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A unified approach to various orthogonalities(*)

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RÉSUMÉ. — Les polynômes orthogonaux vectoriels de dimension d sont un cas particulier des polynômes biorthogonaux. On définit d'abord les polynômes orthogonaux de dimension $d = -1$. On donne des relations de récurrence et une formule de type Christoffel-Darboux. Un théorème de Shohat-Favard est démontré. Ces polynômes sont, en fait, ceux qui apparaissent dans les approximants de Laurent-Padé et dans ceux de Padé en deux points. La liaison avec l'orthogonalité usuelle est explicitée. Ensuite, on montre que les polynômes orthogonaux sur le cercle unité sont un cas particulier des polynômes orthogonaux vectoriels de dimension -1. Ainsi, puisque les polynômes orthogonaux vectoriels de dimension $d = 1$ sont les polynômes habituels, plusieurs résultats connus sur diverses orthogonalités sont retrouvés dans un cadre unifié et parfois généralisés.

ABSTRACT. — Vector orthogonal polynomials of dimension d are a particular case of biorthogonal polynomials. Vector orthogonal polynomials of dimension $d = -1$ are first defined. Recurrence relations and a Christoffel-Darboux-type formula are given. A Shohat-Favard theorem is proved. These polynomials are, in fact, those appearing in Laurent-Padé and two-point Padé approximants. The link with usual orthogonality is explicited. Then it is showed that orthogonal polynomials on the unit circle are a particular case of vector orthogonal polynomials of dimension -1. Thus, since vector orthogonal polynomials of dimension $d = 1$ are the usual ones, many known results about various orthogonalities are recovered in a unified framework and sometimes generalized.

KEY-WORDS : Orthogonal polynomials, biorthogonality.

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1. Biorthogonal polynomials

Let $L_0, L_1, \ldots$ be linearly independent linear functionals on the space of complex polynomials and let us set

$$L_i(x^j) = c_{ij}\,.$$

The number c_{ij} are complex numbers.

Adjacent families of (formal) biorthogonal polynomials with respect to the family $\{L_i\}$ were defined in [4]. They are given by

$$P_k^{(i,j)}(x) = D_k^{(i,j)} \begin{vmatrix} c_{ij} & \cdots & c_{i,j+k} \\ \vdots & \ddots & \vdots \\ c_{i+k-1,j} & \cdots & c_{i+k-1,j+k} \\ 1 & \cdots & x^k \end{vmatrix}$$

where $D_k^{(i,j)}$ is an arbitrary nonzero constant. These polynomials satisfy the biorthogonality conditions

$$L_p\big(x^j P_k^{(i,j)}(x)\big) = 0\,, \quad \text{for } p = i, \ldots, i+k-1\,. \tag{1}$$

Assuming that $P_k^{(i,j)}$ has the exact degree k, which is equivalent to the condition

$$H_k^{(i,j)} = \begin{vmatrix} c_{ij} & \cdots & c_{i,j+k-1} \\ \vdots & \ddots & \vdots \\ c_{i+k-1,j} & \cdots & c_{i+k-1,j+k-1} \end{vmatrix} \neq 0$$

we shall consider in the sequel the monic polynomials $P_k^{(i,j)}$ that is $D_k^{(i,j)} = 1/H_k^{(i,j)}$. It can be proved that these polynomials are related by the following recurrence relations

$$P_k^{(i,j)}(x) = xP_{k-1}^{(i,j+1)}(x) - \frac{L_{i+k-1}\big(x^{j+1} P_{k-1}^{(i,j+1)}\big)}{L_{i+k-1}\big(x^j P_{k-1}^{(i,j)}\big)}\, P_{k-1}^{(i,j)}(x) \tag{2}$$

$$P_k^{(i,j)}(x) = xP_{k-1}^{(i+1,j+1)}(x) - \frac{L_i\big(x^{j+1} P_{k-1}^{(i+1,j+1)}\big)}{L_i\big(x^j P_{k-1}^{(i+1,j)}\big)}\, P_{k-1}^{(i+1,j)}(x) \tag{3}$$

$$P_k^{(i,j)}(x) = P_k^{(i+1,j)}(x) - \frac{L_i\left(x^j P_k^{(i+1,j)}\right)}{L_i\left(x^j P_{k-1}^{(i+1,j)}\right)} P_{k-1}^{(i+1,j)}(x) \tag{4}$$

$$P_k^{(i+1,j)}(x) = P_k^{(i,j)}(x) - \frac{L_{i+k}\left(x^j P_k^{(i,j)}\right)}{L_{i+k}\left(x^j P_{k-1}^{(i+1,j)}\right)} P_{k-1}^{(i+1,j)}(x) \tag{5}$$

with $P_0^{(i,j)}(x) = 1$ and $P_{-1}^{(i,j)}(x) = 0$.

As proved in [5], these polynomials can also be represented as a contour integral. They have applications in many questions of numerical analysis.

Vector orthogonal polynomials of dimension $d \in \mathbb{N}$ were introduced by Van Iseghem [11]. They correspond to the case where the linear functionals L_i are related by

$$L_i(x^{j+1}) = L_{i+d}(x^j) \tag{6}$$

that is

$$L_i(x^{j+n}) = L_{i+nd}(x^j)\,.$$

Vector orthogonal polynomials have applications in the simultaneous approximation of several series by rational functions which generalizes Padé approximants. When $d = 1$, the usual formal orthogonal polynomials are recovered [1]. In that case, if we set $L_0(x^j) = c_{0j} = c_j$, then

$$c_{ij} = c_{i+j}$$

and the determinants $H_k^{(i,j)}$ are the usual Hankel determinants $H_k^{(i+j)}$. The usual formal orthogonal polynomials are very much related to Padé approximants as extensively explained in [1].

We shall now study the case $d = -1$.

We shall prove that the corresponding vector orthogonal polynomials of dimension -1 satisfy a three-term recurrence relation and a Christoffel-Darboux-type identity. A Shohat-Favard theorem will be proved. The link with the usual orthogonality ($d = 1$) will be explicited and it will be showed that orthogonal polynomials of dimension -1 generalize the usual orthogonal polynomials on the unit circle.

Thus various concepts about orthogonality will be recovered in a unified framework and, sometimes, generalized. Since orthogonal polynomials of dimension -1 are those which appear in Laurent-Padé and two-point Padé approximants, they lead to a unified presentation of all these Padé approximants. The case of vector orthogonal polynomials of dimension $-d$, with $d > 0$, is under consideration.

2. Orthogonality of dimension −1

Let us assume that $d = -1$ in (6). Replacing i by $i+1$, we thus have

$$L_i(x^j) = L_{i+1}(x^{j+1}). \tag{7}$$

That is $c_{ij} = c_{i+1,j+1}$ and it follows that

$$P_k^{(i,j)}(x) \equiv P_k^{(i+1,j+1)}(x). \tag{8}$$

Thus, the polynomials $P_k^{(i,j)}$ only depend on the difference $i-j$. This remark must be kept in mind in the sequel.

From (3), we have

$$P_k^{(i,j)}(x) = xP_{k-1}^{(i,j)}(x) - \frac{L_i\left(x^{j+1}P_{k-1}^{(i,j)}\right)}{L_i\left(x^j P_{k-1}^{(i+1,j)}\right)} P_{k-1}^{(i+1,j)}(x). \tag{9}$$

Replacing i by $i+1$ in (2) and using (8) gives

$$P_k^{(i+1,j)}(x) = xP_{k-1}^{(i,j)}(x) - \frac{L_{i+k}\left(x^{j+1}P_{k-1}^{(i,j)}\right)}{L_{i+k}\left(x^j P_{k-1}^{(i+1,j)}\right)} P_{k-1}^{(i+1,j)}(x). \tag{10}$$

Replacing j by $j+1$ in (4) and using (8) gives

$$P_k^{(i,j+1)}(x) = P_k^{(i,j)}(x) - \frac{L_i\left(x^{j+1}P_k^{(i,j)}\right)}{L_i\left(x^{j+1} P_{k-1}^{(i,j)}\right)} P_{k-1}^{(i,j)}(x). \tag{11}$$

Replacing j by $j+1$ in (5) and using (8) gives

$$P_k^{(i,j)}(x) = P_k^{(i,j+1)}(x) - \frac{L_{i+k}\left(x^{j+1}P_k^{(i,j+1)}\right)}{L_{i+k}\left(x^{j+1} P_{k-1}^{(i,j)}\right)} P_{k-1}^{(i,j)}(x). \tag{12}$$

Using alternately (5) and (9) (or (9) and (10)) allows to compute simultaneously the families $\{P_k^{(i,j)}\}$ and $\{P_k^{(i,j+1)}\}$. Similarly, using alternately (2) and (11) (or (11) and (12)) allows to compute simultaneously the families $\{P_k^{(i,j)}\}$ and $\{P_k^{(i+1,j)}\}$. Such a possibility is much more difficult to exploit for general biorthogonal polynomials [7]. For polynomials of dimension -1 the relations (2), (4) and (5) still hold thus providing other possible recursive schemes.

From the preceding relations, we shall now deduce a three-term recurrence relation for the family $\{P_k^{(i,j)}\}$ when i and j are fixed.

Thus, to simplify our notations, we shall set in (9)

$$\lambda_k = \frac{L_i\left(x^{j+1} P_{k-1}^{(i,j)}\right)}{L_i\left(x^j P_{k-1}^{(i+1,j)}\right)}$$

and in (5)

$$\mu_k = \frac{L_{i+k}\left(x^j P_k^{(i,j)}\right)}{L_{i+k}\left(x^j P_{k-1}^{(i+1,j)}\right)} .$$

Thus (5) and (9) now write as

$$P_k^{(i+1,j)}(x) = P_k^{(i,j)}(x) - \mu_k P_{k-1}^{(i+1,j)}(x) \tag{13}$$

$$P_k^{(i,j)}(x) = x P_{k-1}^{(i,j)}(x) - \lambda_k P_{k-1}^{(i+1,j)}(x) \tag{14}$$

and we obtain

$$P_{k+1}^{(i,j)}(x) = x P_k^{(i,j)}(x) - \lambda_{k+1}\left[P_k^{(i,j)}(x) - \mu_k P_{k-1}^{(i+1,j)}(x)\right] .$$

Replacing $P_{k-1}^{(i+1,j)}$ by its expression from (14), we have

$$P_{k+1}^{(i,j)}(x) = \left(x + B_{k+1}\right) P_k^{(i,j)}(x) - C_{k+1} x P_{k-1}^{(i,j)}(x) \tag{15}$$

with

$$B_{k+1} = -\lambda_{k+1}\left(1 + \frac{\mu_k}{\lambda_k}\right) \quad \text{and} \quad C_{k+1} = -\frac{\lambda_{k+1}\mu_k}{\lambda_k} .$$

We have thus obtained a three-term recurrence relation for the polynomials $P_k^{(i,j)}$ when i and j are fixed. Let us now express B_{k+1} and C_{k+1} in terms of the polynomials involved in (15) alone.

We have

$$L_p\left(x^j P_{k+1}^{(i,j)}\right) = L_p\left(x^{j+1} P_k^{(i,j)}\right) + B_{k+1} L_p\left(x^j P_k^{(i,j)}\right) - C_{k+1} L_p\left(x^{j+1} P_{k-1}^{(i,j)}\right) .$$

Using (7), this relation writes

$$L_p\big(x^j P_{k+1}^{(i,j)}\big) = L_{p-1}\big(x^j P_k^{(i,j)}\big) + B_{k+1} L_p\big(x^j P_k^{(i,j)}\big) - C_{k+1} L_{p-1}\big(x^j P_{k-1}^{(i,j)}\big) .$$

Thus, thanks to the biorthogonality conditions (1), we have

$$L_p\big(x^j P_{k+1}^{(i,j)}\big) = 0 \quad \text{for } p = i+1, \ldots, i+k-1 .$$

For $p = i$, using (7) again, we obtain

$$0 = L_i\big(x^{j+1} P_k^{(i,j)}\big) + B_{k+1} L_i\big(x^j P_k^{(i,j)}\big) - C_{k+1} L_i\big(x^{j+1} P_{k-1}^{(i,j)}\big)$$

and for $p = i + k$, we have

$$0 = L_{i+k-1}\big(x^j P_k^{(i,j)}\big) + B_{k+1} L_{i+k}\big(x^j P_k^{(i,j)}\big) - C_{k+1} L_{i+k-1}\big(x^j P_{k-1}^{(i,j)}\big) .$$

Due to the biorthogonality conditions (1), this system reduces to

$$\begin{aligned} C_{k+1} &= \frac{L_i\big(x^{j+1} P_k^{(i,j)}\big)}{L_i\big(x^{j+1} P_{k-1}^{(i,j)}\big)} \\ B_{k+1} &= \frac{C_{k+1} L_{i+k-1}\big(x^j P_{k-1}^{(i,j)}\big)}{L_{i+k}\big(x^j P_k^{(i,j)}\big)} . \end{aligned} \tag{16}$$

Let us remark that

$$\mu_k = -\frac{C_{k+1}\lambda_k}{\lambda_{k+1}} = -\frac{\lambda_k}{L_i\big(x^{j+1} P_{k-1}^{(i,j)}\big)} \; \frac{L_i\big(x^{j+1} P_k^{(i,j)}\big)}{\lambda_{k+1}} .$$

We shall set

$$\alpha_k = \frac{L_i\big(x^{j+1} P_k^{(i,j)}\big)}{\lambda_{k+1}}$$

and thus

$$\mu_k = -\frac{\alpha_{k-1}}{\alpha_k} .$$

Obviously λ_k, μ_k and B_{k+1}, C_{k+1} and α_k depend on i and j but this dependence was not indicated for simplicity.

Thus we finally proved the theorem 1.

THEOREM 1.— *Vector orthogonal polynomials of dimension* -1 *satisfy the three-term recurrence relation*

$$P_{k+1}^{(i,j)}(x) = (x + B_{k+1})P_k^{(i,j)}(x) - C_{k+1}xP_{k-1}^{(i,j)}(x), \quad k = 0, 1, \dots$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$ *and*

$$C_{k+1} = \frac{L_i\left(x^{j+1}P_k^{(i,j)}\right)}{L_i\left(x^{j+1}P_{k-1}^{(i,j)}\right)}$$

$$B_{k+1} = \frac{C_{k+1}L_{i+k-1}\left(x^jP_{k-1}^{(i,j)}\right)}{L_{i+k}\left(x^jP_k^{(i,j)}\right)}.$$

Let us now give a Christoffel-Darboux-type formula for these polynomials. Let R_k be any polynomial of degree k. We shall define R_k^* by

$$R_k^*(x) = x^k\overline{R}_k(x^{-1})$$

where the bar means that all the coefficients are replaced by their conjugates.

Let us compute first the quantity

$$A_k = P_k^{(i,j)}(x)\overline{P_k^{(i+1,j)*}(y)} - P_k^{(i+1,j)}(x)\overline{P_k^{(i,j)*}(y)}.$$

Using (13), we have

$$\begin{aligned} A_k &= P_k^{(i,j)}(x)\left[\overline{P_k^{(i,j)*}(y)} - \mu_k\overline{y}\overline{P_{k-1}^{(i+1,j)*}(y)}\right] \\ &\quad - \left[P_k^{(i,j)}(x) - \mu_kP_{k-1}^{(i+1,j)}(x)\right]\overline{P_k^{(i,j)*}(y)} \\ &= -\mu_k\overline{y}P_k^{(i,j)}(x)\overline{P_{k-1}^{(i+1,j)*}(y)} + \mu_kP_{k-1}^{(i+1,j)}(x)\overline{P_k^{(i,j)*}(y)}. \end{aligned}$$

Now by (14), we obtain

$$A_k = -\mu_kx\overline{y}P_{k-1}^{(i,j)}(x)\overline{P_{k-1}^{(i+1,j)*}(y)} + \mu_kP_{k-1}^{(i+1,j)}(x)\overline{P_{k-1}^{(i,j)*}(y)}.$$

Let us now compute the sum $\mu_k^{-1}A_k + A_{k-1}$. Using the preceding identity, we immediately have

$$\mu_k^{-1}A_k + A_{k-1} = (1 - x\overline{y})P_{k-1}^{(i,j)}(x)\overline{P_{k-1}^{(i+1,j)*}(y)}\,.$$

Using $\mu_k = -\alpha_{k-1}/\alpha_k$ we thus obtain a Christoffel-Darboux-type formula.

THEOREM 2

$$\alpha_{k+1}\left[P_{k+1}^{(i+1,j)}(x)\overline{P_{k+1}^{(i,j)*}(y)} - P_{k+1}^{(i,j)}(x)\overline{P_{k+1}^{(i+1,j)*}(y)}\right] =$$

$$= (1 - x\overline{y})\sum_{n=0}^{k}\alpha_n P_n^{(i,j)}(x)\overline{P_n^{(i+1,j)*}(y)}$$

$$= \alpha_k\left[P_k^{(i+1,j)}(x)\overline{P_k^{(i,j)*}(y)} - x\overline{y}P_k^{(i,j)}(x)\overline{P_k^{(i+1,j)*}(y)}\right]\,.$$

This is the relation given by Bultheel [6, p. 95]. The reason is that, in fact, these polynomials are, apart a multiplying factor, those used in Padé and two-point Padé approximants. It is also possible to define polynomials of the second kind (or associated) and the results of [6] are valid.

We shall now prove a Shohat-Favard type theorem for vector orthogonal polynomials of dimension -1, that is the reciprocal of theorem 1 namely that if a family of polynomials satisfies a three-term recurrence relation of the form given in the theorem 1, then it is a family of vector orthogonal polynomials of dimension -1 whose moments $L_i(x^j)$, satisfying $L_{i+1}(x^{j+1}) = L_i(x^j)$, can be computed.

Let us first remark that the family of linear functionals $\{L_i\}$ is defined apart a multiplying factor. Indeed if $L_i(x^j) = c_{ij}$ and if P_k is such that $L_i(P_k) = 0$ for $i = 0, \ldots, k-1$ then we also have $L_i'(P_k) = 0$ for $i = 0, \ldots, k-1$ where $L_i' = aL_i$ that is $L_i'(x^j) = ac_{ij}$ where a is any number different from zero. Thus $L_0(1)$ can be set to an arbitrary nonzero value.

Thus let $\{P_k\}$ be a family of polynomials satisfying

$$P_{k+1}(x) = (x + B_{k+1})P_k(x) - C_{k+1}xP_{k-1}(x)\,, \quad k = 0, 1, \ldots$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$.

We shall see that, under some assumptions, there exists a family of functionals, uniquely defined apart a multiplying factor, such that $L_{i+1}(x^{j+1}) = L_i(x^j)$ and $\forall\, k$, $L_i(P_k) = 0$ for $i = 0, \ldots, k-1$. Thus $\{P_k\}$ will be the family of vector orthogonal polynomials of dimension -1 with respect to the family $\{L_i\}$.

We have

$$P_1(x) = x + B_1$$

and

$$L_0(P_1) = 0 = L_0(x) + B_1 L_0(1) .$$

Since the L_i's are defined apart a multiplying factor, then $L_0(1)$ is arbitrary and we shall take $L_0(1) = 1$ (if we take $L_0(1) = 0$ then $\forall\, i$, j, $L_i(x^j) = 0$). Thus this first orthogonality condition gives the value of $L_0(x)$. Now, for $i = 0$ and 1

$$L_i(P_2) = 0 = L_i(xP_1) + B_2 L_i(P_1) - C_2 L_i(xP_0) .$$

For $i = 0$, $L_0(P_1) = 0$ and the preceding relation gives the value of $L_0(x^2)$ since $L_0(1)$ and $L_0(x)$ are known.

When $i = 1$, we first set, following (7), $L_1(x) = L_0(1)$ and $L_1(x^2) = L_0(x)$. Thus $L_1(xP_1) = L_0(P_1)$ and $L_1(xP_0) = L_0(P_0)$ are known and the preceding relation allows to compute the value of $L_1(P_1)$ if $B_2 \neq 0$. Thus, since $L_1(x) = L_0(1) = 1$, we obtain the value of $L_1(1)$.

By induction, let us assume that the quantities $L_i(x^j)$ are known for $i = 0, \ldots, k-1$ and $j = 0, \ldots, k$. We have for $i = 0, \ldots, k$

$$L_i(P_{k+1}) = 0 = L_i(xP_x) + B_{k+1} L_i(P_k) - C_{k+1} L_i(xP_{k-1}) .$$

When $i = 0$, $L_0(P_k) = 0$ and $L_0(x^{k+1})$ can be computed since $L_0(x^j)$ is known for $j = 0, \ldots, k$.

Now we set $L_k(x^j) = L_{k-1}(x^{j-1})$ for $j = 1, \ldots, k+1$ and $L_i(x^{k+1}) = L_{i-1}(x^k)$ for $i = 1, \ldots, k$. Thus the right hand side of the preceding relation is zero for $i = 1, \ldots, k-1$ since $L_i(xP_k) = L_{i-1}(P_k) = 0$, $L_i(P_k) = 0$ and $L_i(xP_{k-1}) = L_{i-1}(P_{k-1}) = 0$.

For $i = k$, $L_k(xP_k) = L_{k-1}(P_k) = 0$ and $L_k(xP_{k-1}) = L_{k-1}(P_{k-1})$ is known. Thus $L_k(P_k)$ can be computed if $B_{k+1} \neq 0$. Since $L_k(x^j)$ is known for $j = 1, \ldots, k+1$ then $L_k(1)$ can be obtained from the above formula.

Let us also remark that if $B_1 = 0$ then $\forall\, i$, j, $L_i(x^j) = 0$. Thus we proved the theorem 3.

THEOREM 3.— *Let $\{P_k\}$ be a family of polynomials satisfying*

$$P_{k+1}(x) = (x + B_{k+1})P_k(x) - C_{k+1}xP_{k-1}(x), \quad k = 0, 1, \ldots$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$.

If $B_{k+1} \neq 0$ for $k = 0, 1, \ldots$, then $\{P_k\}$ is the family of monic vector orthogonal polynomials of dimension -1 with respect to a uniquely determined (apart a multiplying factor) family of linear functionals $\{L_i\}$ satisfying $L_{i+1}(x^{j+1}) = L_i(x^j)$ whose moments $L_i(x^j)$ can be computed and are not all zero.

Remark.— Obviously it is possible to obtain similarly the linear functionals such that

$$L_p(x^j P_k) = 0, \quad \text{for } p = i, \ldots, i + k - 1$$

where i and j are fixed non negative integers.

Remark.— From the expressions of B_{k+1} and C_{k+1} it is easy to see that B_{k+1} is different from zero if and only if $C_{k+1} \neq 0$ and $L_{k-1}(P_{k-1}) \neq 0$. The condition $L_{k-1}(P_{k-1}) \neq 0$ insures the existence of the vector orthogonal polynomial P_k. C_{k+1} is different from zero if and only if $L_0(xP_k) \neq 0$. Defining the functional L_{-1} by $L_{-1}(x^j) = L_0(x^{j+1})$, we have $L_0(xP_k) = L_{-1}(P_k)$. Again it is easy to see that the condition $L_{-1}(P_k) \neq 0$ insures the existence of the vector orthogonal polynomial P_{k+1}. The family $\{L_i\}$ is said to be definite if and only if $\forall\, k$, $L_k(P_k) \neq 0$. This condition insures the existence of all the polynomials $\{P_k\}$. In particular, we have $L_0(P_0) = L_0(1) \neq 0$ and we recover a condition discussed above. The condition $B_{k+1} \neq 0$ must be compared with the condition for the usual Shohat-Favard theorem about the ordinary orthogonal polynomials (that is the vector orthogonal polynomials of dimension $d = 1$) which is $C_{k+1} \neq 0$ (see [1, p. 155]).

Let us now consider the polynomials

$$\widetilde{P}_k(x) = x^k P_k(x^{-1}).$$

If the polynomials P_k satisfy the recurrence relation of theorem 3, then we have

$$x^{k+1} P_{k+1}(x^{-1}) = x^{k+1}(x^{-1} + B_{k+1}) P_k(x^{-1}) - C_{k+1} x^{k+1} x^{-1} P_{k-1}(x^{-1})$$

that is

$$\widetilde{P}_{k+1}(x) = \left(1 + B_{k+1}x\right)\widetilde{P}_k(x) - C_{k+1}x\widetilde{P}_{k-1}(x), \quad k = 0, 1, \ldots$$

with $\widetilde{P}_{-1}(x) = 0$ and $\widetilde{P}_0(x) = 1$.

Thus, by theorem 3, the polynomials $\widetilde{P}_k$ (which are no more monic) form a family of vector orthogonal polynomials of dimension -1. More precisely, if we set

$$U_k(x) = \frac{\widetilde{P}_k(x)}{B_1 \cdots B_k}$$

then the polynomials U_k are monic and they satisfy

$$U_{k+1}(x) = \left(x + B_{k+1}^{-1}\right)U_k(x) - C_{k+1}B_k^{-1}B_{k+1}^{-1}xU_{k-1}(x), \quad k = 0, 1, \ldots$$

with $U_{-1}(x) = 0$ and $U_0(x) = 1$.

Let us denote by $\{M_i\}$ the family of linear functionals such that $\forall\, k$

$$M_i(U_k) = 0, \quad \text{for } i = 0, \ldots, k-1.$$

Thus, from the relations of theorem 1, we have

$$\begin{aligned} M_0(xU_k) &= C_{k+1}B_k^{-1}B_{k+1}^{-1}M_0(xU_{k-1}) \\ M_k(U_k) &= C_{k+1}B_k^{-1}M_{k-1}(U_{k-1}), \quad k = 1, 2, \ldots \end{aligned}$$

with $M_0(xU_0) = -1/B_1$ and $M_0(U_0) = 1$.

Replacing B_k, B_{k+1} and C_{k+1} by their expressions we obtain relations between the two families of linear functionals

$$\begin{aligned} \frac{M_0(xU_k)}{M_0(xU_{k-1})} &= \frac{L_k(P_k)L_0(xP_{k-2})}{L_{k-2}(P_{k-2})L_0(xP_{k-1})} \\ \frac{M_k(U_k)}{M_{k-1}(U_{k-1})} &= \frac{L_0(xP_k)L_{k-1}(P_{k-1})L_0(xP_{k-2})}{L_0(xP_{k-1})L_{k-2}(P_{k-2})L_0(xP_{k-1})}. \end{aligned}$$

An open problem is to express the quantities $M_i(x^j)$ in terms of the $L_i(x^j)$'s. Another open problem is to study if the Christoffel-Darboux type formula of theorem 2 implies the recurrence relations of the vector orthogonal polynomials of dimension -1 as is the case with the usual orthogonal polynomials of dimension $d = 1$ [2], [3].

3. Orthogonality on the real line

Let us now explain the link with the usual formal orthogonality. We previously saw that for $d = -1$:

$$c_{ij} = c_{i-1,j-1}\,.$$

Thus we have

$$c_{ij} = \begin{cases} c_{0,j-i} & \text{if } j - i \geq 0 \\ c_{i-j,0} & \text{if } i - j \geq 0. \end{cases}$$

In both cases, we can define quantities c_{ij} with negative indexes by

$$c_{ij} = \begin{cases} c_{-1,j-i-1} = \ldots = c_{i-j,0} & \text{if } j - i \geq 0 \\ c_{i-j-1,-1} = \ldots = c_{0,j-i} & \text{if } i - j \geq 0 \end{cases}$$

and set

$$c_{ij} = c_{i-j}$$

since c_{ij} depends only on the difference $i - j$.

Let c be the linear functional on the space of complex Laurent polynomials defined by

$$c(x^i) = c_i\,, \quad i \in \mathbb{Z}$$

and $c^{(n)}$ by

$$c^{(n)}(x^i) = c_{n+i}\,, \quad n\,,\, i \in \mathbb{Z}\,.$$

Orthogonality with respect to c will be called orthogonality on the real line since it generalizes this case.

Thus, if R_k is any polynomial of degree k, we have

$$L_p\big(R_k(x)\big) = c\big(x^p R_k(x^{-1})\big) = c\big(x^p x^{-k}\overline{R}_k{}^*(x)\big) = c^{(-k)}\big(x^p \overline{R}_k{}^*(x)\big)$$

where R_k^* is defined as above.

It follows that the polynomial $P_k^{(i,j)}$ satisfies, for $p = i, \ldots, i + k - 1$

$$L_p\left(x^j P_k^{(i,j)}\right) = c^{(-k)}\left(x^p x^{-j} \overline{P}_k^{(i,j)*}\right) = c^{(-k-j)}\left(x^p \overline{P}_k^{(i,j)*}\right) = 0\,.$$

Thus $\overline{P}_k^{(0,j)*}$ is the polynomial of degree k of the family of formal orthogonal polynomials with respect to $c^{(-k-j)}$.

This connection was already pointed out by Bultheel [6, p. 98 ff]. However, our approach, using only linear functionals, seems simpler than his based on bilinear forms defined on the set of Laurent polynomials.

On the recurrence relations satisfied by these polynomials when some of the determinants $H_k^{(0,j)}$ are zero, see Draux [8].

We also see that we have

$$c\big(x^p x^{-j} P_k^{(i,j)}(x^{-1})\big) = 0\,, \quad \text{for } p = i, \ldots, i+k-1\,.$$

Thus, if we set

$$\widetilde{P}_k^{(i,j)}(x) = x^k P_k^{(i,j)}(x^{-1})$$

we have

$$c\big(x^{p-k-j}\,\widetilde{P}_k^{(i,j)}(x)\big) = 0\,, \quad \text{for } p = i, \ldots, i+k-1\,.$$

Thus, for any i and j such that $i - j = n + k$, $\widetilde{P}_k^{(i,j)}$ is identical to $P_k^{(n)}$, the usual orthogonal polynomial of degree k of the family of orthogonal polynomials with respect to $c^{(n)}$ (normalized by the condition $P_k^{(n)}(0) = 1$) that is $\forall\ j \geq 0$, it holds

$$c^{(n)}\big(x^p P_k^{(n)}(x)\big) = c\big(x^p \widetilde{P}_k^{(n+k+j,j)}(x)\big) = 0\,, \quad \text{for } p = 0, \ldots, k-1\,.$$

This connection can be seen more directly from the determinantal formulae of the orthogonal polynomials.

4. Orthogonality on the unit circle

We shall now show that orthogonality on the unit circle is a particular case of vector orthogonality of dimension -1.

Let us assume that $c(\overline{x}^i) = \overline{c}_i$ and that $\overline{x} = 1/x$ (or, in other words, that x is on the unit circle) then we have

$$c(\overline{x}^i) = \overline{c}_i = c(x^{-i}) = c_{-i}\,.$$

Conversely, if this relation holds, then it is equivalent to assume that $\overline{x} = 1/x$ and the usual orthogonality on the unit circle is recovered [9], [10]. Since our notion generalizes this case, it will be called orthogonality on the unit circle.

We shall now examine this case in more details.

Let us set

$$P_k^{(i,j)}(x) = a_0 + \cdots + a_k x^k$$

where obviously the coefficients depend on i, j and k.

Thanks to the orthogonality conditions (1), we have $a_k = 1$ and

$$\begin{aligned} a_0 c_{i-j} + \cdots + a_k c_{i-j-k} &= 0 \\ &\vdots \\ a_0 c_{i+k-j-1} + \cdots + a_k c_{i-j-1} &= 0\,. \end{aligned} \tag{17}$$

Thus (17) can be written as

$$\begin{aligned} \overline{a}_0 \overline{c}_{i-j} + \cdots + \overline{a}_k \overline{c}_{i-j-k} &= 0 \\ &\vdots \\ \overline{a}_0 \overline{c}_{i+k-j-1} + \cdots + \overline{a}_k \overline{c}_{i-j-1} &= 0\,. \end{aligned}$$

Since $c_{-n} = \overline{c}_n$, this system is equivalent to

$$\begin{aligned} \overline{a}_0 c_{j-i} + \cdots + \overline{a}_k c_{j-i+k} &= 0 \\ &\vdots \\ \overline{a}_0 c_{j-k-i+1} + \cdots + \overline{a}_k c_{j-i+1} &= 0\,. \end{aligned} \tag{18}$$

Let us now consider the polynomial

$$P_k^{(j+1,i)}(x) = b_0 + \cdots + b_k x^k$$

with $b_k = 1$. The orthogonality conditions (1) give

$$\begin{aligned} b_0 c_{j-i+1} + \cdots + b_k c_{j-i-k+1} &= 0 \\ &\vdots \\ b_0 c_{j-i+k} + \cdots + b_k c_{j-i} &= 0\,. \end{aligned} \tag{19}$$

Comparing (18) and (19), we see that both systems are the same and

$$b_n = \frac{\overline{a}_{k-n}}{\overline{a}_0}, \quad n = 0, \ldots, k.$$

Thus if we set

$$P_k^{(i,j)*}(x) = x^k \overline{P}_k^{(i,j)}(x^{-1}) = \overline{a}_0 x^k + \cdots + \overline{a}_k$$

we have $P_k^{(i,j)*}(0) = \overline{a}_k = 1$ and thus

$$P_k^{(i,j)*}(x) = \overline{P}_k^{(i,j)}(0) P_k^{(j+1,i)}(x). \tag{20}$$

Replacing in (14) gives, for $i = j$

$$P_{k+1}^{(i,i)}(x) = x P_k^{(i,i)}(x) - \frac{\lambda_{k+1}}{\overline{a}_0} P_k^{(i,i)*}(x).$$

Taking $x = 0$, we have

$$P_{k+1}^{(i,i)}(0) = -\frac{\lambda_{k+1}}{\overline{a}_0}$$

and we finally obtain

$$P_{k+1}^{(i,i)}(x) = x P_k^{(i,i)}(x) + P_{k+1}^{(i,i)}(0) P_k^{(i,i)*}(x)$$

which is one of the usual recurrence relations for orthogonal polynomials on the unit circle when $i = j = 0$. The other usual recurrence relations for orthogonal polynomials on the unit circle when $i = j = 0$ could be recovered in a similar way from (14) and (15) (see [9], [10]).

Thus, relations (9), (10) and (15) generalize the usual recurrence relations to adjacent families of orthogonal polynomials on the unit circle.

Using (20) in the Christoffel-Darboux-type formula given in the previous section leads immediately to the usual corresponding formula for orthogonal polynomials on the unit circle.

5. Conclusion

Vector orthogonality of dimension $d = -1$ seems to be a central notion since it generalizes orthogonality on the unit circle and the usual notion of orthogonality which are both recovered as particular cases. Thus, in particular, it opens the way to a unified presentation of Padé, Laurent-Padé and two-point Padé approximants.

Because of the numerous applications of orthogonal polynomials, vector orthogonal polynomials of dimension -1 might also have interesting applications, but they now remain to be discovered.

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