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Linear forms in two logarithms and Schneider's method (III)

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RÉSUMÉ. — Nous appliquons la méthode de Schneider pour obtenir des bornes inférieures pour des formes linéaires en deux logarithmes de nombres algébriques. Ici nous ne considérons que le cas rationnel. Dans la première partie, nous raffinons les estimations que nous avons obtenues dans le second papier de cette série. La fin de cet article est consacrée au cas où l'un de ces logarithmes est égal à $i\pi$.

ABSTRACT. — We apply Schneider's method to get lower bounds for linear forms of two logarithms of algebraic numbers. Here we consider only the rational case. In the first part, we refine the estimates which we proved in the second paper of this series. The end of this paper is devoted to the case when one of these logarithms is equal to $i\pi$.

Introduction

We refine the lower bound which was obtained in our previous paper [M. W. 2] (which will be denoted [*] in the sequel). We consider an homogenous linear combination of two logarithms of algebraic numbers with integer coefficients.

$$b_1 \log \alpha_1 - b_2 \log \alpha_2.$$

We combine the method of [*] with a technique which already appeared in [M. W. 1]. We improve the numerical results, which is relevant in several circumstances (see e.g. [C.K.T.], [C. W.], [C.F.]). We treat the case of linear dependent logarithms which was only tackled in [*] and we pay special attention to the case when one of the algebraic numbers is a root of unity.

Since we use intensively [*], we keep the numerotation of the sections up to §8 and very often we only give the modifications which we introduce here (this is the reason why there is no §4 here).

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§1. A lower bound for linear forms in two logarithms

We first give here a simple statement : better estimates will be proved later (especially theorem 5.11 in §5).

For the convenience of the reader, we recall the definition of Weil's absolute logarithmic height $h(\alpha)$ of algebraic numbers. Namely, if α is algebraic of degree d over $\mathbf{Q}$, with conjugates $\sigma_1\alpha,\dots,\sigma_d\alpha$, and minimal polynomial

$$c_0X^d+\cdots+c_d=c_0\prod_{i=1}^{d}(X-\sigma_i\alpha)\quad,\quad(c_0>0)$$

then

$$h(\alpha)=\frac{1}{d}(\mathrm{Log}c_0+\sum_{i=1}^{d}\mathrm{Log}\ \max\ \{1,|\sigma_i\alpha|\}).$$

Let α_1,α_2 be two non-zero algebraic numbers of exact degrees D_1,D_2. Let D denote the degree over $\mathbf{Q}$ of the field $\mathbf{Q}(\alpha_1,\alpha_2)$. For $j=1,2$, let $\log\alpha_j$ be any non zero determination of the logarithm of α_j.

Further let b_1,b_2 be two positive rational integers such that

$$b_1\log\alpha_1\neq b_2\log\alpha_2.$$

Define $B=\max\ \{b_1,b_2\}$ and choose two positive real numbers a_1,a_2 satisfying

$$a_j\geq 1, a_j\geq h(\alpha_j)+\ \mathrm{Log}2, a_j=f|\log\alpha_j|/D \text{ for } j=1,2 \text{ and } f\geq 2e.$$

Then theorem 5.11 implies the following result.

COROLLARY 1.1.— *Under the above hypotheses, we have*

$$|b_1\log\alpha_1-b_2\log\alpha_2|\geq\exp\{-270D^4\cdot a_1a_2\cdot(7.5+\ \mathit{Log}\ B)^2\}.$$

At the end of this paper, we study $|\beta\log\alpha-i\pi|$.

The fact that we get a sharper estimate than in our previous work [*] comes from two modifications. Firstly, we look more closely at the conditions which are to be verified by the parameters of the auxiliary function. Secondly, we conclude in two steps :

(i) like in [*], we show that the polynomial φ which occurs in the construction of the auxiliary fucntion is zero at integer points (u, v) in a rectangle of average size (this rectangle is not as "big" as in [*]),

(ii) then (like in [M. W. 1]), we prove that $\varphi(u/2, v/2) = 0$ for integer points in a big rectangle.

The plan of this paper is the following :

§2. Auxiliary lemmas

We keep the auxiliary lemmas given in [*], §2, except for the following one.

LEMMA 2.1.— *(Siegel's lemma). Let $\alpha_1, \ldots, \alpha_q$ be algebraic numbers of absolute heights $a_1, \ldots, a_q$ respectively. Define $D = [\mathbf{Q}(\alpha_1, \ldots, \alpha_q) : \mathbf{Q}]$. Let*

$$P_{ij} \in \mathbf{Z}[X_1, \ldots, X_q], (1 \le i \le \nu, 1 \le j \le \mu)$$

be polynomials (not all zero) of degree at most $N_{j,h}$ *in* X_h *(for* $1 \leq h \leq q$*). Define*

$$\mathbf{L_j} = (\Sigma_i L^2(P_{ij}))^{1/2} \text{ and } \gamma_{ij} = P_{i,j}(\alpha_1, \dots, \alpha_q), (1 \leq i \leq v, 1 \leq j \leq \mu).$$

If $\nu > \mu D$*, then there exist rational integers* $x_1, \dots, x_\nu$ *not all of which are zero, such that*

$$\sum_{i=1}^{\nu} \gamma_{i,j} x_i = 0, \ (1 \leq j \leq \mu),$$

and $\max|x_i| \leq ((V_1 \dots V_\mu)^{D/(\nu-\mu D)}$*, where* $V_j = L_j \cdot \exp\left(\sum_{h=1}^{q} N_{jh} a_h\right)$.

Proof.— Apply [B. V.] theorem 12.

§3. Interpolation formula

We replace lemma 3.2 of [*] by the following result.

LEMMA 3.2.— *Let* β *be a rational number,* $\beta = b_1/b_2, b_1, b_2 \in \mathbf{Z}$, $(b_1, b_2) = 1$.
Let U *and* V *be two positive integers. Put*

$$\Gamma = \{u + v\beta; (u, v) \in \mathbf{Z} \times \mathbf{Z}, |u| \leq U, |v| \leq V\}$$

and

$$\Delta = \min_{\gamma \in \Gamma} \prod_{\gamma' \in \Gamma, \gamma' \neq \gamma} |\gamma' - \gamma|.$$

We suppose
(H) the points $(u + v\beta), |u| \leq 4U$ *and* $|v| \leq 4V$ *are pairwise distinct.*
Then we have

$$\Delta \geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{-(14\pi^2/27)(V+1)^3 b_2^{-2}\}.$$

Proof.— The proof is the same as in [*] except that we notice that our new hypothesis (H) implies now that each value of x_v can be obtained only once, where – for $v \neq v_0$ fixed – we denote by x_v be the minimum of $|u - u_0 + \beta(v - v_0)|$.

Thus, arguing as in [*], we get

$$\Delta \geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{-(7\pi^2/36)(1+2^2+\cdots+(2V)^2)b_2^{-2}\}$$
$$\geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{-(7\pi^2/108)(2V+1)^3 b_2^{-2}\}.$$

This implies lemma 3.2.

§4 Zero estimate

There is no section 4 here because we shall just apply the zero estimate of [*] §4.

§5. The main result

5.1. *Common notations and hypotheses for §§5, 6 and 7*

Let α_1, α_2 be two non-zero algebraic numbers of respective degrees equal to D_1 and D_2, the total degree of the field we are working in is $D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}], \log \alpha_j$ is any non-zero determination of the logarithm of $\alpha_j, l_j = |\log \alpha_j|$, for $j = 1, 2$.

Moreover let $\beta = b_1/b_2$ be a rational number, where $b_1, b_2 \in \mathbf{Z}$, $0 < b_1, b_2$, and $(b_1, b_2) = 1$, such that

$$\Lambda = \beta \log \alpha_1 - \log \alpha_2$$

does not vanish.

We define many parameters as follows.

a) *parameters depending on* α_1 *and* α_2 (namely $a_1, a_2, a_1', a_2', a', \sigma, \sigma_1, \sigma_2, f, \nu$) :

We define

$a_j' = h(\alpha_j)(j = 1, 2), a' = \max\{a_1', a_2'\}, a'' = \max\{\mathrm{Log}|\alpha_1|, \mathrm{Log}\ |\alpha_2|\}$, so that $a'' \leq Da'$,

$\sigma_j = a_j'/a_j, j = 1, 2$, (so that $0 \leq \sigma_j \leq 1$) , $\sigma = (\sigma_1 + \sigma_2)/2$ (so that $0 \leq \sigma \leq 1$).

Now we notice that for any non-zero algebraic number α and any non-zero determination $\log\alpha$ of its logarithm we have

$$|\log\alpha| \geq \exp(-\deg\,(\alpha)(1+h(\alpha))),$$

therefore we can choose a real number f with $1 \leq f \leq 2e^{D(a'+1)}$, such that the numbers $a_j = fD^{-1}l_j$ satisfy

$$a_j \geq 1/D, \text{ for } j = 1, 2.$$

We also assume

$a_j \geq a'_j$ for $j = 1, 2$.

Finally we put $\nu = 1$ if $D > 1$ and $\nu = 0$ if $D = 1$.

b) *parameters depending on* β (namely B, G, G_0, G', μ, ρ) :

We put

$$B = \max\{b_1, b_2\},$$

$$G_0 = 0.59 + \text{Log}\, B + \text{Log Log}\, 2B,$$

$$0.09 + \text{Log}\, B + \text{Log Log}\, 2B = (1+\mu)\, \text{Log B}.$$

Let ρ be a positive number (further conditions on ρ will be required in §5.3 below), we define

$G' = 1 + \text{Log}\,(0.5 + \rho/l_1), G = G_0 + \max\{0.41, G'/D\}$.

c) *the parameters* $Z, \theta, \theta_0, \theta_1, \theta_2$:

Let

$$\theta \geq 10$$

be a real number and let Z be a positive number which satisfies

$1 \leq Z \leq \min\{DG/\theta, Da_1, Da_2, \text{Log}\,(ef)\}$ (as usual e is defined by $\text{Log}\, e = 1$),

(5.0) $$Z \leq \sqrt{D}G/10.$$

We put

$\varepsilon = Z/\text{Log}\,(ef)$ (so that $\varepsilon \leq 1$).

Further we define

$\theta_1 = \theta(D\, a_2/Z), \theta_2 = \theta(D\, a_1/Z)$, (so that $\theta_1, \theta_2 \geq \theta$), $\theta_0 = \max\{\theta_1, \theta_2\}$.

d) *the main parameter* U :

As in [*], we define

$$U = D^4 a_1 a_2 G^2 Z^{-3}.$$

Notice that (5.0) and the conditions on Z imply

$$U \geq \max\{\theta DG, 100D^2a_1, 100D^2a_2, \theta^2 Da_1, \theta^2 Da_2, 10D^{3/2}G\}.$$

5.2 *Notations and hypotheses for* §*5 and* §*6*

Let $c_0, c_1, c, \mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}, C, \eta^*, \mu, \rho, p^*, \xi, \xi_0$ be positive real numbers. Assume

(5.1) $c_0\theta_0 \geq 190, c\theta_1 \geq 20, c\theta_2 \geq 20, c_1\theta \geq 12, c_1 \leq c, 5c \leq c_0 - 1/\theta,$
$2c_1 + c_0/c\theta + 2(1 - 1/\theta c_0) \leq 4c\xi, 6c_1 \leq c_0 - 1/\theta, 1 \leq \mathcal{X} \leq e,$
$(2c - 1/\theta)\xi \geq c_1,$

(5.2) $(c_0 - 1/\theta)(c_1 - 1/\theta) \geq 3.65(c + 1/\theta)^2, c_0(c_1 + 1/\theta) \leq 4.85(c - 1/\theta)^2,$

(5.3)

$$\eta^* = \frac{(2c + 1/\theta_1)(2c + 1/\theta_2)}{c_0(2c_1 - 1/\theta) - (2c + 1/\theta_1)(2c + 1/\theta_2)}, \rho = \frac{(2c + 1/\theta_1)(2c + 1/\theta_2)}{(c_0 - 1/\theta_0)(c_1 - 1/\theta)},$$

(5.4) $$p^* = \eta^* \{c_0 + (c_1 - 1/2\theta)(\sigma_1(c - 1/2\theta_1) + \sigma_2(c - 1/2\theta_2))\}$$

(5.5.i) $(c_0(c_1 - 1/\theta) - (2c + 1/\theta)^2)\xi \leq c_0(2c_1 - 1/\theta)(c_0 + (2c_1 - 1/\theta)c\sigma),$

(5.5.ii) $(2c + 1/\theta_1)(2c + 1/\theta_2)\xi \geq p^* + c_0 + (2c_1 - 1/\theta)(\sigma_1(c - 1/2\theta_1)$
$+ \sigma_2(c - 1/2\theta_2)) + 0.05\nu,$

(5.5.*iii*) $4c^2\mathcal{X}_0^2\xi_0 \geq 2(p^* + c_0) + 4\sigma c c_1 \mathcal{X} + 2c_1\sigma/\theta + (0.44 + 2D^{-1}\text{Log } \mathcal{X}_0)/G,$

where $\xi_0 = \varepsilon^{-1} + \dfrac{1}{Z}\text{ Log}\left(\dfrac{4cf\mathcal{X}_0^2}{\mathcal{X}c_1 e(1 + 1/2\theta\mathcal{X}c)Z}\right) - \dfrac{e^{-C}}{Z}, 1 \leq \mathcal{X}_0 \leq 1.5,$

(5.6.i) $C \geq p^* + c_0 + (4c^2\mathcal{X}_0^2/Z)\text{ Log }(2e) + 2\mathcal{X}_0(3c_1\sigma + 1/D)c + 3c_1\sigma/\theta -$
$0.48c_0/G + 0.09,$

(5.6.ii) $C \geq 2(p^* + c_0 + 0.09) + (4c^2\mathcal{X}_0^2/Z)\text{ Log }(\mathcal{X}e/\mathcal{X}_0) + 2\sigma c c_1(2\mathcal{X} +$
$\mathcal{X}_0) + 2c\mathcal{X}_0/D + 3c_1\sigma/\theta + 0.2c_0/G + (2/DG)(4\text{ Log }(e\mathcal{X}/\mathcal{X}_0)$
$-\text{ Log }(19.7c\theta\mathcal{X}_0))^+,$

[as usual, for a real number x we define $x^+ = \max\{x, 0\}$]

(5.6.*iii*) $$C \geq 10c \text{ and } CD \geq 200,$$

(5.7) $$0 < \xi \leq \varepsilon^{-1} + \frac{1}{Z}\operatorname{Log}\frac{2cf(1-1/2\theta c)^2}{Zc_1e^2} - \frac{e^{-C}}{Z},$$

(5.8) $$\mathcal{X}_2 = \sqrt{c_0c_1}/c, \mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2, \qquad [\text{by}(5.2), \mathcal{X}_2 \geq \sqrt{3}]$$

(5.9) $$\mathcal{X} \geq \sqrt{c_0c_1}/c + (1+\sqrt{c_0\theta})/(2c\theta),$$

(5.10) either α_1 and α_2 are multiplicatively independent or

$$\mathcal{X} \geq \sqrt{c_0c_1}/c + 1/(\theta c) + c_1/c.$$

5.3 *Statement of the main result.*

THEOREM 5.11.— *Under the above hypotheses, we have* $|\Lambda| > e^{-CU}$.

The rest of §5 is devoted to the proof of this inequality. Therefore we assume $\operatorname{Log}|\Lambda| \leq -CU$ and we shall eventually reach a contradiction.

5.4. *The parameters*

We define L_0, L_1, M_1, M_2 by

$$L_0 = [c_0D^3a_1a_2GZ^{-3}], L_1 = [c_1DGZ^{-1}],$$
$$M_1 = [cD^2Ga_2Z^{-2}], M_2 = [cD^2Ga_1Z^{-2}].$$

We put

$$2L_1 + 1 = x_1DGZ^{-1}, \text{ so that } |x_1 - 2c_1| \leq 1/\theta,$$
$$2M_1 + 1 = y_1D^2Ga_2Z^{-2}, \quad 2M_2 + 1 = y_2D^2Ga_1Z^{-2},$$

so that $|y_j - 2c| \leq 1/\theta_j$ for $j = 1, 2$.

The following inequalities

(5.12) $$L_0 \geq (c_0 - 1/\theta_0)D^3a_1a_2GZ^{-3} \geq c_0\theta - 1, L_0 \geq 190,$$
$$L_1 \geq (c_1 - 1/\theta)DGZ^{-1} \geq c_1\theta - 1, L_1 \geq 12,$$

$$M_1 \geq (c - 1/\theta_1)D^2Ga_2Z^{-2}, M_2 \geq (c - 1/\theta_2)D^2Ga_1Z^{-2},$$

and $M_1, M_2 \geq 20, M_1 \geq L_1$ are all consequences of the definition of Z and of (5.1).

By lemma 2.2 and the definition of $a_i's$, we have

$$(*) \qquad |\Lambda| \geq 2^{-D} \cdot \exp(-b_1 D a_1' - b_2 D a_2') \cdot B^{-1}.$$

This shows first that the numbers $u + v\beta, |u| \leq 4M_1$ and $|v| \leq 4M_2$ are pairwise distinct (here and in the sequel the letters u and v represent rational integers); otherwise $b_1 < 8M_1$ and $b_2 < 8M_2$, which implies

$$|\Lambda| \geq \exp(-D(1 + 8M_1 a_1 + 8M_2 a_2) - G) \geq \exp(-(16c + 2)U/10D).$$

and contradicts the assumption $|\Lambda| \leq e^{-CU}$, since $C \geq 10c$ and $CD \geq 200$.

We have $|a_1 b_1 - a_2 b_2| \leq fB \cdot e^{-CU}$, and the hypotheses on f, C and U imply

$$a_1 b_1 \leq (1 + e^{-CU/2}) \cdot a_2 b_2 \leq 1.001 \cdot a_2 b_2.$$

We also remark that $M_2 \leq b_2/33$: if not (*) implies the estimate

$$|\Lambda| \geq \exp(-3b_2 D a_2) \leq \exp(-99 M_2 a_2 D),$$

which contradicts $|\Lambda| \leq e^{-CU}$, since $C > 10c, \theta \geq 10$ and $M_2 a_2 D \leq cU/\theta$. This remark is used in the proof of proposition 5.19.

Moreover, we remark also that (*) and (5.0) imply

$$(**) \qquad B \geq \max\{(C/2 - 0.1)\theta^2, 49DC\},$$

so that $\operatorname{Log} B \geq 9.2, G > 12.49, \omega(2B) < 0.09 + \operatorname{Log}\operatorname{Log} 2B$ (see lemma 2.7 of [*] for the definition of $\omega(x)$) and also $U \geq 186 D \operatorname{Log} D$.

5.5 *The auxiliary function.*

We denote by $\{\xi_1, \ldots, \xi_D\}$ a basis of $\mathbf{Q}(\alpha_1, \alpha_2)$ over $\mathbf{Q}$, where $\xi_d = \alpha_1^{d_1} \alpha_2^{d_2}$, $0 \leq d_j < D_j (j = 1, 2)$ and $d_1 + d_2 < D$. This implies the estimate

$$\max\{h(\xi_j); 1 \leq j \leq D\} \leq \min\{(D_1 - 1)a_1' + (D_2 - 1)a_2', (D - 1)a'\}.$$

As in [*], we shall construct an auxiliary function of the form

$$F(z) = \sum_{h=0}^{L_0} \sum_{k=-L_1}^{k=L_1} p_{h,k} \Delta_h(z) \alpha_1^{kz}, \text{ where } p_{h,k} = \sum_{d=1}^{D} p_{h,k,d} \xi_d, p_{h,k,d} \in \mathbf{Z}$$

and $\Delta_h(z)$ is defined in lemma 2.4 of [*]. For rational integers u and v we put

$$\varphi(u,v)=\sum_{h=0}^{L_0}\sum_{k=-L_1}^{L_1} p_{h,k}\Delta_h(u+v\beta)\alpha_1^{ku}\alpha_2^{kv}.$$

PROPOSITION 5.14.— *There exist rational integers $p_{h,k,d}$, not all zero, such that*

$$\varphi(u,v)=0 \textit{ for } -M_1\le u\le M_1, -M_2\le v\le M_2,$$

with $P:=\text{Log }(\max|p_{h,k}|)$

$$\le \eta(L_0(G_0+0.016)+((x_1/4)(\sigma_1y_1+\sigma_2y_2)+0.013\nu)U/D)$$

$$\le (p+0.013\eta\nu)U/D-\eta(G-G_0-0.016)L_0,$$

where $p=\eta(c_0+(x_1/4)(\sigma_1y_1+\sigma_2y_2))$ *and* $\eta=y_1y_2/(c_0x_1-y_1y_2)$. *Moreover*

$$P_1:=\text{Log }(\Sigma_{h,k,d}|p_{h,k,d}|)\le P+0.006\nu U/D+0.051L_0.$$

and

$$\text{Log }(\Sigma_{h,k}|p_{h,k}|)\le P_2:=P_1+(D-1)a''\le P_1+0.01\nu U.$$

Remark.— By (5.2), we have $0.7\le\eta\le\eta^*\le1.22$; moreover, $p\le p^*$.

Proof of proposition 5.14.

We have to solve in $\mathbf{Z}$ a linear system of $(2M_1+1)(2M_2+1)$ equations in the $D(L_0+1)(2L_1+1)$ unknowns $p_{h,k,d}$. We use lemma 2.1. By definition, we have

(5.15)

$$(2M_1+1)(2M_2+1)/\left((L_0+1)(2L_1+1)-(2M_1+1)(2M_2+1)\right)=\eta.$$

With the notations of lemma 2.1, we have

$$i\to(h,k,d), j\to(u,v), N_{j,1}=2L_1|u|+D_1-1, N_{j,2}=2L_1|v|+D_2-1,$$

$$P_{ij}=\Delta_h(u+v\beta)b_2^{L_0}\Omega(b_2,L_0)X_1^{L_1|u|+ku+d_1}X_2^{L_1|v|+kv+d_2},$$

with $d_1+d_2<D$. By lemma 2.4 of [*], using the inequality $\beta a_1\le1.001a_2$ as well as our assumption (5.1) we get

$$L(P_{ij})\le 2(X^h/h!)b_2^{L_0}\Omega(b_2,L_0), X=\max\{1.1|u+v\beta|,L_0/2\}=L_0/2.$$

Notice that

$$\sum_{h=0}^{L_0} \frac{X^h}{h!} \leq e^X,$$

so that $(\Sigma_i L^2(p_{ij}))^{1/2} \leq 2(D(2L_1+1))^{1/2}(\sqrt{e}b_2)^{L_0}\Omega(b_2, L_0)$ and

$$V_{u,v} \leq 2(D(2L_1+1))^{1/2}(\sqrt{e}b_2)^{L_0}\Omega(b_2, L_0). \\ \cdot \exp\{2L_1|u|h(\alpha_1) + 2L_1|v|h(\alpha_2) + (D-1)a'\}.$$

Now we have

$$\sum_{u=-M_1}^{M_1} (2L_1|u|)h(\alpha_1) = 2L_1M_1(M_1+1)h(\alpha_1).$$

and $2M_1(M_1+1)h(\alpha_1) \leq (1/2)(2M_1+1)^2 a_1'$.

Hence, since a similar result holds for the summation over v,

$$\Sigma_{u,v}\text{Log } V_{u,v} \leq (2M_1+1)(2M_2+1)\{\text{Log }(2(D(2L_1+1))^{1/2}(\sqrt{e}b_2)^{L_0} \\ \Omega(b_2, L_0)) + a_1'L_1(M_1+1/2) + a_2'L_1(M_2+1/2) + (D-1)a'\}$$

An application of lemma 2.1 shows that there is a non trivial solution with

$$\text{Log }(\max|p_{h,k,d}|) \leq \eta(L_0G_0 + L_1(a_1'(M_1+1/2) + a_2'(M_2+1/2)) + (D-1)a'' \\ + \text{Log }(2(D(2L_1+1))^{1/2}),$$

where $G_0 = 0.59 + \text{ Log } B + \text{ Log Log } B \leq G - 0.41$.

Moreover

$$\text{Log }(\Sigma|p_{h,k,d}|) \leq \text{ Log }(\max|p_{h,k,d}|) + \text{ Log }(D(L_0+1)(2L_1+1))$$

and

$$\text{Log }(\Sigma|p_{h,k}|) \leq \text{ Log }(\max|p_{h,k,d}|) + \text{ Log }(D(L_0+1)(2L_1+1)) + (D-1)a'.$$

We have

$$L_1(a_1'(M_1+1/2) + a_2'(M_2+1/2)) \leq (x_1/4)(y_1\sigma_1 + y_2\sigma_2)(U/D).$$

By (5.0) and (5.1), we have also

$$\text{Log }(2L_1+1) \leq \text{ Log }((25/72)L_0) \leq 0.023L_0, \text{ Log } 2 \leq 0.004L_0, \\ \text{Log }(L_0+1) \leq 0.028L_0, (D-1)a' \leq 0.01\nu U/D, \text{ Log } D \leq 0.006\nu U/D.$$

Now it is easy to get proposition 5.14.

5.6 *The extrapolation.*

Put $M_{1,0} = [\mathcal{X}_0 c D^2 a_2 G Z^{-2} + 0.5]$ and $M_{2,0} = [\mathcal{X}_0 c D^2 a_1 G Z^{-2} + 0.5]$. In this section we prove that

$$(\#) \quad \varphi(u,v) = 0 \text{ for } -M_{1,0} \le u \le M_{1,0}, -M_{2,0} \le v \le M_{2,0}.$$

By construction, this is true for $-M_1 \le u \le M_1$ and $-M_2 \le v \le M_2$. We proceed almost exactly like in [*], and we give only the details wich are different.

Define $N = M_{1,0} + M_{2,0} - M_1 - M_2$ and $\mathcal{X}_n$ by

$$M_1^{(n)} = \mathcal{X}_n c a_2 D^2 G Z^{-2}, 1 \le n \le N, \text{ so that } 1 \le \mathcal{X}_n \le \mathcal{X}_0.$$

We proove, by induction on $n, (0 \le n \le N)$, that

$$(P)_n \qquad \varphi(u,v) = 0 \text{ for } |u| \le M_1^{(n)} \text{ and } |v| \le M_2^{(n)}.$$

As already seen, this is true for $n = 0$, while $(P)_N$ is nothing else than (#).

We consider the set

$$\Gamma_{n-1} = \{z_1, \ldots, z_m\} = \{u + v\beta; |u| \le M_1^{(n-1)}, |v| \le M_2^{(n-1)}\},$$

where $m = (2M_1^{(n-1)} + 1)(2M_2^{(n-1)} + 1)$, and a point $z_0 \in \Gamma_n, z_0 \notin \Gamma_{n-1}$.

Since $\beta a_1 \le a_2(1 + e^{-CU/2})$ we get $M_1^{(n-1)} + 1 \ge \beta M_2^{(n-1)} - e^{-CU/3}$.

Define $R_1 = M_1^{(n)} + M_2^{(n)}\beta, R = m/(L_1 l_1)$.

PROPOSITION 5.19.— *We have*

$$|F(z_0)| \le E_1 + E_2$$

where

$$\text{Log } E_1 \le -m \text{ Log } (R/eR_1) + 1 + \text{ Log } (\Sigma_{h,k}|p_{h,k}|) + L_0(1 + \text{Log } (1/2 + R/L_0)),$$

so that

$$(5.20) \ \text{Log } E_1 \le P_2 + L_0 G' + 1.5 + L_0 \text{ Log } (m/(2M_1+1)(2M_2+1)) - m\xi Z$$

and

$$\text{Log } E_2 \le \text{ Log } (\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}) + m \text{ Log } (eR_1/(M_1^{(n-1)} + 1)) + 2M_2^{(n-1)} \text{ Log } (eB/2M_2^{(n-1)}) + 0.027 M_2^{(n-1)},$$

so that

$$\text{Log } E_2 \le \text{Log } \max_{\gamma\in\Gamma_{n-1}} |F(\gamma)| + m\text{Log } 2e + 2M_2^{(n-1)} \text{ Log } (eb_2/2M_2^{(n-1)})$$

$$+ 0.03M_2^{(n-1)}. \tag{5.21}$$

Proof. — Thanks to the interpolation formula, we have

$$\text{Log } E_1 \le -m \text{ Log } (R/R_1) + \text{ Log } (R/(R-R_1)) + \text{ Log } |F|_R;$$

this implies the first upper bound for Log E_1. The second one follows like in [*], except that now

$$R_1/(M_1^{(n-1)}+1) \le 2 + e^{-CU/3}. \tag{5.25}$$

The interpolation formula implies also

$$\text{Log } E_2 \le \text{Log } (\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}) - (4M_2^{(n-1)}+2)\text{Log } (M_1^{(n-1)}!)$$

$$+5.12(M_2^{(n-1)}+1)^3 b_2^{-2} + \text{ Log } m + (m-1) \text{ Log } R_1 + \text{ Log } (2R^4/(R^4 - R_1^4)),$$

where Log $(2R^4/(R^4 - R_1^4)) < 2.04$,

$$m \le (41/20)^2 M_1^{(n)} M_2^{(n-1)} = (41/20)^2 (M_1^{(n)}/R_1) R_1 M_2^{(n-1)}$$

$$< 4.3 R_1 M_2^{(n-1)},$$

$$5.12(M_2^{(n-1)}+1)^3 b_2^{-2} \le 6(M_2^{(n-1)})^3 b_2^{-2} < 0.01 M_2^{(n-1)},$$

$$(4M_2^{n-1)}+2) \text{ Log } (M_1^{(n-1)}!) \ge$$

$$m \text{ Log } ((M_1^{(n-1)}+1)/e) + (2M_1^{(n-1)}+1) \qquad \text{Log } (2\pi/e),$$

$$\text{Log } ((2M_2^{(n-1)})!) \ge 2M_2^{(n-1)} \text{ Log } (2M_2^{(n-1)}/e) + 0.5 \text{ Log } (4\pi M_2^{(n-1)}),$$

so that

$$\text{Log } E_2 \le \text{ Log } (\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}) - m \text{ Log } ((M_1^{(n-1)}+1)/eR_1)$$

$$+2M_2^{(n-1)} \text{ Log } (eB/2M_2^{(n-1)}) + 0.5 \text{ Log } (9M_2^{(n-1)}) + 0.01 M_2^{(n-1)}.$$

This implies the first upper bound of Log E_2. The second one follows from (5.25). This completes the proof of proposition 5.19.

PROPOSITION 5.29.— *Put* $\lambda = \max\{1/2, |\gamma|/L_0\}$. *For* $\gamma = u + v\beta \in \Gamma_N$, *we have*

$$|F(\gamma) - \varphi(u,v)| \le E_3$$

where

$$Log\ E_3 \le -CU + P_2 + \lambda L_0 + Log\ (L_1 M_{2,0}) + L_1 M_{1,0} Da_1' + 3Da_2 + (L_1 M_{2,0} + 1) Da_2' + 1 \le (-C + 2cc_1 \mathcal{X}_0 \sigma + c_1\sigma/\theta + 0.04)U + P_2 + 0.55 L_0.$$

Proof.— We first show that for $-L_1 \le k \le L_1$ and $-M_{2,0} \le v \le M_{2,0}$, we have (5.30) $|\alpha_1^{\beta k v} - \alpha_2^{kv}| \le$
$\exp\{-CU + 3Da_2 + \mathrm{Log}\ L_1 M_{2,0} + (L_1 M_{2,0} + 1) Da_2' + L_1 M_{2,0} Da_2/CU\}$.

Indeed,

$$|\alpha_1^{\beta k v} - \alpha_2^{kv}| \le |\alpha_1^{\beta} - \alpha_2| \cdot L_1 M_{2,0} \cdot \max\{1, |\alpha_1^{\beta}\alpha_2|^{-1}\}.$$
$$\cdot \exp((L_1 M_2^* - 1) \cdot \ \mathrm{Log}\ (\max\{|\alpha_1^{\beta}|, |\alpha_2|, |\alpha_1^{\beta}|^{-1}, |\alpha_2|^{-1}\})).$$

We have $|\mathrm{Log}\ |\alpha_2|| \le Da_2'$ and $|\alpha_1^{\beta}| \le |\alpha_2| + e^{-CU/2}$ so that $\mathrm{Log}\ |\alpha_1^{\beta}| \le Da_2' + e^{-CU/2}$.

Clearly, $|\alpha_1^{\beta}| \ge \exp(-Da_2') - \exp(-CU + 2Da_2 + e^{-CU})$ and this leads to

$$\mathrm{Log}\ |\alpha_1^{\beta}| \ge -Da_2' - Da_2/(CU). \tag{5.31}$$

These lower bounds give

$$\mathrm{Log}\ (|\alpha_1^{\beta}\alpha_2|^{-1}) \le 2Da_2' + Da_2/CU, \tag{5.32}$$
$$\mathrm{Log}\ (\max\{|\alpha_1^{\beta}|, |\alpha_2|, |\alpha_1^{\beta}|^{-1}, |\alpha_2|^{-1}\}) \le Da_2' + Da_2/CU,$$

and (5.30) follows since $|\alpha_1^{\beta} - \alpha_2| \le \exp\{-CU + 3Da_2\}$.

We have $|\gamma| \le 2.03 \mathcal{X}_0 c D^2 G Z^{-2} a_2$, thus $\lambda = 0.5$ by (5.1). Moreover,

$\mathrm{Log}\ (L_1, M_{2,0}) \le \mathrm{Log}\ (L_0 c_1/(c_0 - 1/\theta)) + \mathrm{Log}\ (L_0 c(\mathcal{X}_0 + 1)/(c_0 - 1/\theta)) \le 0.04 L_0$.

The conclusion follows like in [*].

PROPOSITION 5.34.— *For* $\gamma = u + v\beta \in \Gamma_N$, *either* $\varphi(u,v)$ *is equal to zero or*

$$|\varphi(u,v)| \ge E_4$$

with
$-Log\ E_4 \leq (D-1)P_1 + (c_0 + 4cc_1\mathcal{X}_0\sigma + 2c_1\sigma/\theta + 0.01\nu)U - DL_0(G-G_0) - L_0/2.$

Proof. — From Liouville inequality, we deduce

$$\begin{aligned}-\text{Log }|\varphi(u,v)| &\leq (D-1)P_1 + (D-1)\lambda L_0 + (1+\mu)DL_0\text{Log }B \\ &\quad +2DL_1(a_1'M_1^{(n)} + a_2'M_2^{(n)}) + D(D-1)a' \\ &\leq (D-1)P_1 + 0.01\nu U + DL_0G_0 + 2DL_1(a_1'M_1^{(n)} + a_2'M_2^{(n)}) - L_0/2,\end{aligned}$$

and the result follows.

PROPOSITION 5.35. — *Assume that in (5.21) we have*

$$\max_{\gamma\in\Gamma_{n-1}} |F(\gamma)| \leq E_3$$

then

$$E_1 + E_2 + E_3 < E_4.$$

Proof. — We use (5.5) to check

$$E_1 < E_4/3, \tag{5.36}$$

and then we shall use (5.6) (i) to check

$$\max\{E_2, E_3\} < E_4/3. \tag{5.37}$$

Put $m_0 = (2M_1+1)(2M_2+1)$. The first inequality is true when (we know that $\lambda = 0.5$)

$$\begin{aligned}m\xi Z \geq DP_1 &+ 2DL_1(a_1'M_1^{(n)} + a_2'M_2^{(n)}) + (D-1)a'' + D(D-1)a' \\ &+3 + DL_0(G_0 + G'/D) + L_0\text{Log }(m/m_0).\end{aligned}$$

It is again sufficient to check this inequality for $n = 1$ and (5.36) is implied by

$$m_0\xi Z - (DP_1 + (c_0 + 0.02\nu)U + 2DL_1((M_1+1)a_1' + M_2a_2') + 3) \geq 0.$$

Remark that $M_1 \geq L_1$ implies

$$2L_1((M_1+1)a_1' + M_2a_2') \leq (x_1/2)(y_1\sigma_1 + y_2\sigma_2)U/D.$$

We have also $3 \leq 0.016 L_0$ and

$$P_1 \leq (p + 0.013\eta\nu + 0.006\nu)U/D - \eta(G - G_0 - 0.1)L_0$$
$$\leq (p + 0.022\nu)U/D - \eta(G - G_0 - 0.1)L_0$$

Therefore (5.36) is a consequence of

$$H := y_1 y_2 \xi - (p + c_0 + (x_1/2)(y_1\sigma_1 + y_2\sigma_2) + 0.05\nu) \geq 0,$$

[recall that $p = \eta(c_0 + (x_1/4)(y_1\sigma_1 + y_2\sigma_2))$ and $\eta = y_1y_2/(c_0x_1 - y_1y_2)$].

We see that $\partial H/\partial x_1 \geq 0$, so that the worst value for x_1 is $2c_1 - 1/\theta$:

$$\partial H/\partial x_1 > ((y_1y_2)^2/2 - 1)(y_1\sigma_1 + y_2\sigma_2)/2 \geq 0 \text{ (since } y_1, y_2 > 2).$$

Whereas

$$\frac{\partial H}{\partial y_1} = y_2\xi - \frac{y_2 c_0 x_1 (c_0 + x_1(y_1\sigma_1 + y_2\sigma_2)/4)}{(c_0x_1 - y_1y_2)^2} - x_1\sigma_1(\eta + 2)/4.$$

As easily verified, $\partial^2 H/\partial y_1^2 \leq 0$ and $\partial(y_2^{-1}\partial H/\partial y_1)/\partial y_2 \leq 0$, so that $\partial H/\partial y_1 \leq 0$ when

$$(c_0x_1 - y^2)^2\xi \leq c_0x_1(c_0 + x_1 y\sigma/2), \text{ where } y = 2c - 1/\theta.$$

This proves that (5.5.i) implies $\partial H/\partial y_1 \leq 0$ (and also $\partial H/\partial y_2 \leq 0$). Now, by condition (5.5.ii) we have $H \geq 0$.

We now prove (5.37). It is sufficient to check

$$DP_1 + (c_0 + 6cc_1\mathcal{X}_0\sigma + 3c_1\sigma/\theta + 0.04)U + m\text{Log}\,(2e) + D(D-1)a'$$
$$+(D-1)a'' + 2M_{2,0}\text{Log}\,(3B/2M_{2,0}) - DL_0(G - G_0) + 0.05L_0 < CU.$$

We have

$$M_{2,0}\text{Log}\,(3B/2M_{2,0}) \leq 2M_{2,0}(G - 1.9) - 2M_{2,0}\text{Log}\,2M_{2,0}$$
$$\leq 2\mathcal{X}_0 cU/D - 10M_{2,0}.$$

And now m is bounded by

$$m = (2M_{1,0} + 1)(2M_{2,0} - 1) \leq (2M_{1,0} - 1)(2M_{2,0} - 1) + 4M_{2,0}$$
$$\leq 4\mathcal{X}_0^2 c^2 U/Z + 4M_{2,0}.$$

This shows that (5.37) is a consequence of (5.6.i).

We have proved (#).

5.7 *End of the proof.*

We shall prove that the non-zero polynomial $\Delta_{h,k} p_{h,k} \Delta_h(X) Y^{L_1+k}$ vanishes at the points

$(u/2 + \beta v/2, \alpha_1^{u/2} \alpha_2^{v/2}), (u,v) \in \mathbf{Z} \times \mathbf{Z}, |u| \leq M_1^*, |v| \leq M_2^*,$

where $M_1^* = [\mathcal{X} c D^2 a_2 G Z^{-2} + 0.5]$ and $M_2^* = [\mathcal{X} c D^2 a_1 G Z^{-2} + 0.5]$.

According to proposition 4.1 of [*] (zero estimate) and the obvious analog of proposition 5.43 of [*], we will obtain a contradiction; and this will prove theorem 5.11.

Consider a point $\gamma = (u + v\beta)/2$, with $|u| \leq M_1^*, |v| \leq M_2^*$.

We suppose that γ does not belong to Γ_N (otherwise $\varphi(u/2, v/2) = 0$ by the preceeding section) and we apply the interpolation formula for γ and the points of Γ_N with

$$R_1 = (M_1^* + \beta M_2^*)/2, R = m^*/L_1 l_1,$$
$$\text{where } m^* = \mathrm{Card}(\Gamma_N) = (2M_{1,0} + 1)(2M_{2,0} + 1).$$

We get

$$|F(\gamma)| \leq E_1^* + E_2^*,$$

where

$$\mathrm{Log}\ E_1^* \leq -m^* \mathrm{Log}\ (R/eR_1) + 1 + \log(\Sigma_{h,k} |p_{h,k}|) + L_0(1 + \mathrm{Log}\ (1/2 + R/L_0)),$$

and

$$\mathrm{Log}\ E_2^* \leq \mathrm{Log}\ (\max\{|F(\gamma)|; \gamma \in \Gamma_N\}) + m^* \mathrm{Log}\ (eR_1/(M_{1,0}+1))$$
$$+ 2M_{2,0} \mathrm{Log}\ (eB/2M_{2,0}) + 0.027 M_{2,0}.$$

Now,

$$\frac{R}{L_0} \leq \frac{(2c\mathcal{X}_0 + 1/2\theta)^2 (e^{G'-1} - 0.5)}{\rho(c_0 - 1/\theta_0)(c_1 - 1/\theta)} \leq \mathcal{X}_0^2 (e^{G'-1} - 0.5)$$

so that

$$\mathrm{Log}\ \left(\frac{R}{L_0} + \frac{1}{2}\right) \leq G' - 1 + 2\mathrm{Log}\ \mathcal{X}_0,$$
$$\mathrm{Log}\ (R/eR_1) \geq \xi_0 Z$$

and

$$R_1 \leq (\mathcal{X}/\mathcal{X}_0)(M_{1,0} + 1)(1 + e^{-CU/3}).$$

This leads to

$$\text{Log } E_1^* \leq -m^*\xi_0 Z + P_2 + (G' + 2\text{Log } \mathcal{X}_0)L_0 + 1$$

and

$$\begin{aligned}\text{Log } E_2^* &\leq \text{Log } E_3 + m^*\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 2\mathcal{X}_0 cU/D + 2M_{2,0}\text{Log } (eB/2M_{2,0}) \\ &\leq (-C + 2cc_1\mathcal{X}_0\sigma + 2c_1\sigma/\theta + 0.05)U + m^*\text{Log } (e\mathcal{X}/\mathcal{X}_0) \\ &\quad +2\mathcal{X}_0 cU/D - 2M_{2,0}\text{Log } (4M_{2,0}) + P_1 + 0.55L_0.\end{aligned}$$

Notice that $m^* \leq 4\mathcal{X}_0^2 c^2 U + 4M_{1,0} + 4M_{2,0} + 1$. Without loss of generality, we may suppose that $M_{1,0} \leq M_{2,0}$, and then we have

$$\begin{aligned}&m^*\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 2M_{2,0}\text{Log } (eB/2M_{2,0}) \\ &\leq (4\mathcal{X}_0^2 c^2 U + 8M_{2,0} + 1)\text{Log } (e\mathcal{X}/\mathcal{X}_0) + \\ &\quad 2M_{2,0}(G - \text{Log Log } 2B - \text{Log } (2M_{2,0})) \\ &\leq 4\mathcal{X}_0^2 c^2 U\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 8M_{2,0}\text{Log } (e\mathcal{X}/\mathcal{X}_0) + \\ &\quad 2M_{2,0}(G - \text{Log } (19.7M_{2,0})) + 2 \\ &\leq 4\mathcal{X}_0^2 c^2 U\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 2\mathcal{X}_0 U/D + \\ &\quad 2M_{2,0}(4\text{Log } (e\mathcal{X}/\mathcal{X}_0) - \text{Log}(19.7M_{2,0})) + 2 \\ &\leq 4\mathcal{X}_0^2 c^2 U\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 2\mathcal{X}_0 U/D + \\ &\quad 2(U/DG)(4\text{Log } (e\mathcal{X}/\mathcal{X}_0) - \text{Log } (19.7c\mathcal{X}_0))^+ + 2.\end{aligned}$$

(we have used the notation $x^+ = \max\{x, 0\}$ for real numbers), this gives

$$\begin{aligned}\text{Log } E_2^* &\leq (-C + 2cc_1\mathcal{X}_0\sigma + c_1\sigma/\theta + 4\mathcal{X}_0^2 c^2\text{Log } (e\mathcal{X}/\mathcal{X}_0) + 2\mathcal{X}_0 c/D + 0.09)U \\ &\quad +P_1 + 0.55L_0 + 2(U/DG)(4\text{Log } (e\mathcal{X}/\mathcal{X}_0) - \text{Log } (17c\theta\mathcal{X}_0))^+.\end{aligned}$$

Moreover, $|\varphi(u/2, v/2) - F(\gamma)| \leq E_3^*$ with

$$\begin{aligned}\text{Log } E_3^* &\leq -CU + P_2 + \lambda^* L_0 + \text{Log } (L_1 M_2^*) + L_1 M_1^* D(a_1'/2) \\ &\quad +(L_1 M_2^* + 1)D(a_1'/2) + 4Da_2 + 1,\end{aligned}$$

where $\lambda^* = \max\{0.5, |\gamma|/L_0\} = 0.5$ (by (5.1) and $\mathcal{X} \leq e$), thus

$$\text{Log } E_3^* \leq (-C + p/D + \mathcal{X}cc_1\sigma + \mathcal{X}c_1\sigma/\theta + 0.09 + c_0/2DG)U.$$

Finally, if $\varphi(u/2, v/2) \neq 0$ then $|\varphi(u/2, v/2)| \geq E_4^*$, where

$$\begin{aligned}-\text{Log } E_4^* &\leq (2D-1)\text{Log } (\Sigma_{h,k}|p_{h,k,d}|) + 2DL_1(a_1'M_1^* + a_2'M_2^*) \\ &\quad +2DL_0(\lambda^* + \text{Log } 2 + \text{Log } B + \text{Log Log } 2B + 0.09) + 2D(D-1)a' \\ &\leq (2D-1)P_1 + (2c_0 + 4cc_1\sigma\mathcal{X} + 2c_1\sigma/\theta + 0.02)U + 2DL_0(G - G_0 + \text{Log } 2).\end{aligned}$$

Now it is easy to verify that condition (5.5.iii) implies $E_1^* < E_4^*/3$ and that condition (5.6.ii) implies $\max\{E_2^*, E_3^*\} < E_4^*/3$. This proves that

$$\varphi(u/2, v/2) = 0.$$

This completes the proof of theorem 5.11.

6. Numerical examples

We use the notations and hypotheses of §5.1 and 5.2, and we produce suitable values for the constant C, so that the assumptions (5.1) to (5.10) have been checked. Therefore the conclusion

$$|\Lambda| > \exp(-CU)$$

of theorem 5.11 holds.

Here we choose $f = 2e$, and $\varepsilon = \sigma = 1$ (the worst values for σ and ε). we proceed essentially as in [*] : we fix $\theta \geq 10$ and $Z \geq 1$, then we choose c and c_1, and for those values we seach a suitable c_0 (if it exists), for this we have to solve a quadratic equation and then to verify the conditions (5.1). From (5.6), with the value of $\mathcal{X}$ given by (5.9) and (5.10), we deduce a suitable value for C.

The results are given in fig. 1 and 2 in the case of multiplicatively independent numbers. These results improve those of [*], for example in figure 1 for $Z = 1$ and $\theta = 14$, we got $C = 530$ and now we have $C = 258$.

In fig. 1 we fix $Z = 1, \theta$ varies, and we display the optimal value of C together with the corresponding choices of c, c_1 and c_0.

θ	14	15	16	17	18	19	20
C	258	257	255	254	253	252	251
c_0	28.25	28.51	28.04	28.33	27.98	27.98	28.12
c_1	1.45	1.44	1.44	1.42	1.42	1.41	1.41
c	3	2.99	2.99	2.98	2.98	2.97	2.97

figure 1 : multiplicatively independent numbers, $Z = 1$.

In fig. 2, both Z and θ vary and we display the optimal value of CZ^{-3}. At the end of each row we display the range for (c, c_1) corresponding to the given row. For instance, at the end of the first row in fig. 2 the indication

$$2.93 \leq c \leq 3.01, 1.4 \leq c_1 \leq 1.45$$

means that for $Z = 1$ and for the given values of θ (with $10 \le \theta \le 100$) we always choose c and c_1 in these intervals.

Z \ θ	12	13	14	15	20	30	50	100	c_1	c
1	263	261	259	257	251	245	243	236	1.4; 1.45	2.93; 3.01
1.1	234	232	231	229	225	220	215	212	1.51; 1.59	3.33; 3.42
1.2	209	208	207	206	202	198	194	191	1.62; 1.67	3.73; 3.81
1.3	188	187	186	185	182	178	175	173	1.72; 1.76	4.12; 4.2
1.4	170	169	168	167	164	161	159	156	1.8; 1.88	4.5; 4.59
1.5	153	152	152	151	149	146	144	142	1.91; 1.97	4.88; 4.93
2	96	96	95	95	94	92	91	91	2.3; 2.36	6.58; 6.66
3	43	42.8	42.7	42.6	42.2	41.7	41.4	41	2.87; 2.93	9.1; 9.17
5	12.6	12.5	12.5	12.5	12.4	12.3	12.2	12.1	3.49; 3.55	11.68; 11.72
8	3.4	3.4	3.4	3.4	3.3	3.3	3.3	3.3	3.832; 3.83	12.92; 12.95

figure 2 : multiplicatively independent numbers, values of C/Z^3.

Figures 3 and 4 correspond respectively to figures 1 and 2 for multiplicatively dependent numbers.

θ	14	15	16	17	18	19	20
C	270	268	267	266	265	264	263
c_0	28.92	29.25	28.72	29.2	28.74	28.41	28.99
c_1	1.41	1.39	1.39	1.38	1.38	1.38	1.37
c	2.99	2.98	2.98	2.97	2.97	2.97	2.96

figure 3 : dependent numbers, $Z = 1$.

Z	1	1.1	1.2	1.3	1.4	1.5	2	3	5	8
C/Z^3	273	243	217	194	175	158	98	44	12.7	3.5
c_0	29.52	35.15	40.64	46.65	53.04	59.59	88.78	136.89	186.74	209.05
c_1	1.41	1.51	1.62	1.72	1.8	1.88	2.28	2.84	3.46	3.84
c	3	3.4	3.8	4.19	4.57	4.94	6.64	9.15	11.71	12.94

figure 4 : multiplicatively dependent numbers, values of C/Z^3 if $\theta \ge 12$.

7. A consequence of the main result

With the notations and hypotheses of §5.1, we shall deduce from theorem 5.11 :

COROLLARY 7.1.— *Take $f = 2e, \theta = 11$ and suppose that Λ is not zero then*

$$|\Lambda| > \exp\{-1770\, U\}.$$

Proof.— We suppose $|\Lambda| < \exp\{-1000\, U\}$. This implies Log $B > 10.8$: by Liouville estimate, if Log $B \leq 10.8$ then

$$|\Lambda| > 2^{-D} \cdot B^{-1} \cdot \exp(-Db_1 a_1 - Db_2 a_2) > B^{-1} \cdot \exp(-2Da_2 - D),$$

with $2DBa_2 + \text{Log } B + D \leq (2e^{10.8} + 12)Da_2 < 98100Da_2$, and $Da_2 \leq \theta^{-2}U$.

Thus $G > 14.24$.

We get the result with the constant C by dividing the interval $[1, \infty[$ in small intervals like in [*]. We check condition (5.5) in the worst case, namely with ε and σ replaced by 1.

The numerical values we obtain are displayed in fig. 5 below. For instance, in the range $1 \leq Z < 2$, one can choose $c_1 = 2.26, c = 6.65$, (and $c_0 = 89.21$, a value which is not given in the table), and one gets $C = 1000$.

Z 1	2	3	4	5	6	7	11	13	15	∞
C	1000	1330	1610	1670	1720	1740	1770	1760	1750	1730
c_1	2.26	2.84	3.28	3.44	3.6 : 3.68	3.68; 3.75	3.78; 4	4.03	4.06	4.06
c	6.65	9.16	11.1	11.72	12.3; 12.5	12.5; 12.71	12.7; 13.22	13.24	13.24	13.22

figure 5 : values of C in intervals on $Z, \theta \geq 11$.

8. Proof of corollary 1.1

We assume that the hypotheses of corollary 1.1 are fulfilled, and we shall prove the conclusion by considering several cases. As we may, we assume $a_1 b_1 \leq a_2 b_2$.

a) Assume Log $B \leq 10.7$. Then we prove the estimate in corollary 1.1 with the constant 268 instead of 270. For this we use lemma 2.2 of [*] :

$$|\Lambda| \geq 2^{-D} b_2^{-1} |b_1 \log \alpha_1 - b_2 \log \alpha_2| \geq \exp\{-D\text{Log } 2 - 2DBa_2 - \text{Log } B\}.$$

Since Log $B \leq 10.7$ we have $2B + \text{Log } 2 + \text{Log } B < 268(7.5 + \text{Log } B)^2$, hence $2DBa_2 + D\text{Log } 2 + \text{Log } B \leq 268Da_2(7.5 + \text{Log } B)^2$, which proves our claim.

b) From now on we assume Log $B \geq 10.7$. There is no loss of generality to assume that b_1 and b_2 are relatively prime. We are going to use theorem 5.11 with

$$f \geq 2e \text{ and } G = \text{Log } B + \text{Log Log } 2B + \max\{1, 0.59 + G'\},$$

where $G' \geq \text{Log } (e/2 + 2e/l_1), l_j = |\log \alpha_j| \quad (j = 1, 2)$.

In [*] we proved that $l_i \geq \exp(-Da_i)$ for $i = 1, 2$ and $2l_1 \geq \exp(-Da_2)$.

c) Assume $l_1 \geq 1/22.3$. In this case we take $Z = 1$,

$$G' = 1 + \text{Log } (0.5 + 4/3l_1) < 4.41,$$

and then we may choose $G = 5 + \text{Log } B + \text{Log Log } 2B$.

We prove the inequality of corollary 1.1. We use the estimates of §6 with admissible choices of θ.

Put $F = (5 + \text{Log } B + \text{Log Log } 2B)/(7.5 + \text{Log } B)$.

To prove our claim we consider the following seven cases :

$10.7 \leq \text{Log } B < 14.3$, then $\theta \geq 18, C = 264, F < 1.01$ and $F^2C < 270$,
$14.3 \leq \text{Log } B < 16.25$, then $\theta \geq 22, C = 262, F < 1.014$ and $F^2C < 270$,
$16.25 \leq \text{Log } B < 19.1$, then $\theta \geq 24, C = 260, F < 1.0183$ and $F^2C < 270$,
$19.1 \leq \text{Log } B < 20.95$, then $\theta \geq 27, C = 259, F < 1.021$ and $F^2C < 270$,
$20.95 \leq \text{Log } B < 24.77$, then $\theta \geq 29, C = 258, F < 1.0229$ and $F^2C < 270$,
$24.77 \leq \text{Log } B < 30.6$, then $\theta \geq 33, C = 257, F < 1.0248$ and $F^2C < 270$,
$30.6 \leq \text{Log } B$, then $\theta \geq 39, C = 256, F < 1.026$ and $F^2C < 270$.

d) Now on we assume $l_2 \leq l_1 < 1/22.3$. With the present notations

$$l_1 \geq \exp(-Da_i), \text{ for } i = 1, 2,$$

so that we have $Da_1 > 3.1$. Looking again at Liouville estimate, we see that if Log $B \leq 11.97$ then

$$|\Lambda| \geq \exp\{-269D^2a_1a_2(7.5 + \text{Log } B)^2\}.$$

Hence we may suppose Log $B > 11.97$. Besides one may choose

$G' = 2.41 + Z_0, G = 3 + \text{Log } B + \text{Log Log } 2B + Z_0$ where $Z_0 = \text{Log } (1/l_1)$.

Thus $G \geq 20.6$ and

$$3.1 < Z_0 \leq \min\{Da_1, Da_2, \text{Log } (ef)\}.$$

We take $Z = \min\{G/11, Z_0\}$. We obviously have

$$1 \leq Z \leq \min\{DG/11, Da_1, Da_2, \text{Log } (ef), \sqrt{D}G/8\}.$$

Now, from corollary 7.1 we deduce

$$|b_1 \log \alpha_1 - b_2 \log \alpha_2| \geq \exp\{-1770 D^4 a_1 a_2 G^2 Z^{-3}\}.$$

If $G \geq 11 Z_0$ then $Z = Z_0 > 3.1$,

$$G^2/Z^3 < (5 + \text{Log } B + \text{Log Log } 2B)^2/27 < (1.1^2/27)(7.5 + \text{Log } B)^2,$$

and we have proved our claim.

Whereas if $G < 11Z_0$ then by the results of figure 5 :

. if $Z \leq 3$ then $C = 1330$ and $1330 G^2/Z^3 < 1330 \cdot 11^3/20.6$,

. if $Z > 3$ then $C = 1770$ and $1770 G^2/Z^3 < 1770 \cdot 11^3/33$,

and the result follows in each case.

Now the proof of corollary 1.1 is complete.

9. Examples

a) *class number one*

J. M. Cherubini and R. V. Wallisser [C.W.], applied an estimate of linear forms in logarithms taken from [M. W. 1] to compute all the imaginary quadratic fields of class number one.

The linear form which is used by these authors is

$$\Lambda = p\text{Log }(5 + 2\sqrt{6}) - 2q\text{Log }(2 + \sqrt{3}), \quad p, q \in \mathbf{Z}.$$

Put $\alpha_1 = 5 + 2\sqrt{6}, \alpha_2 = 2 + \sqrt{3}$; then $D = 4, l_1 = 2.29243\ldots$ and $l_2 = 1.31695\ldots$ We take $a_1 = l_1/2, a_2 = l_2/2, f = 1, Z = \text{Log } 2e, \varepsilon = 1, \sigma = 1$ and $G = 1 + \text{Log } 4\Delta + \text{Log Log } 8\Delta$.

If we choose $c_0 = 403.51, c_1 = 2.91$ and $c = 16.58$ then the same computation than in §6 gives $C = 3455.5$. This leads to $d > -5.1 \cdot 10^{17}$, whereas the lower bound of [M. W. 1] gave only $d > -10^{34}$, and we got $d > -2.5 \cdot 10^{19}$ in [*].

b) *quotient of two pure powers*

Let x, y, p, q be positive rational integers with $x^p \neq y^q$. Let X, Y, B be positive real numbers satisfying $X \geq \max\{x, 3\}, Y \geq \max\{y, 3\}, B \geq \max\{p, q\}$. We prove that

$$|x^p y^{-q} - 1| > \exp\{-1905 \text{Log } X \text{ Log } Y (8 + \text{Log } B)^2\}.$$

If x and y are multiplicatively dependent the result is obvious (see the end of §5 of [*]). Now we assume that x and y are multiplicatively independent and consider two cases :

i) If Log $B \leq 13.8$, we have $|x^p y^{-q} - 1| \geq y^{-q} \geq \exp\{-B \text{Log } Y$, and the assumption Log $B \leq 13.8$ implies $B < 2072(8 + \text{Log } B)^2$. Now we have

$$\text{Log } X \geq \text{Log } 3 \quad \text{and} \quad 2072/\text{Log } 3 < 1887.$$

Therefore, we get the conclusion.

ii) If Log $B > 12.33$ we use theorem 5.11 with $Z = \varepsilon = f = D = 1$.

We choose now $G = 1.0021(8 + \text{Log } B), \theta = 21.86, c_0 = 212.77$, $c_1 = 1.99, c = 10.02$ and find $C = 1896$ and the result follows easily.

c) *ray class field*

In this section we present a work of J. Cougnard and V. Fleckinger which uses some linear form in two logarithms. They consider the ray class field K, extension of $k = \mathbf{Q}(\sqrt{-19})$, associated to $\mathcal{P}_7$, the principal ideal of k generated by $\left(1 + \dfrac{1+\sqrt{-19}}{2}\right)$. They show that the ring of integers O_K of the field K does not have any basis over O_k composed of the powers of some element of O_K.

They reduce this problem, via the study of the integers points of some elliptic curve, to the computation of the integers for which a certain linear form in two logarithms is very small. Namely, they want to find all the rational integers b_1 and b_2 such that

$$|b_1 \text{Log }(\alpha_1) - b_2 \text{Log }(\alpha_2)| \leq \exp(-0.0367B),$$

where $B = \max\{|b_1|, |b_2|\}$ and α_1 is a real root of the polynomial

$$P = X^9 - 16X^8 - 38X^7 - 179X^6 + 41X^5 - 237X^4 + 307X^3 - 120X^2 + 19X - 1,$$

and α_2 a root of the reciprocal polynomial with respect to P, with

$$l_1 = \text{Log } \alpha_1 = 2.92112\ldots, l_2 = \text{Log } a_2 = 2.24999\ldots.$$

Using Baker's estimate they prove $B < 10^{212}$. Here, we apply theorem 5.11 to this linear form.

Firstly, using the method of computation of the measure of a polynomial described in [CMP], we see that $M(P) < 344.56$, so that

$$h(\alpha_1) = h(\alpha_2) < 5.843/9.$$

We put $D = 9, f = 5.843, a_1 = fl_1/D, a_2 = fl_2/D, \sigma = 0.852$, $Z = 1.97$ (it is easy to verify that this value of Z satisfies the inequalities $Z \leq \min\{Da_1, Da_2, \text{Log }(e, f)\}$). We have $G' < 0.2D$ so that we can take $G = 1 + \text{Log } B + \text{Log Log } 2B$.

We suppose $\text{Log } B \geq 23.17$, then $G \geq 27$ and $\theta > 120$. Applying theorem 5.11 we find the constant $C < 1235$ (for $c_0 = 140.56, c_1 = 2.58, c = 9.11$). This gives

$$0.0367B \leq 1235 \cdot D^4 \cdot a_1 \cdot a_2 \cdot (1 + \text{Log } B + \text{Log Log } 2B)^2/Z^3,$$

so that $\text{Log } B < 23.23$, and $B < 1.3 \cdot 10^{10}$.

10. The case of a root of unity

We consider here the special case when one of the numbers α_i is a root of unity. We choose the following notations ζ is a root of unity, $\zeta = e^{i\pi/m}$, and α is a non zero algebraic number.

We put $D = [\mathbf{Q}(\zeta, \alpha) : \mathbf{Q}]$. We choose $\log \zeta = i\pi/m$, and $\log \alpha$ is any non-zero determination of the logarithm of $\alpha, l = |\log \alpha|$. As before $\beta = b_1/b_2$ is a rational number, $b_1, b_2 \in \mathbf{Z}, 0 < b_1, b_2, (b_1, b_2) = 1$ such that

$$\Lambda = i\beta\pi/m - \log \alpha$$

does not vanish. We put $B = \max\{b_1, b_2\}, a' = h(\alpha)$ [notice that $h(\zeta) = 0$].

We denote by $a, G, G', Z, \theta, f, \rho$ positive real numbers which satisfy the following relations :

$$1 \leq f \leq 2e^{D(a'+1)}, \theta \geq 10, a = fD^{-1}l, a \geq 1/D, f \geq m/\pi,$$
$$G' = 1 + \text{Log }(0.5 + \rho m/\pi), G_0 = 0.59 + \text{Log } B + \text{Log Log } 2B,$$
$$G = G_0 + \max\{0.41, G'/D\},$$
$$1 \leq Z \leq \min\{DG/\theta, f\theta/m, Da, \text{Log }(ef), \sqrt{D}G/10\}.$$

With the present notations, we have $\theta_1 = \theta fl/Z, \theta_2 = \theta f\pi/mZ$.

We define μ, ε as before and put now $\sigma = h(\alpha)/2a$ (so that $0 \le \sigma \le 1/2$), and

$$U = D^3 f(\pi/m) a G^2 Z^{-3}.$$

Now suppose that $c_0, c_1, c, \mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}, C, \eta^*, \mu, \rho, p^*, \xi, \xi_0$ are positive real numbers which satisfy the hypotheses (5.1) to (5.9), and replace (5.10) by

$$(5.10)' \qquad \mathcal{X} c \ge \sqrt{c_0 c_1} + \frac{1}{2} \cdot \max\left\{ \frac{c_1 Z m}{f\pi} + \frac{1}{\theta_2}, \sqrt{c_0/\theta} + \frac{1}{\theta} \right\}.$$

THEOREM 10.1.— *Under the hypotheses of this paragraph, we have*

$$|\Lambda| > \exp(-C' D^3 f a G^2 Z^{-3}),$$

where $C' = C\pi/m$.

Proof of theorem 10.1.

We suppose that $|\Lambda| < e^{-CU}$, and we show that this leads to a contradiction. We first remark that this implies $|\alpha| = 1$. Indeed, if $|\alpha| \ne 1$ then by Liouville inequality,

$$|\Lambda| \ge |Re(\Lambda| = |Re(\log\alpha)| = |\mathrm{Log}\ |\alpha|| \ge 2^{-D} \exp\{-h(|\alpha|)\},$$

and $h(|\alpha|) \le h(\alpha)$, which contradicts $|\Lambda| < e^{-CU}$. Thus $D \ge 2$. Remark also that now we have $a'' = 0$.

An easy proof (given at the end of §5 of [*]) shows that α is not a root of unity.

Then the proof of theorem 10.1 follows exactly that of theorem 5.11, except at the very end, when we apply the zero-estimate.

Recall that $\mathcal{X}_2 = \sqrt{c_0 c_1}/c$ and $\mathcal{X}_1 = \mathcal{X} - \mathcal{X}_2$. We define the integers U_2 and V_2 by

$$\mathcal{X}_2 c D^2 a G Z^{-2} - 0.5 < U_2 \le \mathcal{X}_2 c D^2 a G Z^{-2} - 0.5,$$
$$\mathcal{X}_2 c D^2 a_0 G Z^{-2} - 0.5 < V_2 \le \mathcal{X}_2 c D^2 a_0 G Z^{-2} - 0.5, \text{ where } a_0 = f\pi/m,$$

and U_1, V_1 by

$$U_1 = M_1^* - U_2, V_1 = M_2^* - V_2$$

[recall that $M_1^* = [\mathcal{X} c D^2 a Z^{-2}] + 0.5$ and $M_1^* = [\mathcal{X} c D^2 a_0 Z^{-2}] + 0.5$].

Since α is not a root of unity, we have

$$\text{Card } \{\zeta^u\alpha^v; |u| \le L_1, |v| \le L_1\} = 2m(2V_1+1)$$

and $(2V_1+1) \ge 2M_2^* - 2(\mathcal{X}_2 cD^2 a_0 Z^{-2} + 0.5) + 1 > (2\mathcal{X}_1 c - 1/\theta_2) D^2 a_0 G Z^{-2}$; thus the condition

$$\text{Card } \{\zeta^u\alpha^v; |u| \le L_1, |v| \le L_1\} > 2L_1$$

is implied by (5.10)'.

To conclude, we have to verify that $(2U_1+1)(2V_1+1) > L_0$. We have

$$\begin{aligned}(2U_1+1)(2V_1+1)/L_0 &> (2\mathcal{X}_1 cD^2 a_0 G Z^{-2} - 1)(2\mathcal{X}_1 cD^2 a G Z^{-2} - 1)/L_0 \\ &\ge (2\mathcal{X}_1 c - 1/\theta)^2 D^4 a_0 a G^2 Z^{-4}/L_0 \ge (2\mathcal{X}_1 c - 1/\theta)^2 DG/cZ \\ &\ge (2\mathcal{X}_1 c - 1/\theta)^2 \theta / c;\end{aligned}$$

and the conclusion follows again from (5.10)'.

This completes the proof of theorem 10.1.

11. Numerical examples for theorem 10.1

We keep the notations and hypotheses of §10 and we give suitable values for the constant C'. The results are given in figures 6, 7 and 8 below.

In figure 6 we fix $m = 1, f$ varies, and we give a value of $C'f$ and the corresponding choices of c_0, c_1 and c.

In figure 7, we suppose $f \ge 5$ and $m = 1$, and we give the values of the product $5C'$ for D in the range $2 \le D \le 10$.

Figure 8 displays some cases with $m \ge 2$ and $f = 5$.

f	1	1.5	2	2.5	3	3.5	4	5	6	8
$C'f$	4448	3644	3246	3008	2854	2750	2675	2586	2521	2469
c_0	152.05	81.76	53.56	39.76	31.45	26.09	22.13	15.79	13.72	9.97
c_1	2.09	1.98	1.9	1.79	1.72	1.65	1.58	1.6	1.45	1.38
c	8.88	6.27	4.95	4.13	3.58	3.18	2.88	2.48	2.16	1.78

figure 6 : case of a root of unity and $Z = 1$.

D	2	3	4	5	6	7	8	9	10	11
$5C'$	2575	2523	2494	2477	2466	2457	2450	2446	2443	2440
c_0	17.05	16.55	16.76	16.61	16.52	16.46	16.41	16.38	16.34	16.34
c_1	1.51	1.51	1.51	1.51	1.51	1.51	1.51	1.51	1.51	1.51
c	2.45	2.44	2.44	2.44	2.44	2.44	2.44	2.44	2.44	2.44

figure 7 : case of a root of unity, $m = Z = 1$, $2 \leq D \leq 11$

D	m	$5C'$	c_0	c_1	c
2	2	1290	16.66	1.56	2.46
2	3	861	16.74	1.55	2.46
4	2	1248	16.38	1.54	2.45
4	4	625	16.44	1.54	2.45
6	2	1234	16.67	1.53	2.44
6	3	823	16.69	1.53	2.44
6	9	281	16.84	1.52	2.44

figure 8 : some examples with $m > 1$ and $f = 5$

12. A consequence of theorem 10.1.

PROPOSITION 12.1.— *Under the hypotheses of theorem 10.1, we have*

$$|\Lambda| > \exp(-2555 \cdot (f/2e) \cdot a \cdot (6.5 + \log B)^2) \text{ for } f \geq 2e.$$

For $f \geq 2e$ we can take $C' = 2545$ (choose $c_0 = 15, c_1 = 1.53, c = 2.32$).

The estimate of Euler function given in the appendix enables us to get $G'/D \leq 0.76$ (the worst case is $D = 2$ and $m = 3$). Thus we can always take $G = 1.35 + \text{Log } B + \text{Log Log } 2B$. It is easy to verify that $G < 1.00212(6.5 + \text{Log } B)$. Hence the result.

13. A corollary of theorem 10.1

COROLLARY 13.1.— *Let α be an algebraic number of degree D, b a positive rational integer and ζ a root of unity of order m.*

Let $a = \max\{1/D, h(\alpha), f|\log \alpha|/D\}$, suppose $f \geq m/\pi$ then

$$|\alpha^b - \zeta| \geq \exp\{-5000 D^3 f a (5 + \text{Log } b)^2\}.$$

The proof is a straightforward application of the numerical estimates of §12.

14. An example of a measure of irrationality

Let $\alpha = (3 + i\sqrt{7})/4$, then $2\alpha^2 - 3\alpha + 2 = 0$ and $\alpha = e^{i\theta}$ with

$$\theta = \text{Arccos } (3/4)$$
$$= 0.72273424781341561117837735264133333620252184864 24\ldots$$

We want to get a lower bound for $|\frac{\theta}{\pi} - \frac{p}{q}|$, where p and q are rational integers.

One verifies that the number

$$\lambda := \theta/\pi$$
$$= 0.23005345616261588521378056770514289300991139527071410205\ldots$$

has the following expansion as continued fraction

$$[0; 4, 2, 1, 7, 1, 1, 2, 1, 5, 1, 27, 8, 6, 2, 4, 3, 4, 1, 2, 6, 1, 3, 2, 3, 538, 2, 7, 101, 1, 2, 3, 3, 2, 3, 3, 1, 124, 1, 5, 1, 1, 1, 14, 2, 14, 1, \ldots].$$

This implies

$$|\lambda - \frac{p}{q}| > \frac{1}{540q^2} \text{ for } 0 < q < 5 \cdot 10^{27}.$$

We put $\Lambda = ip\pi - \theta iq$ and apply theorem 10.1 with $D = 2, h(\alpha) = (\text{Log } 2)/2$, $m = 1, l_1 = \pi, l_2 = \theta, f = 1/\theta = 1.383.., \sigma = a' = \text{Log}2/2, \varepsilon^{-1} = \text{Log } (ef)$, since $G'/2 \leq (1 + \text{Log } (0.5 + 1.22/\pi))/2 < 0.45$ we can take

$$G = 1.05 + \text{Log } q + \text{Log Log } 2q < 1.082\text{Log } q,$$

then $G \geq 68.9$ and we get (take $c_0 = 67.26, c_1 = 1.81, c = 6.68$, then $C'f = 4179$).

PROPOSITION .— *The number* $\lambda = \text{Arccos } (3/4)/\pi$ *has the following measure of irrationality*

$$|\lambda - \frac{p}{q}| > \exp(-19\,600(\text{Log } q)^2), \ \textit{for } q \geq 2.$$

Remark : Our measure of irrationality of λ is not the best known. Baker proved that there exists a constant c such that $|\lambda - p/q| > q^{-c}$ for $q \geq 2$. But the best known estimate of linear form of logarithms with Baker's method which comes from [B G M M S] gives

$$|\lambda - p/q| > c'q^{-c} \text{ with } c' = \exp(-1.3 \cdot 10^{14}) \text{ and } c = 2.4 \cdot 10^{14},$$

so that our result is better for Log $q < 10^{10}$.

Added in proof. — Our main result, theorem 5.11, is not very good when one of the logarithms is very small. A remedy is to introduce a new parameter $q = a_1 b_1/(a_2 b_2)$. This leads to several minor changes which concern essentially η_0, η, the conditions (5.6.i) and (5.6.ii) : for example, the term Log $(2e)$ in (5.6.i) is replaced by Log$((1+q)e)$.

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APPENDIX

A lower bound for the Euler function

For the proof of the proposition 12.1, we have used the following estimate

PROPOSITION A.1.— *For all positive rational integers one has*

$$n < 2.685 \cdot \varphi(n)^{1.161}.$$

Suppose that the decomposition of n in prime factors is $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ with $p_1 < \dots < p_k$, then

$$\frac{n}{\varphi(n)} = \frac{p_1}{p_1 - 1} \cdots \frac{p_k}{p_k - 1}.$$

Put $\lambda = 1 - \text{Log } 4/\text{Log } 5$, then $p_i/(p_i - 1) \leq 5/4 \leq p_i^\lambda$ for $k \geq 3$. Thus,

$$n/\varphi(n) \leq 3p_3^\lambda \cdots p_k^\lambda < 2.341 p_1^\lambda \dots p_k^\lambda \leq 2.341 n^\lambda,$$

so that, $n < (2.341\varphi(n))^{1/(1-\lambda)} < 2.685\varphi(n)^{1.161}$.

Remark.— It is clear that, for any $\theta > 1$, the method used to prove the proposition permits to compute a sharp constant c_θ such that $n \leq c_\theta \cdot \varphi(n)^\theta$ for all n; for example

$$n \leq 3.046 \cdot \varphi(n)^{1.1}.$$

In fact our proof shows that the maximum value of the quotient $n/\varphi(n)^\theta$ is reached for the integer $2.3.5.\dots p_j$, where p_j is the largest prime p such that $p/(p-1) \leq p^{1-1/\theta}$.

Moreover, this method can also be applied to many multiplicative arithmetical functions.

COROLLARY A2.— *If a number field of degree D contains a root of unity of order k then*

$$k < 2.685 D^{1.161}.$$

The estimate $n \ll \varphi(n)^{1+\varepsilon}$, for any fixed $\varepsilon > 0$, is not the best possible for n large : the following result holds.

PROPOSITION A3.— *For $D \geq 2$ we define the function*

$$\varphi_{-1}(D) = \max\{N \geq 1; \varphi(N) \leq D\}.$$

Then, for any $\varepsilon > 0$, there exists an integer $D_0(\varepsilon) \geq 2$ such that, for all $D \geq D_0$, we have

$$\varphi_{-1}(D) \leq (e^C + \varepsilon) D \, Log \; Log \; D, \tag{A.1}$$

where C is Euler's constant (so that $e^C = 1.78107\ldots$). Moreover,

$$\varphi_{-1}(D) \leq 4D \, Log \; Log \; (D+7) \; for \; D \geq 2. \tag{A.2}$$

Inequality (A.1) is an easy consequence of inequality (3.42) in theorem 15 of [RS], namely

$$\frac{n}{\varphi(n)} < e^C \cdot \text{Log Log } n + \frac{2.50637}{\text{Log Log } n}. \tag{A.3}$$

Indeed this inequality is more precise than the first estimate of proposition A3 and permits to compute admissible values for $D_0(\varepsilon)$. For example :

$$\varphi_{-1}(D) \leq 2\text{Log Log } D \text{ for } D > e^{34.1}. \tag{A.4}$$

Proof of (A.4) : Consider $D > e^{34.1}$ and $N \geq 1$ such that $\varphi(N) = D$. Denote by $\omega(N)$ the number of different prime factors of N.

If $\omega(N) \geq 15$ then $\text{Log Log} N \geq 3.71, \text{Log Log } D \geq 3.66$ and (A.3) gives

$$N/D \leq e^C \cdot \text{Log Log } N + 2.51/3.71.$$

So that $N < 1.97 \cdot D \cdot \text{Log Log } N < D^{21/20}$, since $1.97 \text{Log Log} N < N^{1/20}$. This implies

$$\text{Log Log } N < \text{Log Log } D + 0.05,$$

and inequality (A.4) follows easily.

If $\omega(N) < 15$ then $N \leq 7.06 \cdot D$ and (A.4) is also true.

Another special case of (A.1) is

$$\varphi_{-1}(D) \leq 3.24D \quad \text{Log Log } D \quad \text{for } D \geq 48. \tag{A.5}$$

The proof of (A.5) is similar to that of (A.4) : Consider integers $D \geq 48$ and $N \geq 1$ such that $\varphi(N) = D$.

If $\omega(N) \geq 5$ then $N \geq 2 \times 3 \times 5 \times 7 \times 11 = 2310$ and $D \geq 480$. Then (A.3) implies

$$N/D \leq (e^C + 2.51(\text{Log Log } 2310)^{-2}) \cdot \text{Log Log } N.$$

By proposition A.1, we have $N < 2.7 \times D^{1.161} < D^{1.148}$, since $D \geq 480$; so that

$$\text{Log Log } N \leq \text{Log Log } D + 0.35,$$

and inequality (A.4) follows in this case.

If $\omega(N) < 5$ then $N \leq \frac{35}{8} D$ and (A.4) is also true.

Now, we prove (A.2). From (A.5) we have only to consider $D < 48$. Let $D < 48$ and suppose that $\varphi_{-1}(D) = N$.

Since $D < 48$, we have $\omega(N) \leq 3$. Thus $N \leq \frac{15}{4} D$, so that (A.2) is true for $D \geq 12$.

Finally, if $D \leq 12$ then a direct study shows that (A.2) is still true.