = \eta\Bigg(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\Bigg).$$

Proof.— By $K \subseteq \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P$, we have

$$\eta(K) \leq \eta\Bigg(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\Bigg).$$

We show the converse inequality. Fix a metric d on M. Let S be an $(n, \varepsilon, \mathcal{A}, \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P)$-separating set with the maximum cardinality.

$$s_n^{\mathcal{A}}\Bigg(\varepsilon\,,\, \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\Bigg) = \# S\,.$$

Let $\mathcal{W} = \{W_k\}_{k=1}^{C(\varepsilon)}$ be a finite open cover of M such that $\operatorname{diam}(\mathcal{W}) < \varepsilon$ with the minimum cardinality. Fix $W_k \in \mathcal{W}$ and $U_{i_0} \in \mathcal{U}$. We define S' as follows:

$$\begin{aligned} S' = \big\{ z \in S \cap W_k \cap U_{i_0}\,;\; & P_{0,z} \cap K \neq \emptyset\,, \\ & \text{where } P_{0,z} \text{ is a plaque of } U_{i_0} \text{ containing } z \big\}\,. \end{aligned}$$

We can choose a map κ from S' to K such that for $z \in S'$, $\kappa(z) \in P_{0,z} \cap K$. We show that $\kappa(S')$ is an $(n+1, \varepsilon, \mathcal{A}, K)$-separating set. Let x and y be two points of S'. By $x, y \in W_k$, $d(x,y) < \varepsilon$. Since $d_n^{\mathcal{A}}(x,y) \geq \varepsilon$, there exists n-chain $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x,y)$ such that $w(P_{n,x}, P_{n,y}) \geq \varepsilon$. By $x, y \in U_{i_0} \cap U_{i_1}$, $(U_{i_0}, U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_{n+1}^{\mathcal{U}}(\kappa(x), \kappa(y))$. Hence $d_{n+1}^{\mathcal{A}}(\kappa(x), \kappa(y)) \geq \varepsilon$ and κ is injective.

Therefore $\kappa(S')$ is an $(n+1, \varepsilon, \mathcal{A}, K)$-separating set.

$$s_{n+1}^{\mathcal{A}}(\varepsilon, K) \geq \#\kappa(S') = \#S'.$$

Summing over $W_k \in \mathcal{W}$ and $U_{i_0} \in \mathcal{U}$, we have

$$A \cdot C(\varepsilon) \cdot s_{n+1}^{\mathcal{A}}(\varepsilon, K) \geq \#S = s_n^{\mathcal{A}}\left(\varepsilon, \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\right).$$

Therefore

$$\eta(K) \geq \eta\left(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\right). \square$$

COROLLARY 2.12.— *Let $K \subseteq M$. Put $T_i = \varphi_i^{-1}(\{o\} \times B_i)$ and $T = \bigcup_{i=1}^{A} T_i$. Then*

$$\eta\left(\left(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\right) \cap T\right) = \eta(K).$$

Proof.— Put

$$K' = \left(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\right) \cap T.$$

By $K \subseteq \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K')} P$, we have

$$K' \subseteq \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P \subseteq \bigcup_{P \in \mathcal{P}_2^{\mathcal{U}}(K')} P.$$

So

$$\eta(K') \leq \eta\left(\bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(K)} P\right) \leq \eta\left(\bigcup_{P \in \mathcal{P}_2^{\mathcal{U}}(K')} P\right).$$

By proposition 2.11, we have

$$\eta\Bigg(\bigcup_{P\in\mathcal{P}_1^{\mathcal{U}}(K)} P\Bigg) = \eta(K)$$

and

$$\eta\Bigg(\bigcup_{P\in\mathcal{P}_2^{\mathcal{U}}(K')} P\Bigg) = \eta(K')\,.$$

Hence

$$\eta(K') = \eta(K)\,.\ \square$$

To compute $\eta(K)$, we have only to compute $\eta(K')$.

PROPOSITION 2.13.— *Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a foliation on a compact manifold M (resp. M'). Let $\mathcal{F}\times\mathcal{F}'$ be the product foliation on the manifold $M\times M'$. Let K (resp. K') be a subset of M (resp. M'). Then*

$$\eta(K\times K'\,,\,\mathcal{F}\times\mathcal{F}') = \eta(K,\mathcal{F})\cdot\eta(K',\mathcal{F}')\,.$$

Proof.— Let d (resp. d') be a metric on M (resp. M'). We define a metric d'' on $M\times M'$ as follows. For (x,x'), $(y,y')\in M\times M'$,

$$d''\big((x,x')\,,\,(y,y')\big) = \max\big\{d(x,y)\,,\,d'(x',y')\big\}\,.$$

Let n be a natural number and let ε be a positive number. Let $\mathcal{A} = (\mathcal{U}^s,\mathcal{U},\mathcal{U}^b)$ (resp. $\mathcal{A}' = (\mathcal{U}'^s,\mathcal{U}',\mathcal{U}'^b)$) be a triple of foliation covers of $(M,\mathcal{F})$ (resp. $(M',\mathcal{F}')$). Then $\mathcal{A}\times\mathcal{A}' = (\mathcal{U}^s\times\mathcal{U}'^s\,,\,\mathcal{U}\times\mathcal{U}'\,,\,\mathcal{U}^b\times\mathcal{U}'^b)$ is a triple foliation covers of $(M\times M'\,,\,\mathcal{F}\times\mathcal{F}')$.

Let R (resp. R') be an $(n,\varepsilon,\mathcal{A},K)$-spanning (resp. $(n,\varepsilon,\mathcal{A}',K')$-spanning) set with the minimum cardinality. We show that $R\times R'$ is an $(n,\varepsilon,\mathcal{A}\times\mathcal{A}'\,,\,K\times K')$-spanning set. Let (x,x') be a point of $K\times K'$. Then there exists $y\in R$ (resp. $y'\in R'$) such that $d_n^{\mathcal{A}}(x,y)<\varepsilon$ (resp. $d'^{\mathcal{A}'}_n(x',y')<\varepsilon$). Hence we can deduce

$$d''^{\mathcal{A}\times\mathcal{A}'}_n\big((x,x')\,,\,(y,y')\big) < \varepsilon\,.$$

So $R\times R'$ is an $(n,\varepsilon,\mathcal{A}\times\mathcal{A}'\,,\,K\times K')$-spanning set.

$$\eta(K\times K'\,,\,\mathcal{F}\times\mathcal{F}') \le \eta(K,\mathcal{F})\cdot\eta(K',\mathcal{F}')\,.$$

Next we show the converse inequality. Let S (resp. S') be an $(n, \varepsilon, \mathcal{A}, K)$-separating (resp. $(n, \varepsilon, \mathcal{A}', K')$-separating) set with the maximum cardinality. We show that $S \times S'$ is an $(n, \varepsilon, \mathcal{A} \times \mathcal{A}', K \times K')$-separating set. Let (x, x') and (y, y') be two points of $S \times S'$. We may assume $x \neq y$. Then by $d_n^{\mathcal{A}}(x, y) \geq \varepsilon$, we can deduce that

$$d''^{\mathcal{A}\times\mathcal{A}'}_n\big((x, y), (x', y')\big) \geq \varepsilon .$$

So $S \times S'$ is an $(n, \varepsilon, \mathcal{A} \times \mathcal{A}', K \times K')$-separating set.

$$\eta(K \times K', \mathcal{F} \times \mathcal{F}') \geq \eta(K, \mathcal{F}) \cdot \eta(K', \mathcal{F}') . \square$$

Next we give an easier definition of the expansion growth of foliations. In this definition, we use a holonomy pseudogroup of local homeomorphisms of local transverse sections of $(M, \mathcal{F})$. Let $A = (\mathcal{U}^s, \mathcal{U}, \mathcal{U}^b)$ be a triple of foliation covers of $(M, \mathcal{F})$. Put $T_i = \varphi_i^{-1}(\{o\} \times B_i) \subset M$, $T = \bigcup_{i=1}^{A} T_i \subset M$ and $B = \bigcup_{i=1}^{A} B_i \subset \mathbb{R}^q$. We remark that $\Phi_i|_{T_i}$ is a homeomorphism of T_i to B_i where $\Phi_i = \mathrm{pr} \circ \varphi_i$. We define a map ι of B to T such that $\iota(x) = \left(\Phi_i|_{T_i}\right)^{-1}(x) \in T_i \subset T$ for $x \in B_i \subset B$. We note that ι can be extended to the continuous map of $\overline{B}$ to $\overline{T}$.

We can define a pseudogroup of local homeomorphisms of $B \subset \mathbb{R}^q$ induced by a foliation cover $\mathcal{U}$. If $U_i \cap U_{i'} \neq \emptyset$ then there exists a homeomorphism $\phi_{ii'}$ of $\Phi_{i'}(U_i \cap U_{i'})$ to $\Phi_i(U_i \cap U_{i'})$ such that

$$\Phi_i = \phi_{ii'} \circ \Phi_{i'} \quad \text{on} \quad U_i \cap U_{i'} .$$

Put $\mathcal{H}_1 = \{\mathrm{id}_B\} \cup \{\phi_{ii'}\}_{i,i'=1}^{A}$. Then we define $\mathcal{H}_n$ ($n \in \mathbb{N}$) as follows:

$$\mathcal{H}_n = \{f_n \circ \cdots \circ f_1 \,;\, f_l \in \mathcal{H}_1\} .$$

Here the composition map $f_2 \circ f_1$ is defined on

$$\mathrm{domain}(f_1) \cap f_1^{-1}\big(\mathrm{domain}(f_2)\big) .$$

Put $\mathcal{H} = \bigcup_{n \in \mathbb{N}} \mathcal{H}_n$. We call $\mathcal{H}$ a pseudogroup of local homeomorphisms of B induced by a foliation cover $\mathcal{U}$. We note that id_B is an element of $\mathcal{H}_n$ and that

$$\#\mathcal{H}_n \leq 1 + A^2 + A^4 + \cdots + A^{2n} \leq nA^{2n} + 1 .$$

Let x, y be points of B and let n be a natural number. We define $D_n^{\mathcal{H}_1}(x,y)$ as follows:

$$D_n^{\mathcal{H}_1}(x,y) = \max_{f\in\mathcal{H}_n \text{ such that } x,y\in \text{domain}(f)} \left|f(x) - f(y)\right| .$$

We remark that $D_n^{\mathcal{H}_1}$ is not continuous but lower semi-continuous. Let ε be a positive number. x and y are said to be $(n,\varepsilon,\mathcal{H}_1)$-separated if $D_n^{\mathcal{H}_1}(x,y) \geq \epsilon$. Otherwise x and y are said to be $(n,\varepsilon,\mathcal{H}_1)$-close. Let K be a subset of $B \subset \mathbb{R}^q$. $S \subseteq B$ is said to be an $(n,\varepsilon,\mathcal{H}_1,K)$-separating set if S is a subset of K and for any x, $y \in S$, x and y are $(n,\varepsilon,\mathcal{H}_1)$-separated. $R \subseteq B$ is said to be an $(n,\varepsilon,\mathcal{H}_1,K)$-spanning set if for any $x \in K$, there exists $y \in R$ such that x and y are $(n,\varepsilon,\mathcal{H}_1)$-close. By the following lemma, S is a finite set. Let $\mathcal{W} = \{W_k\}_{k=1}^{C(\varepsilon)}$ be a finite open cover of $\overline{B} \subset \mathbb{R}^q$ such that $\text{diam}(\mathcal{W}) < \varepsilon$ and $\mathcal{W}$ has the minimum cardinality.

LEMMA 2.14.— *If S is an $(n,\varepsilon,\mathcal{H}_1,K)$-separating set, then*

$$\#S \leq C(\varepsilon)^{\#\mathcal{H}_n} .$$

Proof.— For any $f \in \mathcal{H}_n$, we can choose a map κ_f of B to $\mathcal{W}$ such that for $x \in B$, if $x \in \text{domain}(f)$ then $f(x) \in \kappa_f(x) \in \mathcal{W}$ and otherwise $x \in \kappa_f(x) \in \mathcal{W}$. Then for x, $y \in S$, there exists $f \in \mathcal{H}_n$ such that x, $y \in \text{domain}(f)$ and $\left|f(x) - f(y)\right| \geq \varepsilon$. Since $\text{diam}(\mathcal{W}) < \varepsilon$, we have $\kappa_f(x) \neq \kappa_f(y) \in \mathcal{W}$. Therefore we can define an injection map of S to $\mathcal{W} \times \cdots \times \mathcal{W}$ ($\#\mathcal{H}_n$ times). So

$$\#S \leq C(\varepsilon)^{\#\mathcal{H}_n} . \square$$

Put

$$s_n^{\mathcal{H}_1}(\varepsilon,K) = \max\{\#S\,;\, S \text{ is an } (n,\varepsilon,\mathcal{H}_1,K)\text{-separating set}\} ,$$
$$r_n^{\mathcal{H}_1}(\varepsilon,K) = \min\{\#R\,;\, R \text{ is an } (n,\varepsilon,\mathcal{H}_1,K)\text{-spanning set}\} .$$

The following proposition is easily seen.

PROPOSITION 2.15

$$0 \leq r_n^{\mathcal{H}_1}(\varepsilon,K) \leq s_n^{\mathcal{H}_1}(\varepsilon,K) \leq C(\varepsilon)^{\#\mathcal{H}_n} .$$

$r_n^{\mathcal{H}_1}(\varepsilon,K)$, $s_n^{\mathcal{H}_1}(\varepsilon,K)$ and $C(\varepsilon)^{\#\mathcal{H}_n}$ are monotone increasing on $n \in \mathbb{N}$ and monotone decreasing on $\varepsilon > 0$. If $K \subseteq K'$ then $r_n^{\mathcal{H}_1}(\varepsilon,K) \leq r_n^{\mathcal{H}_1}(\varepsilon,K')$ and $s_n^{\mathcal{H}_1}(\varepsilon,K) \leq s_n^{\mathcal{H}_1}(\varepsilon,K')$.

Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a monotone decreasing sequence of positive numbers which converges to 0. By the above proposition, $\left(s_n^{\mathcal{H}_1}(\varepsilon_j, K)\right)_{j\in\mathbb{N}}$, $\left(r_n^{\mathcal{H}_1}(\varepsilon_j, K)\right)_{j\in\mathbb{N}}$, $\left(C(\varepsilon_j)^{\#\mathcal{H}_n}\right)_{j\in\mathbb{N}}$ are elements of $\widetilde{\mathcal{I}}$ and we have

$$\left[r_n^{\mathcal{H}_1}(\varepsilon_j, K)\right]_{j\in\mathbb{N}} \leq \left[s_n^{\mathcal{H}_1}(\varepsilon_j, K)\right]_{j\in\mathbb{N}} \leq \left[C(\varepsilon_j)^{\mathcal{H}_n}\right]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}\,.$$

Moreover it is easy to see that

$$\left[C(\varepsilon_j)^{\#\mathcal{H}_n}\right]_{j\in\mathbb{N}} \leq \left[j^{\#\mathcal{H}_n}\right]_{j\in\mathbb{N}} \leq \left[j^{nA^{2n}+1}\right]_{j\in\mathbb{N}} \leq \left[e^{e^n}\right] \in \widetilde{\mathcal{E}}\,.$$

The following theorem shows that $\left[r_n^{\mathcal{H}_1}(\varepsilon_j, K)\right]_{j\in\mathbb{N}}$ and $\left[s_n^{\mathcal{H}_1}(\varepsilon_j, K)\right]_{j\in\mathbb{N}}$ are equal to the expansion growth of the foliation $\mathcal{F}$. By this way, we can compute the expansion growth of foliations more easily.

THEOREM 2.16.— *Let K be a subset of M. Put $\Phi(K) = \bigcup_{i=1}^{A} \Phi_i(K \cap U_i) \subseteq B$. Then*

$$\begin{aligned}\eta(K, \mathcal{F}) &= \left[s_n^{\mathcal{H}_1}\left(\varepsilon_j, \Phi(K)\right)\right]_{j\in\mathbb{N}} = \left[r_n^{\mathcal{H}_1}\left(\varepsilon_j, \Phi(K)\right)\right]_{j\in\mathbb{N}} \\ &\leq \left[j^{\#\mathcal{H}_n}\right]_{j\in\mathbb{N}} \leq \left[e^{e^n}\right] \in \widetilde{\mathcal{E}}\,.\end{aligned}$$

Before proving the above theorem, we show two lemmas.

LEMMA 2.17.— *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x,\ y \in B_i$ and $|x - y| < \delta$ then*

$$w\left(P_{\iota(x)}, P_{\iota(y)}\right) < \varepsilon$$

where $P_{\iota(x)}$ is the plaque of U_i containing $\iota(x)$.

Proof.— The map

$$u : \overline{B}_i \times \overline{B}_i \ni (x, y) \longmapsto w\left(P_{\iota(x)}, P_{\iota(y)}\right) \in [\,0\,,\,+\infty\,)$$

is uniformly continuous. So for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any (x, y), $(x', y') \in \overline{B}_i \times \overline{B}_i$ if $\max\left\{|x - x'|\,,\,|y - y'|\right\} < \delta$ then $\left|w\left(P_{\iota(x)}, P_{\iota(y)}\right) - w\left(P_{\iota(x')}, P_{\iota(y')}\right)\right| < \varepsilon$. Especially putting $x' = y' = y$, if $|x - y| < \delta$ then $w\left(P_{\iota(x)}, P_{\iota(y)}\right) < \varepsilon$. □

LEMMA 2.18.— *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $x, y \in B_i$ *and*

$$w\big(P_{\iota(x)}, P_{\iota(y)}\big) < \delta$$

then $|x - y| < \varepsilon$.

Proof.— Fix a positive number ε. The map

$$v : \left\{(x, y) \in \overline{B}_i \times \overline{B}_i\,;\ |x - y| \geq \varepsilon\right\} \ni (x, y) \longmapsto \max\left\{d\Big(\iota(x)\,,\ \overline{P}^b_{\iota(y)}\Big)\,,\ d\Big(\iota(y)\,,\ \overline{P}^b_{\iota(x)}\Big)\right\} \in [\,0\,, +\infty\,)$$

is continuous. Since domain(v) is compact, there exists $(x_0, y_0) \in \overline{B}_i \times \overline{B}_i$ which gives the minimum value of v. If $v(x_0, y_0) = 0$, then we obtain $\iota(x_0) \in \overline{P}^b_{\iota(y_0)}$. So $x_0 = y_0$. This contradicts $|x_0 - y_0| \geq \varepsilon$. Hence there exists a positive number δ such that $v(x_0, y_0) \geq \delta > 0$. Therefore if $|x - y| \geq \varepsilon$, then

$$w\big(P_{\iota(x)}, P_{\iota(y)}\big) \geq v(x, y) \geq v(x_0, y_0) \geq \delta\,. \;\square$$

Proof of theorem 2.16

By corollary 2.12, we may assume that $K \subseteq T \subset M$. We will show that $\eta(K, \mathcal{F}) \leq \left[r^{\mathcal{H}_1}_n\big(\varepsilon_j\,,\ \Phi(K)\big)\right]_{j \in \mathbb{N}}$. Fix a natural number n and a positive number ε. Take a positive number $\delta < 1$ which satisfies lemma 2.17 and that if $|x' - y'| < \delta$ then $d\big(\iota(x')\,,\ \iota(y')\big) < \varepsilon$. Let $R \subset B$ be an $\big(n + 1,\, \delta,\, \mathcal{H}_1, \Phi(K)\big)$-spanning set with the minimum cardinality. We give $x \in K \subseteq T$. Then there exists $i_0 \in \{1, \ldots, A\}$ such that $x \in T_{i_0}$. Put $x' = \Phi_{i_0}(x) \in B_{i_0}$. Since R is a spanning set, there exists $y' \in R$ such that $D^{\mathcal{H}_1}_{n+1}(x', y') < \delta$. By $|x' - y'| < \delta < 1$, we have $y' \in B_{i_0}$. Put $y = \iota(y') \in T_{i_0} \subset U_{i_0}$. Then $d(x, y) < \varepsilon$. For any $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}^{\mathcal{U}}_n(x, y)$, we have $(U_{i_0}, U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}^{\mathcal{U}}_{n+1}(x, y)$. By $D^{\mathcal{H}_1}_{n+1}(x', y') < \delta$, we have

$$\big|\Phi_{i_n}\big(P_{i_n,x} \cap T_{i_n}\big) - \Phi_{i_n}\big(P_{i_n,y} \cap T_{i_n}\big)\big| < \delta\,.$$

Here we remark that $P_{i_n,x} \cap T_{i_n}$ consists of only one point. So by lemma 2.17, $w(P_{i_n,x}\,,\ P_{i_n,y}) < \varepsilon$. Hence $d^{\mathcal{A}}_n(x, y) < \varepsilon$. So $\iota(R)$ is an $(n, \varepsilon, \mathcal{A}, K)$-spanning set. Therefore

$$r^{\mathcal{A}}_n(\varepsilon, K) \leq \#\iota(R) \leq r^{\mathcal{H}_1}_{n+1}\big(\delta, \Phi(K)\big)\,.$$

$$\eta(K, \mathcal{F}) \leq \left[r^{\mathcal{H}_1}_n\big(\varepsilon_j, \Phi(K)\big)\right]_{j \in \mathbb{N}}\,.$$

Next we will show that $\eta(K,\mathcal{F}) \geq \left[s_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))\right]_{j\in\mathbb{N}}$. Fix a natural number n and a positive number ε. Take a positive number δ which satisfies lemma 2.18. Let $S \subset B$ be an $(n, \varepsilon, \mathcal{H}_1, \Phi(K))$-separating set with the maximum cardinality. Fix $i_0 \in \{1, \ldots, A\}$. Take two points x', $y' \in S \cap B_{i_0}$. Put $x = \iota(x')$ and $y = \iota(y')$. Then by $D_n^{\mathcal{H}_1}(x', y') \geq \epsilon$, there exists $(U_{i_1}, \ldots, U_{i_n}) \in \mathcal{C}_n^{\mathcal{U}}(x, y)$ such that

$$\left|\Phi_{i_n}\left(P_{i_n,x} \cap T_{i_n}\right) - \Phi_{i_n}\left(P_{i_n,y} \cap T_{i_n}\right)\right| \geq \varepsilon .$$

So by lemma 2.18, we have $w(P_{i_n,x}\,,\, P_{i_n,y}) \geq \delta$. Hence $d_n^{\mathcal{A}}(x,y) \geq \delta$. So $\iota(S \cap B_{i_0})$ is an $(n, \delta, \mathcal{A}, K)$-separating set. Therefore

$$s_n^{\mathcal{A}}(\delta, K) \geq \#\iota(S \cap B_{i_0}) = \#(S \cap B_{i_0}) .$$

So

$$A \cdot s_n^{\mathcal{A}}(\delta, K) \geq \#S = s_n^{\mathcal{H}_1}(\varepsilon, \Phi(K)) .$$

$$\eta(K,\mathcal{F}) \geq \left[s_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))\right]_{j\in\mathbb{N}} . \square$$

Ghys, Langevin and Walczak [GLW] defined the entropy $h(\mathcal{F},\mathcal{U})$ of a foliation $\mathcal{F}$ to be

$$h(\mathcal{F},\mathcal{U}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n^{\mathcal{H}_1}(\varepsilon, B) .$$

The choice of the equivalence class of sequences of monotone increasing functions in section 1 was chosen precisely so that the entropy of foliations is well-defined up to multiplication of positive numbers on the equivalence class. This gives the following corollary.

COROLLARY 2.19.— *The entropy of a foliation $\mathcal{F}$ on a compact manifold M is not zero if and only if $\eta(M,\mathcal{F}) \in \widetilde{\mathcal{E}}$ has quasi-exponential growth.*

By the theorem 2.16, we had $\eta(M,\mathcal{F}) \leq \left[e^{e^n}\right]$. But the expansion growth of foliations of class C^1 is dominated by the exactly exponential growth [GLW].

PROPOSITION 2.20.— *If $\mathcal{F}$ is a foliation of class C^1, then*

$$\eta(M, \mathcal{F}) \leq [e^n] .$$

Proof.— Let $C(\varepsilon)$ be as in lemma 2.14. Since $\overline{B} \subset \mathbb{R}^q$ is compact, there exists a positive number α such that

$$C(\varepsilon) \leq \frac{\alpha}{\varepsilon^q} .$$

On the other hand, since $\mathcal{F}$ is a foliation of class C^1, there exists a Lipschitz constant $\beta \geq 1$ such that for any $f \in \mathcal{H}_1$ and for any x, $y \in$ domain(f), $|f(x) - f(y)| \leq \beta|x - y|$. Therefore if $D_n^{\mathcal{H}_1}(x, y) \geq \varepsilon$ then

$$|x - y| \geq \frac{\varepsilon}{\beta^n} .$$

So

$$s_n^{\mathcal{H}_1}(\varepsilon, B) \leq C\left(\frac{\varepsilon}{\beta^n}\right) \leq \alpha \cdot \left(\frac{\beta^n}{\varepsilon}\right)^q$$

$$\eta(M, \mathcal{F}) = \left[s_n^{\mathcal{H}_1}(\varepsilon_j, B)\right]_{j \in \mathbb{N}} \leq \left[\frac{\alpha}{(\varepsilon_j)^q} \cdot \beta^{qn}\right]_{j \in \mathbb{N}} \leq [e^n] . \square$$

3. Codimension 1 case

In this section, we restrict ourselves to a transversely oriented codimension 1 foliation $\mathcal{F}$ of class C^0 on a compact manifold M. Take a 1-dimensional foliation $\mathcal{J}$ transverse to $\mathcal{F}$. (By [HH2], it exists.) Let $\mathcal{A} = (\mathcal{U}^s, \mathcal{U}, \mathcal{U}^b)$ be a triple of foliation covers of $(M, \mathcal{F})$. We may assume that $\mathcal{U}^b = \{(U_i^b, \varphi_i^b)\}_{i=1}^A$ is a bidistinguished foliation cover of $(\mathcal{F}, \mathcal{J})$. We use the notations which we defined in section 2. Put $T_i = \varphi_i^{-1}(\{0\} \times B_i)$. We note that $\mathcal{U}$ can be taken so that for $i \neq i'$, $\overline{T}_i \cap \overline{T}_{i'} = \emptyset$. Put $T = \bigcup_{i=1}^A T_i \subset M$. We identify $T \subset M$ with $B \subset \mathbb{R}$ by the map ι^{-1}. Fix a metric d on M. We may assume that for each i the metric on T_i induced by d coincides with the one induced from the Euclidean metric on $\mathbb{R}$. Let $\mathcal{H}_1$, $\mathcal{H}_n$ and $\mathcal{H}$ be as in section 2.

Now we define the growth of a leaf and the sum of growths of at most a countable number of leaves. For a leaf L and $y \in L \cap T$, put

$$\mathrm{gr}(L) = [\#\mathcal{H}_n(y)] \in \mathcal{E}\,.$$

For a set $\{L_j\}_{j\in\mathbb{N}}$ of at most a countable number of leaves and $y_j \in L_j \cap T$, put

$$\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) = [\#\mathcal{H}_n(y_1, \ldots, y_j)]_{j\in\mathbb{N}} \in \widetilde{\mathcal{E}}\,.$$

We show that $\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j)$ is independent of the choice of y_j and if $\{F_k\}_{k\in\mathbb{N}}$ is a set of at most a countable number of leaves such that $\{L_j\}_{j\in\mathbb{N}} = \{F_k\}_{k\in\mathbb{N}}$ then

$$\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) = \sum_{k\in\mathbb{N}} \mathrm{gr}(F_k)\,.$$

Let $z_k \in F_k \cap T$. For any $j \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$\{y_1, \ldots, y_j\} \subseteq \mathcal{H}_N(z_1, \ldots, z_k)\,.$$

Therefore

$$\#\mathcal{H}_n(y_1, \ldots, y_j) \le \#\mathcal{H}_{2n}(z_1, \ldots, z_k) \quad \text{for any } n \ge N\,.$$

We can take a large positive real number C such that

$$\#\mathcal{H}_n(y_1, \ldots, y_j) \le C \cdot \#\mathcal{H}_{2n}(z_1, \ldots, z_k) \quad \text{for any } n \in \mathbb{N}\,.$$

So

$$\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) \le \sum_{k\in\mathbb{N}} \mathrm{gr}(F_k)\,.$$

The converse inequality is similarly shown. Of course, if $\{L_j\}_{j\in\mathbb{N}} = \{F_1, \ldots, F_n\}$ then

$$\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) = \mathrm{gr}(F_1) + \cdots + \mathrm{gr}(F_N)\,.$$

Moreover it is easy to see that $\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j)$ is independent of the choice of a foliation cover.

We also define the growth of an open connected $\mathcal{F}$-saturated set Y.

$$\mathrm{gr}(Y) = \sum_{L \text{ is a leaf contained in } \delta Y} \mathrm{gr}(L) \in \widetilde{\mathcal{E}}\,.$$

Here δY is the set of border leaves of Y and δY consists of finitely many leaves (*cf.* [D]).

LEMMA 3.1.— *Let $K \subseteq M$ be an $\mathcal{F}$-saturated set. Let $\{L_j\}_{j\in\mathbb{N}}$ be a set of at most countable leaves which satisfies following two conditions.*

(1) $\bigcup_{j\in\mathbb{N}} L_j$ is dense in $\overline{K}$.

(2) Every leaf which is a border leaf of $M - \overline{K}$ is contained in $\{L_j\}_{j\in\mathbb{N}}$.

Then

$$\eta(K) \le \sum_{j\in\mathbb{N}} \mathrm{gr}(L_j)\,.$$

In particular,

$$\eta(M) \le [\#\mathcal{H}_n] \le [e^n]\,.$$

We remark that in codimensions greater than 1, this lemma does not hold.

Proof.— Fix $0 < \varepsilon < \min_{i\neq i'} d(\overline{T}_i, \overline{T}_{i'})$. Let $\{z_k\}_{k\in\mathbb{N}} = \left(\bigcup_{j\in\mathbb{N}} L_j\right) \cap T$. By (1), we can take an integer N satisfying the following (a) and (b).

(a) $\{z_1, \ldots, z_N\}$ is a $(\varepsilon/3)$-dense set in $\overline{K} \cap T$.

(b) If a component of $T - \overline{K}$ has a length more than $\varepsilon/3$, then its endpoints (except endpoints of T) are contained in $\{z_1, \ldots, z_N\}$.

We will show that $\mathcal{H}_n(z_1, \ldots, z_N)$ is an $(n, \varepsilon, \mathcal{H}_1, \overline{K} \cap T)$-spanning set. For any $x \in \overline{K} \cap T$, let y be a point of $\mathcal{H}_n(z_1, \ldots, z_N)$ which gives the minimum value of $d(x, y)$. We may assume that $x \le y$. First we show $D_n^{\mathcal{H}_1}(x, y) < \varepsilon$. Suppose the contrary. There exists $f \in \mathcal{H}_n$ such that $d\big(f(x), f(y)\big) \ge \varepsilon$. Since $\mathcal{U}$ is a distinguished foliation cover of $(\mathcal{F}, \mathcal{J})$, f is defined on $[\,x\,,\,y\,]$. Put

$$J = \left[\frac{2\,f(x) + f(y)}{3}\,,\ \frac{f(x) + 2\,f(y)}{3}\right]\,.$$

If $\overline{K} \cap J \neq \emptyset$ then by (a) there exists $z_k \in [f(x), f(y)) \cap \{z_1, \ldots, z_N\}$. If $\overline{K} \cap J = \emptyset$ then the length of the component of $T - \overline{K}$ containing J is more than $\varepsilon/3$. By (b) and $f(x)$, $f(y) \in \overline{K}$ we obtain $z_k \in [f(x), f(y)) \cap \{z_1, \ldots, z_N\}$. So $f^{-1}(z_k) \in [x, y) \cap \mathcal{H}_n(z_1, \ldots, z_N)$. This contradicts the choice of y. So $D_n^{\mathcal{H}_1}(x,y) < \varepsilon$. Hence we have

$$r_n^{\mathcal{H}_1}(\varepsilon, \overline{K} \cap T) \leq \#\mathcal{H}_n(z_1, \ldots, z_N) \leq N \cdot \#\mathcal{H}_n \leq N(nA^{2n} + 1).$$

Finally,

$$\begin{aligned} \eta(K) = \eta(\overline{K}) &\leq \left[\#\mathcal{H}_n(z_1, \ldots, z_j)\right]_{j \in \mathbb{N}} = \sum_{j \in \mathbb{N}} \mathrm{gr}(L_j) \\ &\leq \left[\#\mathcal{H}_n\right] \leq [e^n]. \ \square \end{aligned}$$

The above lemma is very useful. For example, it is easy to show that if $(M, \mathcal{F})$ is a foliated S^1-bundle over $\mathbb{T}^2$ then $\eta(M, \mathcal{F}) \leq [n^2]$.

Let L be a resilient leaf. A resilient leaf captures itself by a holonomy contraction. It is easy to see that $\eta(L) \geq [e^n]$ (*cf.* [GLM]). By lemma 3.1, we have $\eta(M) \leq [e^n]$. So we have the following lemma.

LEMMA 3.2.— *If L is a resilient leaf, then $\eta(L) = [e^n]$. Especially if Y is an open LMS (local minimal set, cf. [CC]) with holonomy, then $\eta(Y) = [e^n]$.*

Here Y is said to be without holonomy if each leaf contained in Y has trivial germinal holonomy.

Moreover by lemma 3.1, we have the following theorem. By this theorem, we can compute $\eta(Z)$ in case where Z is a semi-proper leaf or an ELMS (exceptional local minimal set (*cf.* [CC])).

THEOREM 3.3.— *Let Z be a $\mathcal{F}$-saturated set such that $\mathrm{Int}(\overline{Z}) = \emptyset$, then*

$$\eta(Z) = \sum_{L \text{ is a border leaf of } M - \overline{Z}} \mathrm{gr}(L).$$

Proof.— Let $\{L_j\}_{j \in \mathbb{N}}$ be a set of all border leaves of $M - \overline{Z}$. Since $\{L_j\}_{j \in \mathbb{N}}$ satisfies the hyperthesis of lemma 3.1, we have the inequality

$$\eta(Z) \leq \sum_{j \in \mathbb{N}} \mathrm{gr}(L_j).$$

We show $\eta(Z) \geq \sum_{j\in\mathbb{N}} \mathrm{gr}(L_j)$. Put $\{z_k\}_{k\in\mathbb{N}} = (\bigcup_{j\in\mathbb{N}} L_j)\cap T$. $\{z_k\}_{k\in\mathbb{N}}$ is written as a union $\{z_k\}_{k\in\mathbb{N}} = \{x_l\}_{l\in\mathbb{N}} \cup \{y_m\}_{m\in\mathbb{N}}$, where the positive (resp. negative) side of L_{x_l} (resp. L_{y_m}) is contained in $M - \overline{Z}$. For any positive integer N, we can find a positive number δ which satisfies the following conditions. For every $x_l \in \{x_1, \ldots, x_N\}$, the δ-neighborhood of x_l is contained in some U_i and its positive side is contained in $M - \overline{Z}$. Moreover we may take $\delta < \min_{i\neq i'} d(\overline{T}_i, \overline{T}_{i'})$ and small for $\mathcal{A}$.

We show that $\mathcal{H}_n(x_1, \ldots, x_N)$ is $(An, \delta, \mathcal{A}, \overline{Z})$-separating set. For v, $w \in \mathcal{H}_n(x_1, \ldots, x_N)$, we show $d^{\mathcal{A}}_{An}(v, w) \geq \delta$. Suppose that $d^{\mathcal{A}}_{An}(v, w) < \delta$. Here we remark that we are working with $d^{\mathcal{A}}_n$ and not with $D^{\mathcal{H}_1}_n$. By $d(v, w) < \delta$, there exists T_i such that $v, w \in T_i$. We may assume that $v \leq w$. We can represent $v = f(x_l)$ such that $f \in \mathcal{H}_n$ and $x_l \in \{x_1, \ldots, x_N\}$. By $d^{\mathcal{A}}_{An}(v, w) < \delta$, we can apply the same discussion as that of lemma 2.5. Hence we obtain $(U_{i_1}, \ldots, U_{i_{An}}) \in C^{\mathcal{U}}_{An}(v, w)$ along f^{-1} such that there exists $x \in P_{An,w}$ such that $d(x_l, x) < \delta$. This implies $x \in M - \overline{Z}$. This contradicts the choice of $\{x_1, \ldots, x_N\}$. So $d^{\mathcal{A}}_{An}(v, w) \geq \delta$ and

$$s^{\mathcal{A}}_{An}(\delta, \overline{Z}) \geq \#\mathcal{H}_n(x_1, \ldots, x_N).$$

Now for all $N \in \mathbb{N}$, we take $N_1, N_2 \in \mathbb{N}$ such that

$$\{z_1, \ldots, z_N\} \subseteq \{x_1, \ldots, x_{N_1}\} \cup \{y_1, \ldots, y_{N_2}\}$$

and take $\delta > 0$ which satisfies the above condition with respect to $\{x_1, \ldots, x_{N_1}\}$ and $\{y_1, \ldots, y_{N_2}\}$. Then,

$$\begin{aligned} 2s^{\mathcal{A}}_{An}(\delta, \overline{Z}) &\geq \#\mathcal{H}_n(x_1, \ldots, x_{N_1}) + \#\mathcal{H}_n(y_1, \ldots, y_{N_2}) \\ &\geq \#\mathcal{H}_n(z_1, \ldots, z_N). \end{aligned}$$

Therefore

$$\eta(Z) = \eta(\overline{Z}) \geq [\#\mathcal{H}_n(z_1, \ldots, z_j)]_{j\in\mathbb{N}} = \sum_{j\in\mathbb{N}} \mathrm{gr}(L_j). \square$$

COROLLARY 3.4.— *Let L be a totally proper leaf (i.e. $\overline{L}$ consists of proper leaves). Then*

$$\eta(L) = \mathrm{gr}(L).$$

Proof.— By theorem 3.3,

$$\eta(L) = \sum_{F \text{ is a leaf of } \overline{L}} \mathrm{gr}(F).$$

It is easy to be seen that

$$\sum_{F \text{ is a leaf of } \overline{L}} \mathrm{gr}(F) \geq \mathrm{gr}(L).$$

We will show that

$$\sum_{F \text{ is a leaf of } \overline{L}} \mathrm{gr}(F) \leq \mathrm{gr}(L).$$

Fix $z \in L \cap T$. Let $\{y_j\}_{j \in \mathbb{N}} = (\overline{L} - L) \cap T$. Fix $j \in \mathbb{N}$. Let F be a leaf containing y_j. We may assume that L accumulates F from the negative side. For each $y \in F \cap T$, there exists $n \in \mathbb{N}$ and $f_y \in \mathcal{H}_n$ such that $y = f_y(y_j) \in \mathcal{H}_n(y_j) - \mathcal{H}_{n-1}(y_j)$. By [D], there exists $(w, y_j] \subset T$ such that $(w, y_j] \subset \mathrm{domain}(f_y)$ for any $y \in F \cap T$. Since L accumulates F from the negative side, we can take $z_j \in (w, y_j) \cap \mathcal{H}_{N_j}(z)$ for a large number N_j. Then,

$$\#\mathcal{H}_n(y_j) \leq \#\mathcal{H}_n(z_j) \leq \#\mathcal{H}_{n+N_j}(z).$$

Now for any $j \in \mathbb{N}$, we can take a large number C such that

$$\begin{aligned} \#\mathcal{H}_n(z, y_1, \ldots, y_j) &\leq \#\mathcal{H}_n(z) + \#\mathcal{H}_n(y_1) + \cdots + \#\mathcal{H}_n(y_j) \\ &\leq C \cdot \#H_{2n}(z) \end{aligned}$$

for any $n \in \mathbb{N}$. Therefore,

$$\sum_{F \text{ is a leaf of } \overline{L}} \mathrm{gr}(F) \leq \mathrm{gr}(L). \ \square$$

Next we consider an open connected $\mathcal{F}$-saturated set Y. If Y is an open LMS with holonomy, then by lemma 3.2, we have $\eta(Y) = [e^n]$. So we consider the case where Y is without holonomy. If Y is trivial at infinity, we can compute the expansion growth of $(M, \mathcal{F})$ on Y.

THEOREM 3.5. — *Let Y be an open connected $\mathcal{F}$-saturated set without holonomy trivial at infinity.*

(1.1) If $Y = M$ and $\mathcal{F}$ has no EMS (exceptional minimal set) then $\eta(Y) = [1]$.

(1.2) If $Y = M$ and $\mathcal{F}$ has an EMS then $\eta(Y) = [n^{r-1}]$.

(2.0) If $Y \neq M$ and $r = 0$ then $\eta(Y) = \eta(\delta Y)$.

(2.1) If $Y \neq M$, $r \geq 1$ and $\mathcal{F}|_Y$ has no EMS then

$$\eta(Y) = \eta(\delta Y) + [n] \cdot \mathrm{gr}(Y)\,.$$

(2.2) If $Y \neq M$ and $\mathcal{F}|_Y$ has an EMS then $\eta(Y) = \eta(\delta Y) + [n^r] \cdot \mathrm{gr}(Y)$.

Here r is the rank of the image of the Novikov transformation of $\pi_1(Y)$ in $\mathrm{Homeo}_+(\mathbf{R})$ *(cf. [I]).*

For example, if $(M, \mathcal{F})$ is a bundle foliation over S^1 or an irrational foliation on $\mathbb{T}^2$ then $\eta(M, \mathcal{F}) = [1]$ by (1.1) of the above theorem. If $(\mathbb{T}^2, \mathcal{F})$ is a Denjoy foliation then $\eta(\mathbb{T}^2, \mathcal{F}) = [n]$ by (1.2) of the above theorem. If $(M, \mathcal{F})$ is a Reeb foliation then $\eta(M, \mathcal{F}) = [n]$ by (2.1) of the above theorem and proposition 2.9. We remark that if Y is not trivial at infinity in the assumption of the above theorem then we see that $\eta(Y)$ is complicated (*cf.* the proof of theorem 3.7).

Here an open connected $\mathcal{F}$-saturated set is said to be trivial at infinity if there exists a nuclear-arm decomposition (*cf.* [D]) such that $\mathcal{F}$ is a product foliation in each arm. If $Y = M$ then the condition where Y is trivial at infinity is vacuous.

We remark that since δY satisfies the assumption of theorem 3.3, $\eta(\delta Y)$ is represented by the sum of growths of border leaves of $M - \partial Y$.

We only prove (2.1) and (2.2). The proof of (1.1), (2.0) and (1.2) are similar and easier to those of (2.1) and (2.2) respectively. We remark that $(Y, \mathcal{F}|_Y)$ in the case (2.0) is a product foliation of a border leaf and an open interval.

Let Y be as in (2.1) or (2.2) of theorem. We fix a nuclear-arm decomposition of Y (*cf.* [D]):

$$Y = X \cup K_1 \cup \cdots \cup K_s$$

where X is a nucleus and K_m ($m = 1, \ldots, s$) is an arm. We may assume that $\mathcal{J}|_{K_m}$ is a product foliation. Since Y is trivial at infinity, we may assume that $\mathcal{F}\big|_{K_m}$ is a product foliation. Moreover we may assume that ∂T does not intersect any K_m.

Let $\{I_i\}_{i\in\mathbb{N}}$ be the set of all components of $Y \cap T$. Put $I_i = (a_i, b_i)$ $(i \in \mathbb{N})$. We may assume that there exists a map κ of $\mathbb{N}$ such that

$$\bigcup_{i=1}^{\kappa(n)} I_i \subset \bigcup_{P\in\mathcal{P}_n^{\mathcal{U}}(X)} P$$

and

$$\bigcup_{i>\kappa(n)} I_i \cap \bigcup_{P\in\mathcal{P}_n^{\mathcal{U}}(X)} P = \emptyset$$

for any $n \in \mathbb{N}$. We can easily see that $[\kappa(n)] = \mathrm{gr}(Y)$. Put

$$\mathcal{G}_1 = \left\{ f\big|_{\mathrm{domain}(f)\cap I_i} \,;\, f \in \mathcal{H}_1\,,\ i \in \mathbb{N} \right\}\,,$$

$$\mathcal{G}_n = \{f_n \circ \cdots \circ f_1 \,;\, f_l \in \mathcal{G}_1\} \quad (n \in \mathbb{N})$$

and

$$\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n\,.$$

Since $\mathcal{F}\big|_Y$ is without holonomy, $\pi_1(Y)$ acts freely on each leaf J of $\mathcal{J}\big|_Y$. So we obtain the subgroup G_J of $\mathrm{Homeo}_+(J)$ induced by the action of $\pi_1(Y)$. Since Y is trivial at infinity, G_J is finitely generated. Since G_J is free, there exists a monotone increasing continuous map h_J of J to $\mathbb{R}$ (resp. S^1) such that the subgroup $G_{\mathbb{R}}$ (resp. G_{S^1}) of $\mathrm{Homeo}_+(\mathbb{R})$ (resp. $\mathrm{Homeo}_+(S^1)$) induced by h_J and G_J is a subgroup of translations of $\mathbb{R}$ (resp. rotations of S^1) of rank r (resp. $r-1$). We remark that in the case (2.1), h_J is a homeomorphism. We identify the group of translations of $\mathbb{R}$ with $\mathbb{R}$. We may assume that $G_{\mathbb{R}}$ is generated by positive numbers $\alpha_1, \ldots, \alpha_r$.

By the above consideration, there exists a monotone increasing map h_i $(i \in \mathbb{N})$ of I_i into $\mathbb{R}$ satisfying the following condition. For any $f \in \mathcal{G}$, there exists $\alpha_f \in G_{\mathbb{R}}$ such that

$$h\big(f(x)\big) = h(x) + \alpha_f \quad \text{for any } x \in \mathrm{domain}(f)$$

where $h(x) = h_i(x)$ if $x \in I_i$. Moreover since Y is trivial at infinity, we may assume that $\alpha_f = 0$ for any $f \in \mathcal{G}_1$ except for finitely many elements. So we can take a large real number α such that

$$\alpha > \max_{f \in \mathcal{G}_1} |\alpha_f| \,.$$

Proof of (2.1) of theorem 3.5

We are going to show that

$$\eta(Y) \geq \eta(\delta Y) + [n]\,\mathrm{gr}(Y)\,.$$

Obsviously $\eta(Y) \geq \eta(\delta Y)$. Fix a leaf $F \subseteq \delta Y$. We show that $\eta(Y) \geq [n]\,\mathrm{gr}(F)$. We may assume that the negative side of F is contained in Y and $b_1 \in \partial I_1$ is a point of F. For each $n \in \mathbb{N}$ and for each $b \in \mathcal{H}_n(b_1) - \mathcal{H}_{n-1}(b_1)$, we fix $f_b \in \mathcal{H}_n$ such that $b = f_b(b_1)$. Since Y is trivial at infinity, there exists $c \in I_1$ such that f_b is defined on $[\,c\,,\, b_1\,]$ for any $b \in \mathcal{H}(b_1)$. By $r \geq 1$, there exists a loop γ based on b_1 contained in F such that the holonomy map f_γ of I_1 induced by γ is a contraction to b_1. Here we may assume that f_γ is defined on $[\,c\,,\, b_1\,]$. Take a large natural number N such that $f_\gamma \in \mathcal{H}_N$.

Take a positive real number δ satisfying the following conditions:

$$\delta < d\left(c\,,\, \bigcup_{P \in \mathcal{P}_1^{\mathcal{U}}(f_\gamma(c))} P^b\right),$$

$$\delta < \min_{i \neq i'} d(\overline{T}_i, \overline{T}_{i'})\,,$$

and δ is small for $\mathcal{A}$. For $n \in \mathbb{N}$, put

$$S_n = \left\{ f_b(f_\gamma^l(c))\,;\, b \in \mathcal{H}_n(b_1)\,,\ l = 1,\, \ldots,\, n \right\}.$$

Then we can deduce that S_n is an $\big(A(N+1)n,\, \delta,\, \mathcal{A},\, Y\big)$-separating set. We remark that we are working not with $D_n^{\mathcal{H}_1}$ by with $d_n^{\mathcal{A}}$ (*cf.* lemma 2.5 and theorem 3.3). So

$$s^{\mathcal{A}}_{A(N+1)n}(\delta, Y) \geq \# S_n \geq n \cdot \# \mathcal{H}_n(b_1)\,.$$

Therefore

$$\eta(Y) \geq [n]\,\mathrm{gr}(F)\,.$$

Hence

$$\eta(Y) \geq \eta(\delta Y) + [n]\,\mathrm{gr}(Y)\,.$$

Next we show that

$$\eta(Y) \leq \eta(\delta Y) + [n]\,\mathrm{gr}(Y)\,.$$

Fix a positive real number ε. There exists a large integer n_0 such that $|I_i| < \varepsilon$ for any $i > \kappa(n_0)$. We take a positive real number δ such that for any $i \leq \kappa(n_0)$ and for any x, $y \in I_i$, if $|x - y| \geq \varepsilon$ then $\big|h_i(x) - h_i(y)\big| \geq \delta$. We take points $z_1, \ldots, z_N \in \mathbb{R}$ satisfying the following conditions:

$$\begin{cases} z_1 \leq h_i\Big(a_i + \dfrac{\varepsilon}{2}\Big) & \text{if } |I_i| \geq \varepsilon. \\ z_N \geq h_i\Big(b_i - \dfrac{\varepsilon}{2}\Big) & \text{if } |I_i| \geq \varepsilon. \\ 0 \leq z_{k+1} - z_k < \delta\,. & \\ z_N - z_1 \geq \alpha\,. & \\ h_i(I_i) \cap \{z_1, \ldots, z_N\} \neq \emptyset & \text{for any } i \in \mathbb{N}. \end{cases}$$

Fix a positive integer n. Put

$$R_n = \bigcup_{i=1}^{\kappa(n+n_0)} h_i^{-1}\left(\big\{z_k + l\alpha\,;\, k = 1, \ldots, N,\ l = -n, \ldots, n\big\}\right)\,.$$

Let R'_n be an $(n, \varepsilon, \mathcal{H}_1, \delta Y \cap T)$-spanning set with the minimum cardinality.

We will show that $R_n \cup R'_n$ is an $(n,\ 2\varepsilon,\ \mathcal{H}_1,\ Y \cap T)$-spanning set. Take any point x of $Y \cap T$. Let I_i be a component of $Y \cap T$ containing x.

First we consider the case where $i \leq \kappa(n + n_0)$. Let $y \in R_n$ be a point which gives the minimum value of $|x - y|$. We remark that y is a point of I_i. We may assume that $x \leq y$. We show that $D_n^{\mathcal{H}_1}(x, y) < \varepsilon$. Suppose $D_n^{\mathcal{H}_1}(x, y) \geq \varepsilon$. Then there exists $f \in \mathcal{G}_n$ such that $\big|f(x) - f(y)\big| \geq \varepsilon$. Let I_j be a component of $Y \cap T$ containing $f(x)$ and $f(y)$. By

$$|I_j| > \big|f(x) - f(y)\big| \geq \varepsilon\,,$$

we have $j \leq \kappa(n_0)$. Then

$$\big(f(x), f(y)\big) \cap \Big(a_j + \frac{\varepsilon}{2}\,,\, b_j - \frac{\varepsilon}{2}\Big) \neq \emptyset\,.$$

So

$$\left|h_j\big(f(x)\big) - h_j\big(f(y)\big)\right| \geq \delta$$

and

$$\big(h_j\big(f(x)\big)\,,\, h_j\big(f(y)\big)\big) \cap (z_1, z_N) \neq \emptyset\,.$$

On the other hand, $h_j\big(f(x)\big) = h_i(x) + \alpha_f$ and $h_j\big(f(y)\big) = h_i(y) + \alpha_f$. So $\left|h_i(x) - h_i(y)\right| \geq \delta$. By $f \in \mathcal{G}_n$, we have $|\alpha_f| \leq n\alpha$. So

$$\big(h_i(x)\,,\, h_i(y)\big) \cap (z_1 - n\alpha\,, z_N + n\alpha) \neq \emptyset\,.$$

Therefore there exists a point

$$z \in \left\{z_k + l\alpha\,;\, k = 1, \ldots, N,\, l = -n, \ldots, n\right\} \cap \big(h_i(x)\,,\, h_i(y)\big)\,.$$

So

$$h_i^{-1}(z) \in R_n \cap (x, y)\,.$$

This contradicts the choice of y. Hence $D_n^{\mathcal{H}_1}(x, y) < \varepsilon$.

Next we consider the case where $i > \kappa(n + n_0)$. Obviously, $|x - b_i| < \varepsilon$. By $b_i \in \delta Y \cap T$, there exists $y \in R'_n$ such that $D_n^{\mathcal{H}_1}(b_i, y) < \varepsilon$. We show that $D_n^{\mathcal{H}_1}(x, y) < 2\varepsilon$. Given any $f \in \mathcal{H}_n$ such that x, $y \in \text{domain}(f)$.

By $f \in \mathcal{H}_n$ and the choice of $\kappa(n)$, $f(x)$ is contained in the arm K_m containing x. So f is defined on $[\,a_i\,,\, b_i\,]$ and $\big(f(a_i)\,,\, f(b_i)\big) \in \left\{I_i\right\}_{i > \kappa(n_0)}$. So

$$\left|f(x) - f(b_i)\right| < \left|f(a_i) - f(b_i)\right| < \varepsilon\,.$$

By $D_n^{\mathcal{H}_1}(b_i, y) < \varepsilon$, we have $\left|f(b_i) - f(y)\right| < \varepsilon$. Therefore $\left|f(x) - f(y)\right| < 2\varepsilon$. So $D_n^{\mathcal{H}_1}(x, y) < 2\varepsilon$.

By the above two results, $R_n \cup R'_n$ is an $(n, 2\varepsilon, \mathcal{H}_1, Y \cap T)$-spanning set. Hence

$$\begin{aligned} r_n^{\mathcal{H}_1}(2\varepsilon, Y \cap T) &\leq \#R_n + \#R'_n \\ &\leq r_n^{\mathcal{H}_1}(\varepsilon, \delta Y \cap T) + N(2n + 1) \cdot \kappa(n + n_0)\,. \end{aligned}$$

Then we can take a large positive real number C such that

$$r_n^{\mathcal{H}_1}(2\varepsilon, Y \cap T) \leq r_n^{\mathcal{H}_1}(\varepsilon, \delta Y \cap T) + Cn \cdot \kappa(2n)$$

for any $n \in \mathbb{N}$. So

$$\eta(Y) \leq \eta(\delta Y) + [n]\,\mathrm{gr}(Y)\,. \square$$

Next we prove (2.2) of theorem 3.5. Before proving (2.2), we show the following proposition.

PROPOSITION 3.6.— *Let Y be as in (2.1) or (2.2) of theorem 3.5. If L is a leaf contained in Y, then*

$$\mathrm{gr}(L) = [n^r]\,\mathrm{gr}(Y)\,.$$

Proof.— We show that $\mathrm{gr}(L) \leq [n^r]\,\mathrm{gr}(Y)$. We remark that if $h_i(x) = h_i(y)$ for $x \neq y \in I_i$, then the leaf containing x and the leaf containing y are distinct. Since $\{\alpha_f\,;\, f \in \mathcal{G}_1\}$ is a finite set, we can take a large natural number N such that

$$\{\alpha_f\,;\, f \in \mathcal{G}_1\} \subseteq \left\{\sum_{k=1}^{r} t_k\alpha_k\,;\, t_k = -N, \ldots, N\right\}.$$

Fix a point $c \in L \cap I_1$. Let n be a natural number. Take any point $f(c) \in \mathcal{H}_n(c)$ ($f \in \mathcal{G}_n$). Then we have $f(c) \in \bigcup_{i=1}^{\kappa(n)} I_i$ and $h\big(f(c)\big) = h(c) + \alpha_f$. By $f \in \mathcal{G}_n$, we have

$$\alpha_f \in \left\{\sum_{k=1}^{r} t_k\alpha_k\,;\, t_k = -Nn, \ldots, Nn\right\}.$$

So

$$h\big(f(c)\big) \in \left\{h(c) + \sum_{k=1}^{r} t_k\alpha_k\,;\, t_k = -Nn, \ldots, Nn\right\}.$$

This implies that

$$\#\mathcal{H}_n(c) \leq \big(2Nn + 1\big)^r \cdot \kappa(n)\,.$$

So

$$\mathrm{gr}(L) \leq [n^r]\,\mathrm{gr}(Y)\,.$$

Next we show that $\mathrm{gr}(L) \geq [n^r]\,\mathrm{gr}(Y)$. Fix a leaf F contained in δY. We have only to show that $\mathrm{gr}(L) \geq [n^r]\,\mathrm{gr}(F)$. we may assume that the negative side of F is contained in Y and $b_1 \in \partial I_1$ is a point of F. For any $n \in \mathbb{N}$ and for any $b \in \mathcal{H}_n(b_1) - \mathcal{H}_{n-1}(b_1)$, we fix $f_b \in \mathcal{H}_n$ such that $b = f_b(b_1)$. Since Y is trivial at infinity, there exists $c \in L \cap I_1$ such that f_b is defined on $[\,c\,,\,b_1\,]$ for any $b \in \mathcal{H}(b_1)$. There exists a loop γ based on b_1 contained in F such that the holonomy map f_γ of I_1 induced by γ is a contraction to b_1. Here we may assume that f_γ is defined on $[\,c\,,\,b_1\,]$. Put $\alpha' = h_1\big(f_\gamma(c)\big) - h_1(c)$. If necessary, by replacing γ with γ^m, we can assume that

$$\alpha' > \alpha_1,\,\ldots,\,\alpha_r\,.$$

We take a large positive integer N satisfying the following conditions.

$$f_\gamma \in \mathcal{H}_N$$

and for any $x \in \big[\,c\,,\,f_\gamma(c)\big]$ and for any $k \in \{1,\,\ldots,\,r\}$, there exists a $f \in \mathcal{H}_N$ such that

$$h_1\big(f(x)\big) = h_1(x) + \alpha_k\,.$$

Let α_{k_l} $(l = 1,\,\ldots,\,rn)$ be a element of $\{0,\,\alpha_1,\,\ldots,\,\alpha_r\}$. We can take $t_l \in \{0,\,1\}$ satisfying the following condition.

$$\beta_l = \beta_{l-1} + \alpha_{k_l} - t_l\alpha' \in [\,0\,,\,\alpha'\,]$$

for $l = 1,\,\ldots,\,rn$, where $\beta_0 = 0$. Then there exists $f_1,\,\ldots,\,f_{rn} \in \mathcal{H}_N$ such that

$$f_\gamma^{-t_l} \circ f_l \circ \cdots \circ f_\gamma^{-t_1} \circ f_1(c) \in \big[\,c\,,\,f_\gamma(c)\big]$$

and

$$h_1 \circ f_\gamma^{-t_l} \circ f_l \circ \cdots \circ f_\gamma^{-t_1} \circ f_1(c) = h_1(c) + \beta_l$$

for any $l \in \{0,\,1,\,\ldots,\,rn\}$. Then

$$\begin{aligned} h_1 \circ f_\gamma^{t_1+\cdots+t_{rn}} \circ f_\gamma^{-t_{rn}} \circ f_{rn} \circ \cdots \circ f_\gamma^{-t_1} \circ f_1(c) &= \\ = h_1(c) + \alpha_{k_1} + \cdots + \alpha_{k_{rn}}\,. \end{aligned}$$

The cardinality of the set of elements of the form

$$f_\gamma^{t_1+\cdots+t_{rn}} \circ f_\gamma^{-t_{rn}} \circ f_{rn} \circ \cdots \circ f_\gamma^{-t_1} \circ f_1(c)$$

by this construction is more than n^r. So

$$\#\big(\mathcal{H}_{3Nrn}(c) \cap I_1\big) \geq n^r .$$

Therefore the cardinality of the set of elements of the form

$$f_b \circ f_\gamma^{t_1+\cdots+t_{rn}} \circ f_\gamma^{-t_{rn}} \circ f_{rn} \circ \cdots \circ f_\gamma^{-t_1} \circ f_1(c)$$

$(b \in \mathcal{H}_n(b_1))$ is more than $n^r \cdot \#\mathcal{H}_n(b_1)$. So

$$\#\mathcal{H}_{(3Nr+1)n}(c) \geq n^r \cdot \#\mathcal{H}_n(b_1) .$$

Therefore

$$\mathrm{gr}(L) \geq [n^r]\, \mathrm{gr}(F) .$$

Hence

$$\mathrm{gr}(L) \geq [n^r]\, \mathrm{gr}(Y) . \square$$

Proof of (2.2) of theorem 3.5

Obviously, $\eta(Y) \geq \eta(\delta Y)$. Let Z be an EMS of $\mathcal{F}\big|_Y$. By proposition 3.6, a semi-proper exceptional leaf L contained in Z has growth $[n^r]\, \mathrm{gr}(Y)$. So by theorem 3.3,

$$\eta(Z) \geq \mathrm{gr}(L) = [n^r]\, \mathrm{gr}(Y) .$$

Therefore

$$\eta(Y) \geq \eta(\delta Y) + \eta(Z) \geq \eta(\delta Y) + [n^r]\, \mathrm{gr}(Y) .$$

We show the converse inequality. Let $\{F_k\}_{k\in\mathbb{N}}$ be the set of all border leaves of $M - \partial Y$. By theorem 3.3,

$$\eta(\delta Y) = \sum_{k\in\mathbb{N}} \mathrm{gr}(F_k) .$$

Let $\{L_j\}_{j\in\mathbb{N}}$ be a set of leaves contained in Y such that $\bigcup_{j\in\mathbb{N}} L_j$ is dense in Y. By proposition 3.6, $\mathrm{gr}(L_j) = [n^r]\, \mathrm{gr}(Y)$. Moreover from the proof of proposition 3.6, we can easily deduce that

$$\sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) = [n^r]\, \mathrm{gr}(Y) .$$

$\{F_k\}_{k\in\mathbb{N}} \cup \{L_j\}_{j\in\mathbb{N}}$ satisfies the assumption of lemma 3.1. So

$$\eta(Y) \leq \sum_{k\in\mathbb{N}} \mathrm{gr}(F_k) + \sum_{j\in\mathbb{N}} \mathrm{gr}(L_j) = \eta(\delta Y) + [n^r]\, \mathrm{gr}(Y)\,. \square$$

Finally we will construct foliations which have various expansion growths.

THEOREM 3.7.— *Let g be a map of $\mathbb{N}\cup\{0\}$ to $\mathbb{N}$ such that $g(0)=1$ and*

$$g(n) \leq g(n+1) \leq 2g(n) \quad \textit{for any } n \in \mathbb{N}\cup\{0\}\,.$$

Then there exists a foliated manifold $(M,\mathcal{F})$ such that

$$\eta(M,\mathcal{F}) = \left[n^2 g(n)\right].$$

Proof.— Put $I = [\,0\,,1\,]$. Let f_1 be an orientation-preserving homeomorphism of I such that $f_1(x) > x$ for any $x \in (0,1)$. Take $a \in (0,1)$ and put $b = f_1(a)$. Let f_3 be an orientation-preserving homeomorphism of $[\,a\,,\,b\,]$ such that $f_3(x) > x$ for any $x \in (a,b)$. We define the orientation-preserving homeomorphism f_2 of I as follows:

$$f_2(0) = 0\,, \quad f_2(1) = 1$$

and

$$f_2\big|_{f_1^m([a,b])} = f_1^m \circ f_3^{g(|m|)} \circ f_1^{-m}\big|_{f_1^m([a,b])} \quad \text{for any } m \in \mathbb{Z}\,.$$

Let Σ_2 be a closed surface of genus 2. We can represent the fundamental group of Σ_2 as follows:

$$\pi_1(\Sigma_2) = \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4\,;\, [\gamma_1,\gamma_2][\gamma_3,\gamma_4] = 1 \right\rangle.$$

We define a homomorphism Ψ of $\pi_1(\Sigma_2)$ to $\mathrm{Homeo}_+(I)$ such that

$$\Psi(\gamma_1) = f_1\,, \quad \Psi(\gamma_3) = f_2\,, \quad \text{and} \quad \Psi(\gamma_2) = \Psi(\gamma_4) = \mathrm{id}_I\,.$$

Then we obtain the suspension foliation $\mathcal{F}_\Psi$ of Ψ on $\Sigma_2 \times I$.

We will show that

$$\eta(\Sigma_2 \times I\,, \mathcal{F}_\Psi) \geq \left[n^2 g(n)\right].$$

Fix a fiber I of $\Sigma_2 \times I$. Considering the proof of theorem 2.16, we have only to work with the group of homeomorphisms of the fiber I. Put

$$\mathcal{H}_1 = \left\{ \mathrm{id}_I,\, f_1,\, f_1^{-1},\, f_2,\, f_2^{-1} \right\}.$$

Take a point $c \in (a, b)$. Put

$$\delta = \min\left\{ \left| f_3^k(c) - c \right| ;\, k \in \mathbb{Z} - \{0\} \right\}.$$

Let n be a natural number. Put

$$S_n = \left\{ f_1^l\big(f_3^k(c)\big) \,;\, k = -ng(n),\, \ldots,\, ng(n)\,,\; l = -n,\, \ldots,\, n \right\}.$$

Then we can easily show that S_n is a $(4n, \delta, \mathcal{H}_1, I)$-separating set. So

$$s_{4n}^{\mathcal{H}_1}(\delta, I) \geq \# S_n = (2n+1)\big(2ng(n)+1\big)$$
$$\eta(\Sigma_2 \times I\,,\, \mathcal{F}_\Psi) \geq \left[n^2 g(n) \right].$$

We show the converse inequality. Let ε be a positive number. Let $z_1, \ldots, z_N \in I$ be an $(\varepsilon/2)$-dense set of I. Put

$$R_n = \mathcal{H}_n\big(\{z_1, \ldots, z_N\}\big).$$

Then we can easily show that R_n is an $(n, \varepsilon, \mathcal{H}_1, I)$-spanning set. So

$$r_n^{\mathcal{H}_1}(\varepsilon, I) \leq \# R_n \leq N(2n+1)\big(2ng(n)+1\big)$$
$$\eta(\Sigma_2 \times I\,,\, \mathcal{F}_\Psi) \leq \left[n^2 g(n) \right]. \square$$

We remark that the $\mathcal{F}_\Psi$-saturation of (a, b) is an open connected set without holonomy but not trivial at infinity.

By this theorem, we see that $\eta_l^l(K, \mathcal{F})$ and $\eta_l^{l+1}(K, \mathcal{F})$ $(l \in \mathbb{N})$ defined in corollary 2.8 are non-trivial. For example, if we let $g(n)$ be the integer part of n^α $(\alpha \geq 0)$, then $\eta_1^1(M, \mathcal{F}) = 2 + \alpha$. If we let $g(n)$ be the integer part of e^{n^α} $(0 \leq \alpha \leq 1)$, then $\eta_1^2(M, \mathcal{F}) = \alpha$. If we let $g(n)$ be the integer part of $e^{(\log n)^\alpha}$ $(\alpha \geq 1)$, then $\eta_2^2(M, \mathcal{F}) = \alpha$. So we obtained many numerical topological invariants for foliations.

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