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Asymptotic estimates in Weighted Hölder spaces for a class of elliptic scale-covariant second order operators

PIOTR T. CHRUŚCIEL[(1)(2)]

RÉSUMÉ. — On dérive des estimations a *priori* dans les espaces de Hölder à poids pour des opérateurs elliptiques de second ordre "covariant" sous l'action des transformations de changement d'échelle, $r \to \lambda r$. On démontre l'existence des solutions du "problème de Dirichlet extérieur modulo un nombre fini d'harmoniques sphériques", ce qui est un résultat utile lorsqu'on s'intéresse au problème d'existence de solutions de l'équation avec source indépendamment des valeurs au bord de la solution (l'alternative habituelle de Fredholm ne garantit pas existence de solutions pour toute source). Finalement les résultats obtenus sont utilisés pour établir un développement asymptotique fin de certaines solutions de l'équation de p-Laplace pour des variétés asymptotiquement euclidiennes.

ABSTRACT. — Pointwise asymptotic estimates for solutions of second order elliptic equations of a certain form are derived.

1. Introduction

It as been recently observed [Ki] [JK] that the existence of solutions of the p-Laplace equation.

$$\operatorname{div}\big(|\operatorname{grad} f|^{p-2} \operatorname{grad} f\big) = 0 \tag{1.1}$$

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satisfying certain conditions allows an elementary proof to the positive energy theorem in general relativity. In order to carry out this proof one needs certain fine asymptotic estimates on the behaviour on the solutions of (1.1) at infinity. The proof of such estimates, presented in section 4 of this paper, requires an asymptotic estimate on the behaviour of solutions of linear equations of the form(*):

$$\left[\delta_{ij} + (p-2)\frac{x^i x^j}{r^2}\right] f_{,ij} = \rho\,, \quad \rho = O(r^{\alpha-2})\,. \tag{1.2}$$

A sharp estimate for equations of this type does no seem to exist in the literature unless $p = 2$ [Me]. Equations of this type have been studied by Bagirov and Kondratev [BK] in weighted Sobolev spaces, their results can be used to obtain pointwise estimates via weighted embeddings theorems which, however, are not sharp as far as the pointwise decay (or growth) rates are concerned. If $p = 2$ pointwise estimates of the appropriate type have been obtained by N. Meyers [Me] (cf. also [CSCB] and references therein for a restricted range of decay rates but for more general operators). The object of this paper is to present an elementary argument which leads to a *priori* estimates for solutions of equations of the form:

$$\left[a\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{b}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{c+\Delta_S}{r^2}\right] f = \rho \tag{1.3}$$

$a \in \mathbf{R}^+$, $b, c \in \mathbf{R}$, Δ_S is the Laplace-Beltrami operator on a compact Riemannian manifold S (in the case of interest, eq. (1.2), S is a sphere). In the particular case $a = 1$, $b = n-1$, $c = 0$, $S = S_{n-1}$, (1.3) is just the Laplace equation and we recover the results of Meyers [Me], using an apparently much simpler method. The idea of the proof given here is to solve (1.3) by explicit integrals for a finite number of terms in the harmonic decomposition of ρ and to use the comparison principle for an "energy integral" to show that the remainder also has the required asymptotic properties. When (1.3) is the Laplace equation, it is well known that for ρ decaying as r^{-k}, k-integer, $k \geq n-2$, there will in general be logarithmic terms in f; therefore it is natural to consider a logarithmic weight for ρ as well since in some applications on needs to iterate (1.3). Our main result can loosely be described as follows: if

$$|\rho| \leq C r^{\alpha}(1+\ln r)^q\,,$$

(*) A comma denotes partial differentiation, the summation convention is used throughout.

there always exists a solution of (1.3) satisfying:

$$|f| \leq C' r^{\alpha+2}(1+\ln r)^{q+2}. \tag{1.4}$$

If we know the spectrum of S, the $q+2$ exponent in the logarithm above can be replaced by $q+1$ or by q under certain conditions. In section 2 the asymptotic estimates are derived. In section 3 existence of solutions of (1.3) satisfying (1.4) is established using the results of section 2. In section 4 the results of section 2 are used to obtain the desired asymptotic estimate for solutions of the p-Laplace equation.

2. Asymptotic Estimates

Let $r \in [1,\infty)$, let S be a compact Riemannian manifold(*), let $\Omega = [1,\infty) \times S$, let a twice differentiable function f satisfy:

$$Lf = \left[a\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{b}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{c+\Delta_S}{r^2}\right] f = \rho \tag{2.1}$$

$a \in \mathbf{R}^+$, $b, c \in \mathbf{R}$. Let $\{\varphi_i\}$ be a complete orthonormal set of eigenfunctions of Δ_S, $\Delta_S \varphi_i = -\lambda_i \varphi_i$, φ_i ordered in such a way that $\lambda_i \leq \lambda_j$ for $i < j$. Let $\mathcal{H}_i$ denote an eigenspace of $-\Delta_S$, with the indices ordered with increasing eigenvalues, let $O_k = \bigcup_{i\leq k} \mathcal{H}_i$, let V_k be the orthogonal complement of O_k (in $L_2(S)$). Associated to the eigenvalues of $-\Delta_S$ are the "characteristic decay exponents $\mu_i^{\pm}$":

$$\mu_i^{\pm} = \frac{a - b \pm \sqrt{(b-a)^2 + 4a(\lambda_i - c)}}{2a}. \tag{2.2}$$

For $x = (r_1, p_1)$, $y = (r_2, p_2)$, we set $|x-y|^2 = (r_1 - r_2)^2 + d(p_1, p_2)^2$, where d is the Riemannian distance on S. For $k \in \mathbf{N} \cup \{0\}$, $\alpha, q \in \mathbf{R}$, $\lambda \in (0,1)$, we define:

$$\|f\|_{C^k_{\alpha,q}} = \sum_{|i|\leq k} \sup_{x\in\Omega} \left| D^i f(x) r^{-\alpha+|i|}(1+\ln r)^{-q}\right|$$

$$[f]_{\lambda,\alpha,q} = \sup_{x\in\Omega} \sup_{y\in\{[\frac{1}{2}r(x),\,2r(x)]\times S\}\cap\Omega} \frac{|f(x)-f(y)|\, r^{-\alpha+\lambda}(1+\ln r)^{-q}}{|x-y|^{\lambda}}$$

$$\|f\|_{C^{k,\lambda}_{\alpha,q}} = \|f\|_{C^k_{\alpha,q}} + \sum_{|i|=k} \left[r^k D^i f\right]_{\lambda,\alpha,q},$$

(*) We assume that S has no boundary. All the results presented here will be valid if f is assumed to vanish on ∂S, when non-empty.

$C_\alpha^{k,\lambda} \equiv C_{\alpha,0}^{k,\lambda}$, $C_\alpha^k \equiv C_{\alpha,0}^k$, $C_{\alpha,q} \equiv C_{\alpha,q}^0$, $C^{k,\lambda} \equiv C_0^{k,\lambda}$, $C_\alpha \equiv C_{\alpha,0}^0$. $W_{k,p}$ denotes the standard Sobolev space, $L_p = W_{0,p}$. The letter C denotes a generic constant which may vary from line to line. S is assumed to be equipped with a smooth Riemannian metric for simplicity, $d\mu$ is the Riemannian normalized (to one) measure on S. f will be said harmonic if $Lf = 0$.

We shall need the following elementary result:

LEMMA 2.1.— *Let $\rho = \rho_i(r)\varphi_i$ (no summation over i), with $\Delta_S\varphi_i = -\lambda_i\varphi_i$, let the function f_i be given by:*

a) If $\mu_i^+ \neq \mu_i^-$, let:

$$f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)}\left\{f_i^+ r^{\mu_i^+} + f_i^- r^{\mu_i^-} + r^{\mu_i^+}\int_1^r s^{1-\mu_i^+}\rho_i(s)\,\mathrm{ds} - r^{\mu_i^-}\int_1^r s^{1-\mu_i^-}\rho_i(s)\,\mathrm{ds}\right\}, \quad f_i^\pm \in \mathbf{R}. \tag{2.3}$$

b) If $\mu_i^+ \neq \mu_i^-$ and if $s^{1-\mu_i^+}\rho_i(s) \in L_1([1,\infty))$(*) *let:*

$$f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)}\left\{r^{\mu_i^-}\left(f_i^- - \int_1^r s^{1-\mu_i^-}\rho_i(s)\,\mathrm{ds}\right) - r^{\mu_i^+}\int_r^\infty s^{1-\mu_i^+}\rho_i(s)\,\mathrm{ds}\right\}, \quad f_i^- \in \mathbf{R}. \tag{2.4}$$

c) If $\mu_i^+ \neq \mu_i^-$ and if $s^{1-\mu_i^-}\rho_i(s) \in L_1([1,\infty))$, let:

$$f_i = \frac{1}{a(\mu_i^+ - \mu_i^-)}\left\{r^{\mu_i^-}\left(f_i^- + \int_r^\infty s^{1-\mu_i^-}\rho_i(s)\,\mathrm{ds}\right) - r^{\mu_i^+}\int_r^\infty s^{1-\mu_i^+}\rho_i(s)\,\mathrm{ds}\right\}, \quad f_i^- \in \mathbf{R}. \tag{2.5}$$

d) If $\mu_i^+ = \mu_i^- = \mu_i$, let:

$$f_i = \frac{r^{\mu_i}}{a}\left\{\alpha_i + \beta_i \ln r + \ln r\int_1^r s^{1-\mu_i}\rho_i(s)\,\mathrm{ds} - \int_1^r s^{1-\mu_i}\rho_i(s)\ln s\,\mathrm{ds}\right\}, \quad \alpha_i, \beta_i \in \mathbf{R}. \tag{2.6}$$

(*) (2.4) and (2.5) are clearly special cases of (2.3), it is however useful to write them explicitly for the sake of clarity of the proof of theorem 2.1. A similar remark applies to (2.6)-(2.8).

e) If $\mu_i^+ = \mu_i^- = \mu_i$ and $s^{1-\mu_i}\rho_i(s) \in L_1([1,\infty))$, let:

$$f_i = -\frac{r^{\mu_i}}{a}\left\{\alpha_i + \ln r \int_r^\infty s^{1-\mu_i}\rho_i(s)\,\mathrm{d}s + \int_1^r s^{1-\mu_i}\rho_i(s)\ln s\,\mathrm{d}s\right\}, \quad \alpha_i \in \mathbf{R}. \tag{2.7}$$

f) If $\mu_i^+ = \mu_i^- = \mu_i$ and $s^{1-\mu_i}\rho_i(s)\ln s \in L_1([1,\infty))$, let:

$$f_i = \frac{r^{\mu_i}}{a}\left\{\alpha_i + \int_r^\infty s^{1-\mu_i}\rho_i(s)\ln s\,\mathrm{d}s - \ln r \int_1^r s^{1-\mu_i}\rho_i(s)\,\mathrm{d}s\right\}, \quad \alpha_i \in \mathbf{R}. \tag{2.8}$$

The function $f = f_i\varphi_i$ satisfies $Lf = \rho$.

The following computational lemma shows that under appropriate conditions the functions $\pm(C_1 + \ln r)^q r^\alpha$ are super- and sub-solutions of the equation (2.1).

LEMMA 2.2.— *Let $a \geq 0$, $b, e, \alpha, q \in \mathbf{R}$. There exist positive constants $C_1(a,b,e,\alpha,q)$ and $C_2(a,b,e,\alpha,q)$ such that, if:*

$$e < -\alpha\big[a(\alpha-1)+b\big]$$

then:

$$\left[a\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{b}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{e}{r^2}\right](C_1 + \ln r)^q\, r^\alpha \leq -C_2\,(1+\ln r)^q\, r^{\alpha-2}. \tag{2.9}$$

We shall also need the following well known result:

LEMMA 2.3.— *For $\varphi \in W_{1,2}(S) \cap V_k$ we have:*

$$\int_S (\nabla\varphi)^2\,\mathrm{d}\mu \geq \lambda_{k+1}\int_S \varphi^2\,\mathrm{d}\mu. \tag{2.10}$$

LEMMA 2.4.— *Let $\Omega_0 = [1\,,\,R_0] \times S$ or let $\Omega_0 = [1,\infty) \times S$, let f satisfy in Ω_0:*

$$Lf = \rho\,, \quad \rho \in C_{\alpha-2,q}\,, \quad \sup_{p\in S}|f(1,p)| \leq M\,,$$

and

a) if $\Omega_0 = [\,1\,,\,R_0\,] \times S$ *let* $\sup_{p\in S} |f(R_0,p)| \leq M R_0^{\alpha}(1+\ln R_0)^q$,
b) if $\Omega_0 = [\,1,\infty) \times S$ *let* $f = O(r^{\beta})$, $\beta > \alpha$.

There exists $m_0(a,b,c,\lambda_i,\alpha,\beta)$ *such that the condition:*

$$\forall\,\varphi \in O_{m_0}\,,\ \forall\,r \in [\,1,\infty)\,,\ \int_S f(r,p)\varphi(p)\,\mathrm{d}\mu = 0 \tag{2.11}$$

implies $f \in C_{\alpha,q}(\Omega_0)$*, and:*

$$\|f\|_{C_{\alpha,q}(\Omega_0)} \leq C_3\left(\|\rho\|_{C_{\alpha-2,q}(\Omega_0)} + M\right).$$

If moreover $\rho \in C^{k,\lambda}_{\alpha-2,q}$, $f|_{\{1\}\times S} = \varphi \in C^{k+2\,,\,\lambda}(S)$, *and* $f|_{\{R_0\}\times S} = 0$ *if* $\Omega_0 = [\,1\,,\,R_0\,] \times S$*, then:*

$$\|f\|_{C^{k+2\,,\,\lambda}_{\alpha,q}(\Omega_0)} \leq C_4\left(\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}(\Omega_0)} + \|\varphi\|_{C^{k+2\,,\,\lambda}(S)}\right)$$

for some constants $C_3(a,b,c,\lambda_{m_0+1},\alpha,\beta,q,\lambda)$, $C_4(a,b,c,\Delta_S,k,\alpha,\beta,q,\lambda)$. C_3 *and* C_4 *are* β*-independent if* $\Omega = [\,1\,,\,R_0\,] \times S$.

Proof.— If $\alpha < 0$ or $\beta \leq 0$ let $\widetilde{\beta} = 0$, if $\alpha > 0$ and $\Omega_0 = [\,1\,,\,R_0\,] \times S$ let $\widetilde{\beta} = \alpha + 1$, if $\alpha > 0$ and $\Omega_0 = [\,1,\infty) \times S$ let $\widetilde{\beta} = \beta$. The function $\widetilde{f} = r^{-\widetilde{\beta}} f$ satisfies an equation of the form (2.1):

$$\widetilde{L}\widetilde{f} = \widetilde{\rho}\,, \quad \widetilde{L} = a\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\widetilde{b}}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\widetilde{c} + \Delta_S}{r^2}$$

with:

$$\widetilde{b} = b + 2\widetilde{\beta}a\,, \quad \widetilde{c} = c + a\widetilde{\beta}^2 + (b-a)\widetilde{\beta}\,, \quad \widetilde{\rho} = r^{-\widetilde{\beta}}\rho\,.$$

Let $F(r) = \int_S \widetilde{f}^2(r,p)\,\mathrm{d}\mu$. We have:

$$\begin{aligned} a\frac{\mathrm{d}^2F}{\mathrm{d}r^2} &= 2\left\{a\int_S\left[\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}r}\right]^2 - \frac{\widetilde{b}}{r}\int_S \widetilde{f}\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}r} - \frac{\widetilde{c}}{r^2}\int_S \widetilde{f}^2 - \frac{1}{r^2}\int_S \widetilde{f}\Delta_S\widetilde{f} + \int_S \widetilde{f}\rho\right\} \\ &\geq -\frac{\widetilde{b}}{r}\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{(2\widetilde{c} - 2\lambda_{m_0+1} + 1)}{r^2}F - \int_S \widetilde{\rho}^2 r^2\,, \end{aligned}$$

and we have used lemma 2.3 and (2.11) to estimate the $\int \widetilde{f}\Delta_S\widetilde{f}$ term, together with $2\widetilde{f}\widetilde{\rho} \geq -\dfrac{\widetilde{f}^2}{r^2} - \widetilde{\rho}^2 r^2$, so that:

$$a\frac{\mathrm{d}^2F}{\mathrm{d}r^2} + \frac{\widetilde{b}}{r}\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{2\widetilde{c} + 1 - 2\lambda_{m_0+1}}{r^2}F \geq -\|\rho\|^2_{C_{\alpha-2\,,\,q}} r^{2\widetilde{\alpha}-2}(1+\ln r)^{2q}\,,$$
$$\widetilde{\alpha} = \alpha - \widetilde{\beta}\,.$$

If $\lambda_{m_0+1} > \{\widetilde{\alpha}[a(\widetilde{\alpha}+1)+\widetilde{b}]+2\widetilde{c}+1\}/2$, lemma 2.2 and the comparison principle yield ($F \to 0$ as $r \to \infty$ if $R_0 = \infty$):

$$\forall\, r \in [1, \infty)\,, \quad F^{1/2}(r) \le C r^{\widetilde{\alpha}} (1+\ln r)^q \left(\|\rho\|_{C_{\alpha-2\,,\,q}} + M \right) . \qquad (2.12)$$

For $4 \le R \le R_0/4$ and $r \in (1/2\,, 2)$ consider:

$$f_R(r, p) = R^{-a}(1+\ln r)^{-q} f(Rr\,, p)\,.$$

f_R satisfies $Lf_R = \rho_R$ with

$$\rho_R(r, p) = R^{-\alpha}(1+\ln R)^{-q} \rho(Rr\,, p)$$

and (2.12) implies:

$$\|f_R\|_{L_2((1/2\,,2)\times S)} \le C \left(\|\rho\|_{C_{\alpha-2\,,\,q}} + M \right) .$$

Interior L_2 estimates [Se] and Schauder theory [GT] lead to:

$$\begin{aligned} \|f_R\|_{C([1/2\,,2]\times S)} &\le C \left(\|\rho\|_{C_{\alpha-2\,,\,q}} + M \right) , \\ \|f_R\|_{C^{k+2,\lambda}([1/2\,,2]\times S)} &\le C \left(\|\rho\|_{C^{k,\lambda}_{\alpha-2\,,\,q}} + M \right) , \end{aligned} \qquad (2.13)$$

(2.13) and L_∞ or $C^{k,\alpha}$ up to boundary estimates [Se] [GT] give:

$$\begin{aligned} \|f\|_{C([1\,,4]\times S)} &\le C \left(\|\rho\|_{C_{\alpha-2\,,\,q}} + M \right) , \\ \|f\|_{C^{k+2,\lambda}([1\,,4]\times S)} &\le C \left(\|\rho\|_{C^{k,\lambda}_{\alpha-2\,,\,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)} \right) . \end{aligned}$$

These estimates combined with the rescaled version of (2.13) yield the claimed result.

We shall state in detail our estimates only when all the μ_i^+ and μ_i^- are distinct (cf. remark 4 below):

THEOREM 2.1.— *Suppose that $\forall\, i$, $\mu_i^+ \ne \mu_i^-$. Let f be a twice differentiable function satisfying:*

$$Lf = \rho\,, \;\; f = O(r^\beta)\,, \;\; \rho = O\big(r^{\alpha-2}(1+\ln r)^q\big)\,, \;\; \alpha < \beta\,, \;\; \sup_{p\in S} |f(1, p)| = M\,,$$

$\alpha, \beta, q \in \mathbf{R}$, and if $\alpha \in \{\Re\mu_i^{\pm}\}$ we shall suppose $q \neq 1$, if $\beta \in \{\Re\mu_i^{\pm}\}$ we shall suppose $q > 0$. f can be written in the form:

$$f = f_H + f_H^- + f_{\log} + f_\rho \tag{2.14}$$

where:

$$Lf_H = Lf_H^- = 0 \tag{2.15}$$

and:

$$f_H = \sum_{\Re\mu_i^+ < \beta} \left(A_i^+ r^{\mu_i^+} + A_i^- r^{\mu_i^-}\right) \varphi_i\,, \quad A_i^{\pm} \in \mathbf{R} \tag{2.16}$$

$$|f_H| \le C(f)\, r^{\mu^+}\,, \quad \mu^+ = \max_{\Re\mu_i^+ < \beta} \Re\mu_i^+ \tag{2.17}$$

$$f_H^- = \sum_{\substack{i:\Re\mu_i^+ \ge \beta \\ \alpha \le \Re\mu_i^- < \beta}} c_i^- r_i^{\mu_i^-} \varphi_i \tag{2.18}$$

$$\begin{aligned} c_i^- &= \int_S f(1,p)\varphi_i(p)\,\mathrm{d}\mu \\ &+ \frac{1}{a(\mu_i^+ - \mu_i^-)} \int_{[1,\infty)\times S} \left(r^{1-\mu_i^+} - r^{1-\mu_i^-}\right) \varphi_i(p)\,\rho(r,p)\,\mathrm{d}r\,\mathrm{d}\mu \end{aligned} \tag{2.19}$$

$$|f_H^-| \le C_H^- \left(M + \|\rho\|_{C_{\alpha-2,q}}\right) r^{\mu^-}\,, \quad \mu^- = \max_{\substack{\Re\mu_i^+ \ge \beta \\ \alpha < \Re\mu_i^- < \beta}} \Re\mu_i^- \tag{2.20}$$

$$C_H^- = C_H^-(a, b, c, \alpha, \beta, \lambda_i)$$

$$f_{\log} = \sum_{i:\Re\mu_i^- = \alpha} f_i^-(r)\varphi_i + \sum_{\substack{i:\Re\mu_i^+ = \alpha \\ \Re\mu_i^- \neq \alpha}} f_i^+(r)\varphi_i \tag{2.21}$$

$$|f_{\log}| \le C_{\log} \|\rho\|_{C_{\alpha-2,q}} r^{\alpha}(1 + \ln r)^{q+1} \tag{2.22}$$

$$|f_\rho| \le C_0 \left(M + \|\rho\|_{C_{\alpha-2,q}}\right) r^{\alpha}(1 + \ln r)^q\,, \tag{2.23}$$

where the constants $C_{\log}$ *and* C_0 *depend on* a, b, c, α, q *and the spectrum of* $S^{(*)}$. *If* $\rho \in C^{k,\lambda}_{\alpha-2,q}$ *then:*

$$\|f_\rho\|_{C^{k+2,\lambda}_{\alpha,q}} \leq C(k,\lambda,\alpha,q,\Delta_S,a,b,c)\left(\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)}\right).$$

$$\|f_{\log r}\|_{C^{k+2,\lambda}_{\alpha,q+1}} \leq C(k,\lambda,\alpha,q,\Delta_S,a,b,c)\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} \tag{2.24}$$

where $\varphi = f\Big|_{\{1\}\times S}$.

Remarks :

1) The coefficients $A_i^{\pm}$ are neither determined by the asymptotic condition $f = O(r^\beta)$ nor by $f\Big|_{\{1\}\times S}$.
2) It should be stressed that all the sums in the statement of theorem 2.1 are over *finite* sets of indices.
3) If any sets of indices over which the summations are perfomed above are empty, then the corresponding term is of course zero. In particular, if $\{\alpha\} \cap \{\Re\mu_i^{\pm}\} = \emptyset$ there are no higher powers of $\ln r$ in f. It must also be noted that though the r dependence of the higher log terms is fairly arbitrary — dictated by the behaviour of ρ — the "angular" dependence of these terms is rigidly fixed.
4) If for some i_0 we have $\mu_{i_0}^+ = \mu_{i_0}^-$ then f will differ from (2.14) by a finite number of terms of the form (2.6)–(2.8), with radial behaviour which can "pick-up" up to two powers of $\ln r$ more than ρ has if $\alpha = \mu_{i_0}^+$, the details are straightforward and are left to the reader.
5) If $\alpha \in \{\Re\mu_i^{\pm}\}$ and $q = -1$ the function $f_{\log}$ will be replaced by $f_{\log(\log)}$, with (2.22) replaced by:
$$\left|f_{\log(\log)}\right| \leq C_{\log\log}(a,b,c,\alpha,\lambda_i)\,\|\rho\|_{C_{\alpha-2,q}}\, r^\alpha \ln(1+\ln r).$$
6) If $\beta \in \{\Re\mu_i^{\pm}\}$ and $q < 0$ the results remain valid except (2.18), where no terms with $\Re\mu_i^+ = \beta$ will appear in the summation.

Proof.— Let m_0 be given by lemma 2.4, let φ_i, $i = 1,\ldots,J$, be an orthonormal basis of O_{m_0}, let:

$$f_i(r) = \int_S f(r,p)\varphi_i(p)\,\mathrm{d}\mu$$
$$\rho_i(r) = \int_S \rho(r,p)\varphi_i(p)\,\mathrm{d}\mu.$$

(*) In fact only a finite number of λ_i's matters.

The function:

$$\tilde{\tilde{f}} = \left(f - \sum_{i=1}^{J} f_i \varphi_i \right)$$

satisfies the equation:

$$L\tilde{\tilde{f}} = \tilde{\tilde{\rho}}\,, \quad \tilde{\tilde{\rho}} = \rho - \sum_{i=1}^{J} \rho_i \varphi_i\,,$$

therefore by lemma 2.4 $|\tilde{\tilde{f}}| \le C r^{\alpha}(1+\ln r)^q \|\rho\|_{C_{\alpha-2,q}}$, and to achieve our proof, it is sufficient to estimate the finite number of terms $\sum f_i \varphi_i$. We have:

$$L(f_i\varphi_i) = \rho_i \varphi_i$$

therefore the f_i's are given by lemma 2.1. The set $I = \mathbb{N} \cap [\,0\,,\,J\,]$ can be partitioned as follows:

$$\begin{aligned}
I_\rho^0 &= \left\{ i \in I : \Re\mu_i^+ \ge \beta\,,\ \Re\mu_i^- \ge \beta \right\} \\
I_\rho &= \left\{ i \in I : \Re\mu_i^+ \ge \beta\,,\ \Re\mu_i^- < \alpha \right\} \\
I_H^- &= \left\{ i \in I : \Re\mu_i^+ \ge \beta\,,\ \alpha < \Re\mu_i^- < \beta \right\} \\
I_H &= \left\{ i \in I : \Re\mu_i^+ < \beta\,,\ \Re\mu_i^- \ne \alpha\,,\ \Re\mu_i^+ \ne \alpha \right\} \\
I_{\log}^1 &= \left\{ i \in I : \Re\mu_i^+ = \alpha\,,\ \Re\mu_i^- \ne \alpha \right\} \\
I_{\log}^2 &= \left\{ i \in I : \Re\mu_i^+ < \beta\,,\ \Re\mu_i^- = \alpha \right\} \\
I_{\log}^3 &= \left\{ i \in I : \Re\mu_i^+ \ge \beta\,,\ \Re\mu_i^- = \alpha \right\}\,.
\end{aligned}$$

If $\alpha \notin \{\Re\mu_i^{\pm}\}$ the sets $I_{\log}^a$, $a = 1,2,3$, are empty and it is sufficient to consider the first three sets above. For $i \in I_\rho^0$ the f_i's are of the form (2.5) with $f_i^- = 0$, let $f_\rho^0 = \sum_{i \in I_\rho^0} f_i \varphi_i$, each term satisfies (2.23) therefore their finite sum also will. For $i \in I_\rho$ the f_i's are of the form (2.4), let $f_\rho^1 = \sum_{i\in I_\rho} f_i\varphi_i$, the constants f_i^- are uniquely determined by ρ and $f\Big|_{\{1\}\times S}$ and again each term satisfies (2.23). For $i \in I_H^-$ the f_i's are of the form (2.5), let $f_\rho^2 = \sum_{i\in I_H^-}$ [integrals in (2.5)] φ_i, $f_H^- = \sum_{i \in I_H^-}$ [the f_i^- terms in (2.5)] φ_i, f_ρ^2 satisfies (2.23), and an elementary calculation gives (2.19). For $i \in I_H$ the f_i's are of the form (2.3), let $f_\rho^3 = \sum_{i \in I_H}$ [integrals in (2.3)] φ_i, let $f_H = \sum_{i\in I_H}$ [$f_i^{\pm}$ terms in (2.3)] φ_i. If $\alpha \notin \{\Re\mu_i^{\pm}\}$ we are done by setting $f_\rho = f_\rho^0 + f_\rho^1 + f_\rho^2 + f_\rho^3 + \tilde{\tilde{f}}$, and from what has been said the $C_{\alpha,q}$ estimates follow. For $i \in I^1_{\log}$ the f_i's are of the form (2.3), and one has to add to f_ρ the function:

$$f^4_\rho = \sum_{i \in I^1_{\log}} \left[\text{integrals in (2.3) with the kernel } s^{1-\mu_i^-}\right] \varphi_i \,,$$

f_H is modified by the appropriate harmonic terms from (2.3), and we set:

$$f^1_{\log} = \sum_{i \in I^1_{\log}} \left[\text{integrals in (2.3) with the kernel } s^{1-\mu_i^+}\right] \varphi_i \,.$$

Similarly,

$$\begin{aligned} f^2_{\log} = & \sum_{\substack{i \in I^2_{\log} \\ \Re\mu_i^- \neq \Re\mu_i^+}} \left[\text{integrals in (2.3) with the kernel } s^{1-\mu_i^-}\right] \varphi_i \,, \\ & + \sum_{\substack{i \in I^2_{\log} \\ \Re\mu_i^- = \Re\mu_i^+}} [\text{integrals in (2.3)}]\, \varphi_i \,, \end{aligned}$$

$$f^3_{\log} = \sum_{i \in I^3_{\log}} \left[\text{integrals in (2.4) with the kernel } s^{1-\mu_i^-}\right] \varphi_i \,,$$

$f_{\log} = f^1_{\log} + f^2_{\log} + f^3_{\log}$. The $C_{\alpha,q}$ estimates for $f_{\log}$, f_ρ, f_H and f_H^- follow directly from the estimate on $\approx\!\!\!\!f$ and the representations (2.3)–(2.5), the higher order $C^{k,\lambda}$ estimates follow from Schauder's theory and a scaling argument, as in the proof of lemma 2.4.

3. An Existence Theorem

THEOREM 3.1.— *Let $\rho \in C^{k,\lambda}_{\alpha-2,q}$, $q > -1$, let $I = \{i : \Re\mu_i^- > \alpha\}$, let $K = \{0\}$ if $I = \emptyset$ or $K = \operatorname{span}\{\varphi_i \,,\ i \in I\}$, and let $\varphi \in C^{k+2,\lambda}(S)$.*

a) There exists a constant $C(a, b, c, k, \lambda, \alpha, q, \Delta_S)$ and a function $f \in C^{k+2,\lambda}_{\alpha,q+2}$ satisfying:

$$Lf = \rho\,, \quad f\Big|_{\{1\}\times S} - \varphi \in K \tag{3.1}$$

and satisfying:

$$\|f\|_{C^{k+2,\lambda}_{\alpha,q+2}} \leq C\left(\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)}\right). \tag{3.2}$$

b) If the decay exponents $\mu_i^{\pm}$ are all distinct ($\forall\, i\,,\ \mu_i^+ \neq \mu_i^-$) or if $\mu_{i_0}^- = \mu_{i_0}^+$ but $\alpha \neq \mu_{i_0}^+$, then $f \in C^{k+2,\lambda}_{\alpha,q+1}$ and:

$$\|f\|_{C^{k+2,\lambda}_{\alpha,q+1}} \leq C\left(\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)}\right). \tag{3.3}$$

c) If $\forall\, i\,,\ \mu_i^+ \neq \mu_i^-$ and $\alpha \notin \{\Re\mu_i^{\pm}\}$, then $f \in C^{k+2,\lambda}_{\alpha,q}$ and:

$$\|f\|_{C^{k+2,\lambda}_{\alpha,q}} \leq C\left(\|\rho\|_{C^{k,\lambda}_{\alpha-2,q}} + \|\varphi\|_{C^{k+2,\lambda}(S)}\right). \tag{3.4}$$

Remarks :

1) We are not making any hypotheses about either the sign or the size of c, which allows for many solutions with the same boundary value and prescribed asymptotics to exist. If, however, $I_1 = \{i : \Re\mu_i^+ \leq \alpha\} = \emptyset$, then f is uniquely determined by (3.1) and (3.2). If $q > 0$ the condition:
$$\forall\, i \in I_1\,, \quad \int_S f(1,p)\varphi_i \,\mathrm{d}\mu = 0$$
can always be imposed and together with (3.1)–(3.2) renders f unique.

2) (3.2) guarantees the existence of a solution with reasonably well controlled asymptotic behaviour even if we do not know anything about the spectrum of Δ_S. The existence result above has also been mentioned in [BK] (cf. remark 2 in [BK]).

3) (3.1) essentially says that we are free to specify $f\big|_{\{1\}\times S}$ up to a finite number of spherical harmonics φ_i, $i \in I$.

4) If $q = -1$ points a) and c) still hold, but b) needs not hold if $\alpha \in \{\Re\mu_i^{\pm}\}$ — in this case one can derive a $\big[1 + \ln(1 + \ln r)\big]$ weighted estimate for f.

5) For $q < -1$, $q \neq -2$, a solution will always exist if I is replaced by $I' = \{i : \Re\mu_i^- \geq \alpha\}$, (3.2) and point c) holding. If the $\mu_i^{\pm}$ are distinct (3.3) will hold as well.

6) If $q=-2$ theorem 3.2 still holds with I replaced by I' (cf. remark 5) if all the $\mu_i^{\pm}$ are distinct or if $\alpha \neq \mu_{i_0}^{+} = \mu_{i_0}^{-}$: in the latter case (3.2) has, however, to be replaced by a $[1+\ln(1+\ln r)]$ weighted $C_\alpha^{k,\lambda}$ estimate.

Proof.— Let m_0 be given by lemma 2.4, let $\{\varphi_i\}$, $i=1,\dots,J$ be any orthonormal basis of O_{m_0}, let:

$$\rho_i(r) = \int_S \rho(r,p)\varphi_i(p)\,\mathrm{d}\mu\,,$$

let $\overline{\rho} = \rho - \sum \rho_i\varphi_i$, by construction $\overline{\rho}\Big|_{\{r\}\times S} \in O_{m_0}^{\perp}$. Let $\overline{C}_{\alpha,q}^{k,\lambda}(R)$ be the Banach space of functions satisfying:

$$\overline{C}_{\alpha,q}^{k,\lambda}(R) = \left\{ f \in C_{\alpha,q}^{k,\lambda}([1\,,\,R]\times S) : \forall r\; f(r,\,\cdot\,) \in O_{m_0}^{\perp} \right\}\,,$$

with the norm induced from $C_{\alpha,q}^{k,\lambda}$. If $c<0$ the problem

$$L(\overline{f}_R) = \overline{\rho}\,,\quad \overline{f}_{R|_{\{1\}\times S}} = P_{O_{m_0}^{\perp}}\varphi\,,\quad \overline{f}_{R|_{\{R\}\times S}} = 0$$

has a unique solution for all $R\geq 1$, where $P_{O_{m_0}^{\perp}}$ is the orthogonal projection (in $L_2(S)$) on $O_{m_0}^{\perp}$. This is also true for $c>0$ which can be established by applying the continuity method [GT] in $\overline{C}_{\alpha,q}^{k,\lambda}(R)$ to:

$$L_t = a\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{b}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{tc+\Delta_S}{r^2}\,,\quad t\in[0\,,\,1]\,,$$

the appropriate estimates and injectivity follow from lemma 2.4. Because the estimates of lemma 2.4 are R independent one can construct, using the family f_R, a solution $\overline{f} : [1\,,\,\infty)\times S \to \mathbf{R}$ of:

$$L\overline{f} = \overline{\rho}$$

which satisfies the estimates of lemma 2.4. By lemma 2.1 we can also find functions f_i satisfying $L(f_i\varphi_i) = \rho_i\varphi_i$, $i=1,\dots,J$, and to achieve the proof we have to show that the f_i's can be chosen in a way consistent with (3.1)–(3.2). We shall analyse the terms $f_i\varphi_i$ in detail only if $\forall\, i$, $\mu_i^+ \neq \mu_i^-$ and if $q>0$, $\alpha \notin \{\Re\mu_i^{\pm}\}$, the general result is obtained along similar lines. For $i\leq J$ such that $\Re\mu_i^+ < \alpha$ we take f_i given by (2.3). We have:

$$f_i^+ + f_i^- = a(\mu_i^+ - \mu_i^-)\int_S \varphi\varphi_i\,\mathrm{d}\mu\,. \tag{3.5}$$

If $\mu_i^+ \in \mathbf{R}$ set $f_i^+ = 0$, f_i^- given by (3.5). If $\Im\mu_i^+ \neq 0$ the requirements $f_i^\pm \in i\mathbf{R}$, $f_i \in \mathbf{R}$ and (3.5) determine $f_i^\pm$ uniquely. For $i \leq J$ for which $\Re\mu_i^+ > \alpha$ and $\Re\mu_i^- < \alpha$ we take f_i to be of the form (2.4) and the coefficient f_i^- is uniquely determined by φ and ρ. Finally for $i \leq J$ for which $\mu_i^+ \in \mathbf{R}$ and $\mu_i^- > \alpha$ we take f_i of the form (2.5) with $f_i^- = 0$ — this does not allow us, however, to fulfill the equality $\int \varphi_i(f|_{\{1\}\times S} - \varphi) = 0$. The case $\Re\mu_i^+ > \alpha$, $\Im\mu_i^+ \neq 0$ is analyzed in a similar way. The $C_{\alpha,q'}$ *a priori* estimates on $f = \overline{f} + \sum f_i\varphi_i$ follow from construction, the higher derivative estimates follow again by a scaling argument.

4. An Asymptotic Estimate For p-Harmonic Functions

In this section we shall suppose that g_{ij} is a Riemannian metric on $\mathbf{R}^n \setminus B(1)$, asymptotically euclidean in the following sense:

$$g_{ij} - \delta_{ij} \in C^{k,\lambda}_{-\alpha}\,, \quad \alpha > 0\,,\ k \geq 1\,, \tag{4.1}$$

$$\exists\, c \in \mathbf{R}^+\,,\ \forall\, X^i \in \mathbf{R}^n\,,\ \forall\, x \in \mathbf{R}^n \setminus B(1)\,, \quad g_{ij}(x)X^iX^j \geq c\sum (X^i)^2\,. \tag{4.2}$$

When $g_{ij} = \delta_{ij}$ the p-Laplace equation in $\mathbf{R}^n \setminus B(1)$:

$$\operatorname{div}\left(|\operatorname{grad} f|^{p-2}\operatorname{grad} f\right) = 0\,, \quad p > 1\,, \tag{4.3}$$

with the condition:

the level surfaces of f are, asymptotically, nested spheres (4.4)

admits as solutions the functions:

$$f_p = \begin{cases} r^{\frac{p-n}{p-1}} & p \neq n \\ \log r & p = n. \end{cases}$$

For $p = \dim M = 3$ it has been shown in [Ch1] (cf. also [Ch2])(*) that under (4.1)–(4.2) there exists a solution f of the problem (4.3)–(4.4) satisfying:

$$f = \log r + \widetilde{f}\,, \quad \widetilde{f} \in C^{1,\lambda}_{-\epsilon} \text{ for some } \epsilon > 0\,,$$

(*) These results are valid with no modification whatever n for $p = n$ and in fact the same methods lead to similar results for other values of p, cf. also [KV].

and $\tilde{f} \in C^{k+1,\lambda}_{-\epsilon}([R_0\,, \infty) \times S_{n-1})$ for some R_0. We shall show that the estimate on the behaviour of $\tilde{f}$ can be made precise. More generally let us consider:

$$f = f_p + \tilde{f}\,, \quad \tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\epsilon}\,, \; \epsilon > 0 \tag{4.5}$$

THEOREM 4.1.— *Let f satisfying (4.5) be p-harmonic in $\mathbf{R}^n \setminus B(1)$, let g_{ij} satisfy (4.1)-(4.2).*

1) if $\alpha < \beta(p,n) = \dfrac{\sqrt{(p-n)^2 + 4(p-1)(n-1)} - n + p}{2(p-1)}$,

then $\tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\alpha}$, *plus eventually a constant if* $\left(\dfrac{p-n}{p-1}\right) - \epsilon \geq 0$ *and* $\left(\dfrac{p-n}{p-1}\right) - \alpha < 0$.

2) if $\alpha = \beta(p,n)$, *then* $\tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\alpha,1}$, *plus eventually a constant if* $\left(\dfrac{p-n}{p-1}\right) - \epsilon \geq 0$ *and* $\left(\dfrac{p-n}{p-1}\right) - \alpha < 0$.

3) if $\alpha > \beta(p,n)$, *then* $\tilde{f} \in C^{k+1,\lambda}_{\frac{p-n}{p-1}-\beta}$, *plus eventually a constant if* $\left(\dfrac{p-n}{p-1}\right) - \epsilon \geq 0$ *and* $\left(\dfrac{p-n}{p-1}\right) - \beta < 0$.

Remarks :

1) In the physically relevant case $n = 3$ one considers α between 0 and 1. If $p = 3$ theorem 4.1 implies:

$$\tilde{f} = O(r^{-\alpha}) \text{ if } \alpha < 1 \text{ or } \tilde{f} = O\left[\frac{\ln r}{r}\right] \text{ if } \alpha = 1\,.$$

2) The estimates of theorem 4.1, case 1), are sharp, which is seen by considering g_{ij} to be the flat metric written in a non-orthonormal asymptotically flat coordinate system, say $y^i = x^i + J^i$, $J^i \in C^{k+1,\lambda}_{1-\alpha}$. The functions f_p given by (4.4), when expressed in terms of the new coordinates, will exhibit the behaviour described in theorem 4.1. When $p = n = 3$ the estimates of case 2) are sharp as well, and it can be shown that the coefficient of the ${}^{\ln r}/_{r}$ term is the mass of the metric.

Proof.— (4.5) and the p-Laplace equation:

$$\left[g^{ij} + (p-2)\frac{\nabla^i f \nabla^j f}{|\nabla f|^2}\right] f_{;ij} = 0$$

imply an equation of the form:

$$\left[\delta_{ij} + (p-2)\frac{x^i x^j}{r^2}\right] \tilde{f}_{,ij} = \rho\,. \tag{4.6}$$

A straightforward though somewhat tedious calculation shows that:

$$\rho \in C^{k-1,\lambda}_{\beta-2}\,, \quad \beta = -\min(\alpha, 2\epsilon) + \frac{p-n}{p-1}\,.$$

(4.6) written in spherical coordinates gives:

$$\left[(p-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\Delta_S}{r^2}\right] \tilde{f} = \rho\,,$$

the characteristic decay exponents are readily calculated to be:

$$\mu_i^{\pm} = \frac{p-n \pm \sqrt{(p-n)^2 + 4(p-1)\lambda_i}}{2(p-1)}\,, \quad \lambda_i = i(i+n-2)\,,$$

(cf. [BGM]) iterating theorem 2.1 if necessary a finite number of times our claim follows.

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