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On absorption probabilities for a random walk between two different barriers

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RÉSUMÉ. — On détermine les probabilités d'absorption de la marche aléatoire stationnaire la plus générale sur les entiers $\{0, 1, \ldots, N\}$, où l'origine (site N) est une barrière partiellement réfléchissante et le site N (site 0) est absorbant.

Nous donnons des expressions explicites, de l'espérance et de la variance du temps d'absorption.

ABSTRACT. — A determination is made of the absorption probabilities of the most general stationary random walk on the integers $\{0, 1, \ldots, N\}$ where by the origin (site N) is a partially reflecting barrier and the site N (site 0) is an absorbing one. Explicit expressions are given for the mean and the variance of the time to absorption.

KEY-WORDS : Absorption time; Absorption probability; generating function; partial fractions.

1. Introduction

Consider a random walk on a line-segment of $N + 1$ sites as shown in figure 1 a) and b). The $N + 1$ sites on the line-segment are denoted by the integers $0, 1, 2, \ldots, N$. Let p be the probability for a particle (per unit time) to move from a site j, $0 < j < N$, to its nearest neighbor on the right, $j + 1$, and q be the probability to move from j to $j - 1$. The probability to stay (for one unit time) at a site j is thus $r = 1 - p - q$. There are two

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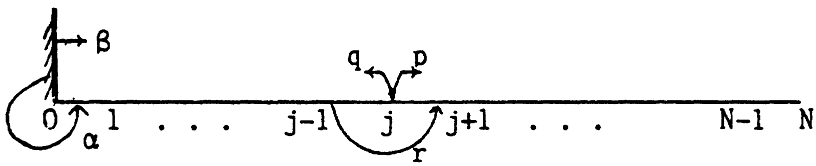
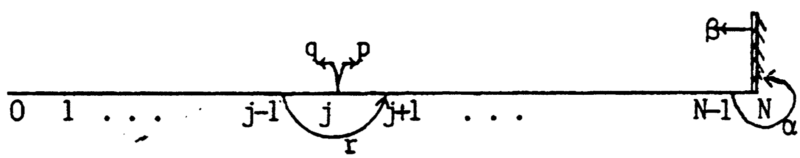
barriers, one of which, site N (site 0), is absorbing and the other, site 0 (site N) is partially-reflecting, that is the probability to stay at a site 0 (site N) is α and the probability to reflect to a site 1 (site $N-1$) is $\beta = 1 - \alpha$, respectively.



a) Partial-reflecting barrier at 0 and absorbing one at N



b) Partial-reflecting barrier at N and absorbing one at 0

Fig. 1 A moving particle on a line segment: right and left jumps are indicated by arrows and stayed at the same site by loops.

The two situations a) and b) in figure 1 can be easily obtained from each other by replacing j with $N - j$ and interchanging p and q, respectively. So we deal with the second one of them.

Let $g_{j0}(t)$ be the probability that the particle is absorbed at 0 at time t given that its initial site was j. Weesakul (1961) [11] has computed the probability $g_{j0}(t)$, in the special cas $r = 0$, $\alpha = p$; however, his calculations contained some errors. Correct formulae can be found in Blasi (1976) [1]. The probability $g_{j0}(t)$ is also given by Hardin and Sweet (1969) [4] in the special two cases $i - q = p$, $\alpha = r$, $\beta = 2p$, and $i - q = p$, $\alpha = 1 - p$, $\beta = p$.

In most text books covering random walks (for example Cox and Miller (1965) [2], and Feller (1968) [3]) the determination of explicit expressions for the absorption probabilities from the generating function is effected by partial fractions; however, it has generally been difficult to obtain (see [4] and [7]). In this note, determination of generalization expression for $g_{j0}(t)$ from the corresponding generating function

$$G_j(z) = \sum_{t=0}^{\infty} g_{j0}(t) z^t, \quad |z| < 1 \tag{1}$$

by partial fraction expansions is presented. This generalization expression apparently is not covered by the literature. Explicit formulae for the mean and the variance of the time to absorption are also given.

2. Partial fraction expansion

The probability $g_{j0}(t)$ that the particle is at location 0 for the first time after t steps given that its initial position was j obeys the following difference equation:

$$g_{j0}(t) = q g_{j-1,0}(t-1) + r g_{j0}(t-1) + p g_{j+1,0}(t-1) \tag{2}$$

for $t = 1, 2, \ldots$ and $j = 1, 2, \ldots, N-1$.

We set

$$g_{00}(t) = \delta_{0,t} = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise,} \end{cases}$$
$$g_{j0}(t) = 0 \quad \text{for } t < j\,.$$

For $j = N$ we have

$$g_{N0}(t) = \beta g_{N-1,0}(t-1) + \alpha g_{N,0}(t-1)\,.$$

Following Neuts (1963) [9] we deduce that (see also [4], [5], [6] and [8])

$$G_j(z) = \frac{q^j z^j T_j(z)}{T_0(z)}, \quad 0 \le j < N \tag{3}$$

and

$$G_N(z) = \beta q^{N-1} z^N \frac{\lambda_1 - \lambda_2}{T_0(z)},$$

where $T_j(z)$ and $\lambda_{1,2}$ are given by

$$T_j(z) = (1-\alpha z)(\lambda_1^{N-j} - \lambda_2^{N-j}) - \beta p z^2 (\lambda_1^{N-j-1} - \lambda_2^{N-j-1}), \tag{4}$$

and

$$\lambda_{1,2} = \frac{1}{2}\left[1 - rz \pm \sqrt{(1-rz)^2 - 4pqz^2}\right]. \tag{5}$$

Both the numerator and the denominator of equation (3) have degree N. If the roots of $T_0(z)$, z_1, z_2, $\ldots$, z_N are distinct, (3) can be decomposed into partial fractions as

$$G_j(z) = \frac{a_1}{z_1 - z} + \frac{a_2}{z_2 - z} + \cdots + \frac{a_N}{z_N - z}, \tag{6}$$

where a's can be determined by

$$\begin{aligned} a_k &= \lim_{z \to z_k} (z - z_k) G_j(z) \\ &= \begin{cases} \dfrac{-(qz_k)^j T_j(z_k)}{{}^{\mathrm{d}}/_{\mathrm{d}z}\left[T_0(z)\right]\Big|_{z=z_k}}, & 0 \le j < N \\ \dfrac{-q^{N-1} z_k^N \beta(\lambda_1 - \lambda_2))}{{}^{\mathrm{d}}/_{\mathrm{d}z}\left[T_0(z)\right]\Big|_{z=z_k}}, & j = N. \end{cases} \end{aligned} \tag{7}$$

In order to determine the roots of the denominator we make the transformation

$$z = \left[r + 2\sqrt{pq}\cos w\right]^{-1}. \tag{8}$$

In terms of the transformation (8),

$$\lambda_{1,2} = \frac{\sqrt{pq}e^{\pm iw}}{r + 2\sqrt{pq}\cos w}, \quad i = \sqrt{-1} \tag{9}$$

and (3) becomes

$$G_j(z) = \frac{\left({}^{q}/_{p}\right)^{j/2} U_j(w)}{U_0(w)}, \quad 0 \le j \le N \tag{10}$$

where the denominator $U_0(w)$ is given by

$$U_0(w) = \sqrt{q}(r-\alpha)\sin Nw + q\sqrt{p}\sin(N+1)w + \sqrt{p}(q-\beta)\sin(N-1)w\,. \tag{11}$$

A study of the function

$$f(w) = \frac{q\sin(N+1)w + (q-\beta)\sin(N-1)w}{\sin Nw}, \tag{12}$$

shows that denominator $U_0(w)$ has N distinct roots w_k, $k = 1, 2, \ldots, N$ if $N \le {}^{\beta}/\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]$. The roots are then

$$z_k = \left[r + 2\sqrt{pq}\cos w_k\right]^{-1}. \tag{13}$$

If $N > \beta/\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]$, there are only $N-1$ distinct roots w_k, $k = 2, 3, \ldots, N$, that give distinct roots z_k of $T_0(z)$. The remaining root of $T_0(z)$ is given by

$$z_1 = \left[r + 2\sqrt{pq}\cosh w_1\right]^{-1}, \tag{14}$$

where w_1 is the unique root of the equation

$$\frac{\sqrt{pq}\sinh(N+1)w}{(\alpha - r)\sinh Nw} + \frac{(q-\beta)\sqrt{p/q}\sinh(N-1)w}{(\alpha - r)\sinh Nw} = 1. \tag{15}$$

From (8) we have

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{2\sqrt{pq}\sin w}{\left[r + 2\sqrt{pq}\cos w\right]^2}, \tag{16}$$

and so

$$a_k = -\frac{(q/p)^{j/2} U_j(w_k)}{\left[\dfrac{\mathrm{d}}{\mathrm{d}w} U_0(w)\dfrac{\mathrm{d}w}{\mathrm{d}z}\right]\Bigg|_{w=w_k}}, \quad 0 \le j \le N. \tag{17}$$

From (6) we can obtain the coefficient of z^t in the expansion of $G_j(z)$ which is

$$g_{j0}(t) = \sum_{k=1}^{N} \frac{a_k}{z_k^{t+1}}, \quad t = 1, 2, \ldots \tag{18}$$

Explicit generalization expression for the absorption probability $g_{j0}(t)$ finally becomes

$$g_{j0}(t) = -2p^{(1-j)/2} q^{(1+j)/2} \sum_{k=1}^{N} \frac{\mathcal{N}}{\mathcal{D}} \left[r + 2\sqrt{pq}\cos w_k\right]^{t-1} \sin w_k \tag{19}$$

where

$$\begin{aligned}
\mathcal{N} &= \sqrt{q}(r-\alpha)\sin(N-j)w_k + q\sqrt{p}\sin(N-j+1)w_k + \\
&\quad + \sqrt{p}(q-\beta)\sin(N-j-1)w_k, \\
\mathcal{D} &= N\sqrt{q}(r-\alpha)\cos(Nw_k) + q(N+1)\sqrt{p}\cos(N+1)w_k + \\
&\quad + (N-1)(q-\beta)\sqrt{p}\cos(N-1)w_k,
\end{aligned}$$

which can be rewritten as:

$$g_{j0}(t) = 2p^{\frac{1-j}{2}} q^{\frac{1+j}{2}} \left[\Pi(w_1) - \sum_{k=2}^{N} \frac{U_j(w_k)}{U_0'(w_k)} \left(r + 2\sqrt{pq}\cos w_k\right)^{t-1} \sin w_k\right], \quad 0 \le j \le N, \tag{20}$$

where

$$\Pi(w_1) = \begin{cases} -\dfrac{U_j(w_1)}{U_0'(w_1)}\left(r + 2\sqrt{pq}\cos w_1\right)^{t-1}\sin w_1 \\ \qquad\qquad\qquad \text{if } N < \dfrac{\beta}{\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]} \\ \dfrac{2j\left(r + 2\sqrt{pq}\right)^{t-1}}{N\left[2N^2 + 3N(2q-\beta)/\beta + 1\right]} \\ \qquad\qquad\qquad \text{if } N = \dfrac{\beta}{\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]} \\ \dfrac{iU_j(iw_1)}{U_0'(iw_1)}\left(r + 2\sqrt{pq}\cosh w_1\right) \\ \qquad\qquad\qquad \text{if } N > \dfrac{\beta}{\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]} \end{cases}$$

$$U_0'(w) = \frac{\mathrm{d}U_0(w)}{\mathrm{d}w}$$

and

$$w_1 = \begin{cases} \cos^{-1}(1 - rz_1)\left[2z_1\sqrt{pq}\right]^{-1} & \text{if } N \le \dfrac{\beta}{\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]} \\ \cosh^{-1}(1 - rz_1)\left[2z_1\sqrt{pq}\right]^{-1} & \text{if } N > \dfrac{\beta}{\left[(\alpha - r)\sqrt{q/p} - 2q + \beta\right]} \end{cases}$$

z_1 with the smallest root in absolute value of $T_0(z)$.

Using the following theorem ([3], p. 277) with appropriate change of notation

THEOREM .— *If $G_j(z)$ is a rational function with a simple root z_1 of the denominator which is smaller in absolute value than all other roots, then the coefficient $g_{j0}(t)$ of z^t is given asymptotically by*

$$g_{j0}(t) \simeq a_1 z_1^{-(t+1)}$$

where a_1 is defined in (17).

We find that

$$g_{j0}(t) \simeq 2p^{(1-j)/2}q^{(1+j)/2}\Pi(w_1)\,, \tag{21}$$

where $\Pi(w_1)$ is defined previously.

We see that with the appropriate change of notation in the special cases considered in the introduction, formula (20) agrees with that of Weesakul [11], Hardin and Sweet [4] and Blasi [1].

3. The mean and the variance of the process

Let Σ_j denotes the time up to absorption at a site 0 when the particle starts at a site j, $0 < j \leq N$; then

$$\Pr(\Sigma_j = t) = g_{j0}(t)$$

and hence

$$\mu = \mathrm{E}[\Sigma_j] = \left[\frac{\mathrm{d}}{\mathrm{d}z}\, G_j(z)\right]_{z=1} . \tag{22}$$

Using (3) and (22), we get

$$\begin{gathered}\mu = \frac{1}{2q+r-1}\left[j + \frac{q(\beta - q + p)}{\beta(2q+r-1)}\,(1 - a^{-j})a^N\right] ,\\ p \neq q\,,\ a = \frac{p}{q}\,,\ \alpha + \beta = 1\,.\end{gathered} \tag{23}$$

When $p = q$, $\lim_{p \to q} \mu$ is evaluated using l'Hospital's rule, and in this case

$$\mu = \frac{i}{\beta} + \frac{j}{2p}\,(2N - j - 1)\,. \tag{24}$$

This result was established by Khan (1984) [6] for the particular case $r = 0$ with appropriate change of notation (see also [10]).

The variance of the absorption time Σ_j can be obtained from the following relation

$$\mathrm{Var}(\Sigma_j) = \left.\frac{\mathrm{d}^2}{\mathrm{d}z^2}\, G_j(z)\right|_{z=1} + \mu(1 - \mu)\,,$$

in the form

$$\begin{gathered}\mathrm{Var}(\Sigma_j) = \left[\delta(1 - a^{-2j})a^N + \delta_j a^{-j} - \delta_0\right] a^N + c_j\,,\\ p \neq q\,,\ a = \frac{p}{q}\,,\ \alpha + \beta = 1\end{gathered} \tag{25}$$

where

$$\delta = \left[\frac{q(\beta - q + p)}{\beta(2q + r - 1)^2}\right]^2, \quad c_j = \left[\frac{p + q - (2q + r - 1)^2}{(2q + r - 1)^3}\right] j$$

and

$$\begin{aligned}\delta_j &= \frac{1}{\left[\beta(2q + r - 1)^2\right]^2} \Big[4q\beta(\beta - q + p)(2p + r - 1)(j - N) + \\ &\quad - 2pq\beta(2\beta + p - q) - q(2p + r - 1)(\beta + p - q)\big(2q - \beta(q - p)\big)\Big], \\ \delta_0 &= \delta_j\big|_{j=0}.\end{aligned}$$

When $q = p$, $\lim_{q \to p} \operatorname{Var}(T_j)$ is evaluated using l'Hospital's rule, and it is found to be

$$\begin{aligned}\operatorname{Var}(T_j) = (6p^2)^{-1} \Big[&N^4 - (N - j)^4 + a_1\big(N^3 - (N - j)^3\big) + \\ &-a_2\big(N^2 - (N - j)^2\big) + a_3 j\Big]\end{aligned} \tag{26}$$

where

$$a_1 = \frac{4p}{\beta} - 2, \quad a_2 = \frac{3p}{\beta}\left(2 - \frac{2p}{\beta} + \beta\right) - 2 \quad \text{and} \quad a_3 = \frac{p}{\beta}(2 - 6p + 3\beta) - 1.$$

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