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An asymptotic condition for variational points of nonquadratic functionals

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RÉSUMÉ. — Soit $u : M \to N$ une application d'une variété riemannienne M dans une autre N. On démontre un énoncé du type Liouville pour certaines fonctionnelles $E(u)$ non quadratiques, moyennant une minoration de la variation de $E(u)$ et une hypothèse sur M.

ABSTRACT. — Let $u : M \to N$ be a map between Riemannian manifolds M and N. We prove a Liouville theorem for certain nonquadratic functionals $E(u)$ under a restriction on the variation of $E(u)$ and a hypothesis on M.

1. Introduction

Let $u : M \to N$ be a map from an n-dimensional Riemannian manifold M to a Riemannian manifold N. Let M be complete, noncompact, simply connected, with infinite injectivity radius (topologically $\mathbb{R}^n$). Define on M a Riemannian n-disc $B = B^n_R(x_0)$ of radius R centered at a point $x_0 \in M$. Assume that for some ℓ there is a smooth embedding of M into $\mathbb{R}^\ell$ (e.g., the exponential map); via this embedding choose local coordinates en B. Let the C^2 metric g on M satisfy $\forall\, y \in B$ the differential inequality

$$\lambda(y) = \left|\partial_1 \left[r\Gamma^k_{1k}(y)\right]\right| \leq K_0 R^{-2} \tag{1.1}$$

Here K_0 is a constant independent of R; $\Gamma^k_{ij}(y)$ is the Christoffel symbol for g_{ij} at y; $r = |y - x_0|$ is the geodesic radial coordinate on B; the index 1 denotes the radial direction; we use the Einstein convention for summing repeated indices. Assume that N is smoothly embedded in $\mathbb{R}^q$ for some q.

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Define the local energy functional

$$E(w,u,p,\Omega) = \int_\Omega w\big(|\mathrm{d}u|^p\big)\,\mathrm{d}v(g)\,,$$

where Ω is a given subdomain of M; $\mathrm{d}u(x) \in T_x^* M$ is the differential at x of the map u; $p \in \mathbb{R}^+$;

$$\mathrm{d}v(g) = \sqrt{\det g_{ij}}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\,;$$

$w(t)$ is a scalar-valued function of $t \in \mathbb{R} \sim \mathbb{R}^-$ satisfying

$$\begin{cases} 0 \le \dot{w}(t) \le K_1 \\ K_2 t \le w(t) \end{cases} \tag{1.2}$$

for independent constants $K_1 \ge 0$ and $K_2 > 0$.

Suppose that E is locally finite on M in the sense that on the restriction $\Omega = M\big|_B$, E is a finite scalar function of the radius R of B for finite R. Also suppose that E satisfies the growth condition

$$E(R) = \int_{B_R} w\big(|\mathrm{d}u|^p\big)\,\mathrm{d}v(g) = \mathrm{o}\big[R^{n-p(K_1/K_2)}\big] \tag{1.3}$$

as $R \to \infty$, where $n \ge p(K_1/K_2)$.

Finally, assume that E satisfies a variational inequality on Ω of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_\Omega w\big(|\psi_t^*\,\mathrm{d}u|^p\big)\,\mathrm{d}v(g) \ge -\int_\Omega f|\xi| w\big(|\mathrm{d}u|^p\big)\,\mathrm{d}v(g)\,, \tag{1.4}$$

where f is a given function on Ω (Theorem 2.1); ψ_t is any 1-parameter family of compactly supported diffeomorphisms of M with $\psi_0 =$ identity and t a small parameter;

$$\xi = \frac{\mathrm{d}\psi(x,t)}{\mathrm{d}t}\bigg|_{t=0}$$

is the initial velocity field of the flow generated by ψ_t.

In this note we give conditions on f and Ω which imply that u is the constant map almost everywhere on M.

The class of functionals considered includes radial perturbations of finite-energy polyharmonic maps. Hypothesis (1.4) is somewhat weaker, however, than the corresponding variational hypothesis used in the study of minimal surfaces and harmonic maps; a map is said to be *r-stationary* on a manifold M [1], [5] if the first r-variation

$$\delta E^r(u) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} E(u \circ \psi_t)$$

vanishes on M for all families ψ_t defined as in (1.4). In our case

$$\begin{aligned}\delta E^r(u) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int w\big(|\psi_t^* \, \mathrm{d}u|^p\big)\,\mathrm{d}v(g)\\ &= -\int w \,\mathrm{div}_M \xi \,\mathrm{d}v(g) + \\ &\quad + p\int w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^{p-2}\big\langle \mathrm{d}u(\nabla_{e_i}\xi)\,,\,\mathrm{d}u(e_i)\big\rangle\,\mathrm{d}v(g)\,,\end{aligned}$$

where $\{e_i\}$ is an orthonormal basis for TM.

Under the hypotheses of the theorems given in Section 2, this object does *not* vanish on M. However, if $p = 2$ and w is the identity, then our choice of the function f in Theorem 2.1 gives (1.4) a geometric interpretation as a decay condition on a generalized r-tension vector field bounded by f (cf. [1], [5]).

Our results are extensions of a theorem by Price for r-stationary harmonic maps on $\mathbb{R}^n$ or $\mathbb{H}^n$ [5]. Theorem 2.2 extends work by Costa and Liao [2], also for harmonic maps. A special case of Theorem 2.1 and some extensions to other types of fields are given in [3] and [4].

2. Theorems

THEOREM 2.1.— *Let u, M, N, B, E, w, and p be defined as in Section 1 and satisfy $\forall\ R > 0$ conditions (1.1)-(1.4), where in (1.4) $\Omega = B$. Let the function f in (1.4) satisfy for all $y \in B$, for some parameter $\tau \in (0, R)$ and for some constant $\varepsilon > 0$*

$$f(r) = f_\tau(r) = \begin{cases} K_3/r\tau & \text{if } 0 \le r \le \tau \\ 0 & \text{if } r > \tau + \varepsilon. \end{cases} \tag{2.1}$$

Then u takes almost every point on M onto a single point of N.

Although $f(x,\tau)$ is o(1) as $\tau \to \infty$, the growth condition (1.3) insures that the righthand side of (1.4) will in general be nonzero even at the limit. If condition (1.3) is strengthened, then Theorem 2.1 is true even for certain maps which do not satisfy (1.4) on all of B.

THEOREM 2.2.— *Let condition (1.3) be replaced by the hypothesis that* $du \in L^{p\left(\frac{k-\varepsilon_0}{k-1-\varepsilon_0}\right)}$ *for some* $\varepsilon_0 \in (0, k-1)$ *and some* $k \in (1,n)$*; but assume only that* u *satisfies (1.4) on* $\Omega = B \sim \Sigma$*, where* Σ *is a compact subset of* B *of Hausdorff codimension* k*. Then the conclusion of Theorem 2.1 holds.*

Theorem 2.1 follows from a technical lemma:

LEMMA 2.3. (Asymptotic monotonicity).— *Assume the hypotheses of Theorem 2.1 with the possible exception of (1.3). Then* u *satisfies almost everywhere on* $\Omega = B_\tau(x_0)$ *the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\frac{\alpha}{\tau}}\tau^{p(K_1/K_2)-n}E(w,u,p,\tau)\right] \geq 0\,, \tag{2.2}$$

where

$$\alpha = 2\left[1 + p({}^{K_1}/{}_{K_2})\right](n-1)K_0 + K_3\,.$$

In order to prove Theorem 2.2 we must prove

LEMMA 2.4.— *The conclusion of Lemma 2.3 holds under the assumptions of Theorem 2.2.*

If the energy over $B_\tau(x_0)$ satisfies (1.3), then certainly the product of the function $\exp(-{}^{\alpha}/{}_{\tau})$ and the energy over $B_\tau(x_0)$ will also satisfy (1.3). Thus inequality (2.2) implies Theorem 2.1. Similarly, since the hypotheses of Theorem 2.2 imply that u is in fact a finite-energy map on all of M, then (1.3) is satisfied and Lemma 2.4 implies Theorem 2.2.

3. Proofs

We prove Lemma 2.3 by modifying arguments by Price [5] (see also [1]). Initially we take $p \geq 2$.

Inequality (1.4) can be written

$$\begin{aligned} &-\int_\Omega w \operatorname{div}_M \xi \, dv(g) + p\int_\Omega w'(|du|^p)|du|^{p-2} \times \\ &\times \left\{ \left\langle du\left(\nabla_{\frac{\partial}{\partial r}} \xi\right), du\left(\frac{\partial}{\partial r}\right) \right\rangle + \left\langle du\left(\nabla_{\frac{\partial}{\partial \theta_i}} \xi\right), du\left(\frac{\partial}{\partial \theta_i}\right) \right\rangle \right\} dv(g) \geq \\ &\geq -\int_\Omega f|\xi| w(|du|^p) | \, dv(g), \end{aligned} \tag{3.1}$$

where $({}^{\partial}/_{\partial r}, {}^{\partial}/_{\partial \theta_i})$, $i = 2, 3, \ldots, n$, is an orthonormal basis for TM. Choose

$$\xi(y) = (y - x_0)\gamma(r),$$

where $\gamma \in C^1$ is chosen so that $\gamma(r) \geq 0$, $\gamma'(r) \leq 0$, $\gamma(r) = 1$ for $r \leq \tau$, and $\gamma(r) = 0$ for $r > \tau + \delta$ with $\delta > 0$ a small constant and $\tau \in (0, R)$. Then

$$\operatorname{div}_M \xi \geq \gamma' r + \gamma[n - (n-1)\lambda r].$$

Also, explicit computation of the inner product $\left\langle \nabla_{r\frac{\partial}{\partial \theta_i}} \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_j} \right\rangle$ gives, by Young's inequality,

$$\left\langle du\left(\nabla_{r\frac{\partial}{\partial \theta_i}} \xi\right), du\left(\frac{\partial}{\partial \theta_i}\right) \right\rangle \leq \gamma\left(|du|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right) + (n-1)\lambda\gamma r|du|^2.$$

Putting these estimates together and cancelling where possible yields in place of (3.1) the inequality (for $\Omega = B$)

$$\begin{aligned} &\int_B w(|du|^p) \left\{\gamma' r + \gamma[n - (n-1)\lambda r] - f\gamma r\right\} dv(g) \leq \\ &\leq p\int_B w'(|du|^p)|du|^{p-2} \left\{ \left|\frac{\partial u}{\partial r}\right|^2 r\gamma' + \gamma|du|^2[1 + (n-1)\lambda r] \right\} dv(g). \end{aligned} \tag{3.2}$$

Let $\gamma(r) = \gamma_\tau(r) = \varphi({}^r/_\tau)$ for φ satisfying

$$\varphi(s) = \begin{cases} 1 & \text{for } s \in [0, 1] \\ 0 & \text{for } s \in [1 + \delta, \infty) \end{cases} \tag{3.3}$$

and $\varphi'(r) \leq 0$. Then

$$\tau\left(\frac{\partial}{\partial\tau}\right)\gamma_\tau(r) = -r\gamma'(r) \geq 0$$

and (3.2) becomes

$$\begin{aligned}
p\int_B w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^{p-2}\left|\frac{\partial u}{\partial r}\right|^2 \tau\left(\frac{\partial\gamma}{\partial\tau}\right)\,\mathrm{d}v(g) \leq \\
\leq \tau\left(\frac{\partial}{\partial\tau}\right)\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) + p\int_B w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^p\gamma\,\mathrm{d}v(g) \\
- n\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) + \int_B w\big(|\mathrm{d}u|^p\big)\big[\gamma(n-1)r\lambda + f\gamma r\big]\,\mathrm{d}v(g) \\
+ p\int_B w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^p\gamma(n-1)r\lambda\,\mathrm{d}v(g).
\end{aligned} \tag{3.4}$$

Using hypotheses (1.2),

$$\begin{aligned}
p\int_B w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^p\gamma\,\mathrm{d}v(g) - n\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) \leq \\
\leq p\int_B K_1|\mathrm{d}u|^p\gamma\,\mathrm{d}v(g) - n\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) \\
\leq \big[p(K_1/K_2) - n\big]\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g).
\end{aligned}$$

A similar inequality can be obtained for the last integral on the right in (3.4), yielding

$$\begin{aligned}
p\int_B w'\big(|\mathrm{d}u|^p\big)|\mathrm{d}u|^{p-2}\left|\frac{\partial u}{\partial r}\right|^2 \tau\left(\frac{\partial\gamma}{\partial\tau}\right)\,\mathrm{d}v(g) \leq \\
\leq \tau\left(\frac{\partial}{\partial\tau}\right)\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) + \big[p(K_1/K_2) - n\big]\int_B w\big(|\mathrm{d}u|^p\big)\gamma\,\mathrm{d}v(g) \\
+ \int_B w\big(|\mathrm{d}u|^p\big)\gamma\Big\{2\tau\big[1 + p(K_1/K_2)\big](n-1)\lambda + fr\Big\}\,\mathrm{d}v(g).
\end{aligned} \tag{3.5}$$

In (3.5) we have used (3.3) to effectively bound r by 2τ. Choose f as in (2.1) and multiply (3.5) by the integrating factor

$$\exp\left\{-\frac{2[1 + p(K_1/K_2)](n-1)K_0 + K_3}{\tau}\right\}\tau^{p(K_1/K_2)-n-1}.$$

Since $\tau < R$ we can use (1.1) to bound λ by ${K_0}/{\tau^2}$ in (3.5) and, using the definition of γ, obtain (2.2).

If $0 < p < 2$ we replace (1.4) by a hypothesis lacking explicit variational meaning, namely, that for all $\varepsilon' > 0$ we have

$$\begin{aligned}
&-\int_\Omega w(|du|^p)\operatorname{div}_M \xi \, dv(g) + \\
&\qquad + p\int_\Omega w'(|du|^p)(|du| + \varepsilon')^{p-2}\langle du(\nabla_{e_i}\xi), du(e_i)\rangle \, dv(g) \geq \\
&\geq -\int_\Omega f|\xi| w(|du|^p)\, dv(g).
\end{aligned} \tag{3.6}$$

Reasoning exactly as in the case $p \geq 2$ but with (3.6) replacing (1.4) we find that the term

$$p\int_B w'(|du|^p)(|du| + \varepsilon')^{p-2}\gamma |du|^2 \big[1 + (n-1)\lambda r\big]\, dv(g)$$

only gets bigger if we replace it by the term

$$\int_B w'(|du|^p)|du|^p \gamma \big[1 + (n-1)\lambda r\big]\, dv(g).$$

Thus we eventually arrive at the inequality

$$\begin{aligned}
&p\int_B w'(|du|^p)(|du| + \varepsilon')^{p-2}\left|\frac{\partial u}{\partial r}\right|^2 \tau \left(\frac{\partial \gamma}{\partial \tau}\right) dv(g) \leq \\
&\quad \leq \tau \int_B w(|du|^p)\left(\frac{\partial \gamma}{\partial \tau}\right) dv(g) + \big[p({K_1}/{K_2}) - n\big]\int_B w(|du|^p)\gamma \, dv(g) \\
&\quad + \int_B w(|du|^p)\gamma\Big\{2\tau\big[1 + p({K_1}/{K_2})\big](n-1)\lambda + fr\Big\}\, dv(g).
\end{aligned} \tag{3.7}$$

Notice that since γ is defined to have compact support in B the right-hand side of (3.7) is bounded independently of ε' if f satisfies (2.1). Thus we can let ε' tend to zero inside the integral on the left. Obviously

$$(|du| + \varepsilon')^{p-2} \leq \left(\left|\frac{\partial u}{\partial r}\right| + \varepsilon'\right)^{p-2}$$

for $p \in (0,2)$, so

$$(|du| + \varepsilon')^{p-2}\left|\frac{\partial u}{\partial r}\right|^2 \leq \left(\left|\frac{\partial u}{\partial r}\right| + \varepsilon'\right)^{p-2}\left|\frac{\partial u}{\partial r}\right|^2,$$

which for $0 < p < 2$ is bounded by $\left|{\partial u}/{\partial r}\right|^p$ as $\varepsilon' \to 0$.

This completes the proof of Lemma 2.3, which in turn proves Theorem 2.1.

We conclure by proving Lemma 2.4 (and thus Theorem 2.2).

We require two classical lemmas :

LEMMA 3.1. (Serrin [6]).— *Denote by $U(\Sigma)$ the class of smooth functions $\overline{\eta}(x)$ which satisfy $0 \leq \overline{\eta} \leq 1$ and vanish in some neighborhood of the compact set Σ. Let Σ have zero s-capacity for $1 \leq s \leq n$. Then there exists a sequence of functions $\overline{\eta}^{(\nu)}$ contained in $U(\Sigma)$ such that $\overline{\eta}^{(\nu)} \to 1$ a.e. and $\|\nabla \overline{\eta}^{(\nu)}\|_{L^s} \to 0$.*

LEMMA 3.2. (Carlson [7]).— *Let the compact set Σ have Hausdooff dimension m, $0 < m < n-1$. Then the s-capacity of Σ is zero, where $s = n - m - \varepsilon_0$ for ε_0 in the interval $(0, n-m-1)$.*

In proving Lemma 2.4 we argue just as in the proof of Lemma 2.3, but we choose

$$\xi(y) = (y - x_0)\overline{\eta}(r)\gamma(r)\,,$$

where $\overline{\eta} = \overline{\eta}^{(\nu)}$ is the sequence of Lemma 3.1. In computing (3.2) we obtain on the right an extra term involving the derivative of $\overline{\eta}^{(\nu)}$; this term can be estimated

$$\begin{aligned}\int_B \overline{\eta}^{(\nu)\prime}(r) r\gamma w\big(|\mathrm{d}u|\big)^p \,\mathrm{d}v(g) &\geq -K_1 \int_B \left|\nabla \overline{\eta}^{(\nu)}\right| r|\mathrm{d}u|^p \,\mathrm{d}v(g) \\ &\geq -K_1 \left\|\nabla \overline{\eta}^{(\nu)}\right\|_{L^{k-\varepsilon_0}(B)} \left\|\mathrm{d}u\right\|_{L^{p(\frac{k-\varepsilon_0}{k-\varepsilon_0-1})}(M)}\,,\end{aligned}$$

which tends to zero as ν tends to infinity. The remainder of the proof of Lemma 2.4 is identical to that of Lemma 2.3.

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