ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 5e série, tome 9, nº 1 (1988), p. 77-101
<http://www.numdam.org/item?id=AFST_1988_5_9_1_77_0>

Annales Faculté des Sciences de Toulouse Vol. IX, n°1, 1988

On the regular solutions for some classes of Navier-Stokes equations

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RÉSUMÉ. — On démontre l'existence et l'unicité de solutions réguliers pour un système de Navier-Stokes. A l'aide d'une hypothèse raisonnable sur l'accroissement du terme non-homogène, on analyse le comportement asymptotique de la solution obtenue. On utilise la méthode de pénalisation de Ebihara associée aux approximations de Galerkin.

ABSTRACT. — In this paper we obtain regular solutions of the Navier-Stokes equations. We shall work in a suitable space of functions where we have uniqueness. Under reasonable assumptions on the growth of the nonhomogeneous term, we obtain the asymptotic behavior of the solutions as $t \to +\infty$. Our approach is based on Ebihara's penalizer and the Galerkin approximations.

0. Introduction

In LIONS [7], TARTAR [12] and TEMAM [13], we can find a methodic study about the weak solutions of the Navier-Stokes equations (1.1), for space dimension n. The uniqueness for the solution of (1.1) in this weak class was proved by LIONS-PRODE [6], when $n = 2$. For $n \geq 3$ and more restriction on the solutions, there exists some results on the uniqueness in LIONS [7] (see Remark 1.2 after Proposition 1.3, Section 1, of this paper) SERRIN [11]. Concerning the regularity of weak solutions we can refer in GIGA [3], RAUTMANN [10], (Cite their references). Some topics are shown in LERAY [4], LIONS [8]. Our objective in this paper is to prove the existence and uniqueness of regular solutions for the Navier-Stokes equations and certain growth of these solutions. Our notations follow LIONS [7] and MIZOHATA [9].

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We can summarize the content of this paper as follows. It contains three sections. The section one is dedicated to fix the notations, to define the concept of regular solutions of the Navier-Stokes system, called (m)-solutions, to obtain certain properties and the uniqueness. At the end of this section we announce the main results, that is, four theorems which will be proved in sections two and three. In the section two, we prove theorems one and two by the Galerkin method. The first one is about the existence of local (m)-solutins of the Navier-Stokes equations and the second theorem about global solutions. Finally in the section three, using an argument of EBIHARA [2], that is, Galerkin method plus a certain penalty term, we prove the last two theorems, which the first, theorem three, gives certain asymptotic behavior of the (m)-solutions. The theorem four gives a characterization of the set of initial data for a particular type of forcing term of the system.

1. Preliminaries and statement of the results

Let Ω be a bounded open set in $\mathbf{R}^n$, $n \geq 2$, with sufficiently smooth boundary Γ and $0 < T < \infty$. We represent by Q the cylinder $\Omega \times [0,T[$ and by $\sum$ the lateral boundary of Q, that is, $\Gamma \times [0,T[$. By $u = u(x,t)$, we represent the velocity vector

$$u = (u_1, u_2, \cdots, u_n),$$

with $u_i = u_i(x,t)$, defined on Q, with values in reals $\mathbf{R}$.

We use the notations :

$$\frac{\partial u}{\partial t} = u' = (u_1', u_2', \cdots, u_n') = \Big(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \cdots, \frac{\partial u_n}{\partial t}\Big)$$
$$\Delta u = (\Delta u_1, \Delta u_2, \cdots, \Delta u_n)$$

$u \cdot \nabla = \sum_{j=1}^{n} u_j \dfrac{\partial}{x_j}$. and $(u \cdot \nabla)u$ is the vector which the i-th component is $\sum_{j=1}^{n} u_j \dfrac{\partial u_i}{\partial x_j}$.

The Navier-Stokes equations are :

$$\frac{\partial u_i}{\partial t} - v\Delta u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = f_i(x,t) - \frac{\partial P}{\partial x_i}$$

$i = 1, 2, \cdots, n$

$$\operatorname{div} u = 0 \text{ in } Q.$$

The initial and boundary conditions are :

$$\begin{aligned} u_i(x,0) &= u_0(x) && \text{in } \Omega \\ u_i &= 0 && \text{on } \Sigma \end{aligned}$$

The problem is that for given $f = (f_1, f_2, \cdots, f_n)$ and $u_o = (u_{o1}, u_{o2}, \cdots, \cdots, u_{on})$ find the vector velocity $u = (u_1, u_2, \cdots, u_n)$ and the pressure $p : Q \to \mathbf{R}$, satisfying the Navier-Stokes equations, the initial and boundary conditions. Note that $v > 0$ is the viscosity. In the vector notation, the Navier-Stokes equations can be written as follows :

$$\begin{aligned} &\frac{\partial u}{\partial t} - v\Delta u + (u \cdot \nabla)u = f - \nabla p \ \text{ in } \ Q \\ &\operatorname{div} u = 0 \ \text{ in } \ Q \\ &u(x,0) = u_o \ \text{ in } \ \Omega \\ &u = 0 \ \text{ on } \ \Sigma \end{aligned} \tag{1.1}$$

We need some functional spaces to formulate our problem about (1.1).

Let $\{w_j\}$ be a Stokes system, that is,

$$-\Delta w_j = \lambda_j w_j + \nabla P_j, \ \operatorname{div} w_j = 0 \ \text{ in } \ \Omega, \ w_j|_\Sigma = 0 \ (j = 1, 2, \cdots)$$

Since Σ is smooth enough, we may assume that $\{w_j\}$ are smooth, say, if Σ is in C^ℓ-class, then $\{w_j\}$ are functions in H^ℓ-class (See [13, p.39]), and they are orthonormal in $[L^2(\Omega)]^n$. By V_k we represent the space of linear combinations of first $w_1, w_2, \cdots, w_k$ and $V \equiv \bigcup_{k=1}^\infty V_k$. And we define

$$\mathcal{V}_\ell \equiv \left\{ v = \sum_{j=l}^\infty c_j w_j ; \sum_{j=l}^\infty \lambda_j^\ell |c_j|^2 < \infty \right\}.$$

This is a closure of V by a topology of inner-product and norm

$$(u, v)_\ell \equiv \sum_{j=l}^\infty \lambda_j^\ell c_j d_j, \ |u|_\ell^2 \equiv (u, u)_\ell, \ \ell = 1, 2, \cdots \tag{1.2}$$

for

$$u = \sum c_j w_j, \ v = \sum d_j w_j \in V.$$

In a standard manner, by $H^k(\Omega)(\overset{\circ}{H}{}^k(\Omega))$ we represent the Sobolev space of order $k \in N$, and by $(\cdot, \cdot\cdot)$, $[L^2(\Omega)]^n$-inner product.

Remark 1.1.— We can prove that in V the norm in (1.2) and the Sobolev norm $\|\cdot\|_\ell$ of $[H^\ell(\Omega)]^n$ are equivalent. For each $\ell, \mathcal{V}_\ell$ is a closed subspace of $[\overset{\circ}{H}{}^1(\Omega) \cap H^\ell(\Omega)]^n$. Note that, we understand the relation for $v \in [H^\ell(\Omega)]^n$, $w = \sum_{j=1}^k c_j w_j \in V_k$,

$$\begin{aligned} (v, w)_\ell &= \sum_{j=1}^{k} c_j \lambda_j^\ell (v, w_j), \ \ell = 1, 2, \cdots \\ &= (P_k[v], w)_{\ell'} \end{aligned} \tag{1.3}$$

because $v = P_k[v] + v'\big(P_k[v] \in V_{k'},\ v' \in V_k^\perp\big)$ where P_k is a projection from $[L^2(\Omega)]^n$ into V_k and $V_k^\perp$ is an orthogonal space of V_k in $[L^2(\Omega)]^n$. Then we see that

$$|(v, w)_\ell| \leq \text{Const } \|v\|_\ell |w|_\ell,$$
$$|P_k[v]|_\ell \leq \gamma(\ell)\|v\|_\ell$$

for $v \in [\overset{\circ}{H}{}^1(\Omega) \cap H^\ell(\Omega)]^n$, $w \in \mathcal{V}_\ell$, for some $\gamma(\ell) > 0$. We sometimes use this number $\gamma(\ell)$. Therefore we can assert :

For $u, v, w \in V$,

$$\begin{aligned} \big|((u \cdot \nabla)v, w)_\ell\big| &\leq \text{Const } \|(u \cdot \nabla)v\|_\ell |w|_\ell \\ &\leq \text{Const } |u|_\ell |v|_{\ell+l} |w|_\ell \qquad \Big(\ell \geq [\tfrac{n}{2}] + 1\Big) \end{aligned} \tag{1.4}$$

by applying Sobolev lemma. Here after we put $W_k \equiv [\overset{\circ}{H}{}^1(\Omega) \cap H^k(\Omega)]^n$, $k = 1, 2, \cdots$.

We also use the well known notation $L^p(0, T, X)$, $C^k(0, t; X)$ for a Banach space $X, 1 \leq p \leq \infty, k$ non negative integers. For $k = 0$ we write $C(0, T; X)$.

Now we are in condition to give the definition of solution for (1.1).

DEFINITION 1.1.— *Let m be a positive integer. A function $u(x,t) : Q \to$ $\mathbf{R}$ is said to be an (m)-solutions for (1.1) in $[0, T[$, if it satisfies :*

(i) $u(x,t) \in L^\infty(0, T; \mathcal{V}_{m+1})$

$$\frac{\partial u}{\partial t}\,(x, t) \in L^\infty(0, t; \mathcal{V}_m)$$

(ii) for every $v \in V$, we have :

$$\Big(\frac{\partial u(t)}{\partial t}, v\Big) + v\big(u(t), v\big)_l + \big((u(t) \cdot \nabla)u(t), v\big) = \big(f(t), v\big)$$

a.e. in $t \in [0, T[$

(iii) $u(0) = u_o$.

By LIONS [7], we know that when $f \in L^2(0, T; L^2(\Omega))$ if there exists a function $u(t) \in L^2(0, T; \mathcal{V}_1) \cap L^\infty(0, t; \mathcal{V}_o)$ such that for any $v \in \mathcal{V}_1$, we have :

$$\left(\frac{\partial u(t)}{\partial t}, v\right) + \upsilon\big(u(t), v\big)_1 + \big((u(t) \cdot \nabla)u(t), v\big) = \big(f(t), v\big),$$

in the distributional sense in $[0, T[$, then we obtain a function $p(x, t) \in L^2(Q)$ which satisfies

$$\frac{\partial u}{\partial t} - \upsilon \nabla u + (u \cdot \nabla)u = f - \nabla p$$

in the distributional sense in Q.

Note that $p(x, t)$ and $p(x, t)$ plus a constant in Ω solve the problem. This means that $p(x, t) + p_o(t)$ is the general solution for (x, t) in Q.

So we have :

PROPOSITION 1.1.— *Let* $m \geq [\frac{n}{2}] + 2$ *and* $f \in L^\infty(0, T; W_{m-1})$. *If there exists an* (m)*-solution* $u(x, t)$ *for (1.1) in* $[0, T[$, *then we get a function* $p(x, t)$ *such that* $\nabla p \in L^\infty\big(0, T; [H^{m-1}(\Omega)]^n\big)$ *and*

$$\frac{\partial u}{\partial t} - \upsilon \nabla u + (u \cdot \nabla)u = f - \nabla p$$

a.e. in Q.

Proof.— From our definition of the (m)-solution $u(x, t)$ and the Sobolev embedding theorems, we obtain a function $p(x, t)$ which satisfies the equality :

$$\frac{\partial u}{\partial t} - \upsilon \nabla u + (u \cdot \nabla)u - f = \nabla p$$

a.e. in Q. Since the left hand side belongs to $L^\infty\big(0, T; [H^{m-1}(\Omega)]^n\big)$ it proves the Proposition 1.1.

PROPOSITION 1.2.— *Let* $m \geq [\frac{n}{2}] + 3$ *and* $f \in C(0, T; W_{m-2})$. *If there exists an* (m)*-solution* $u(x, t)$ *for (1.1) in* $[0, T[$, *then we obtain a function* $p(x, t)$ *such that* $\nabla p \in C\big(0, T; [H^{m-2}(\Omega)]^n\big)$, *and :*

$$\frac{\partial u}{\partial t} - \upsilon \nabla u + (u \cdot \nabla)u = f - \nabla p$$

in Q.

Proof. — If we show that $u'(t) \in C(0,T;\mathcal{V}_{m-2})$, then the assertion will hold applying Sobolev embedding theorems.

We know from the definition of $u(x,t)$ that it belongs to $C(0,t:\mathcal{V}_m)$. In fact, we have, for $0 \leq t,\ s \leq T$:

$$|u(t)-u(s)|_m \leq \left|\int_0^t \frac{\partial u}{\partial \xi}\, d\xi\right|_m \leq \sup_{0\leq\xi\leq T}\left|\frac{\partial u}{\partial t}\right|_m |t-s|.$$

Thus, for a.e. $t,s \in [0,T]$, and $r \leq m-1$, we have :

$$\left(\frac{\partial u}{\partial t}(t), u(s)\right)_r + v\big(u(t),u(s)\big)_{r+1} + \big((u(t)\nabla)u(t),u(s)\big)_r = \big(f(t),u(s)\big)_r,$$

because $u(t)$ is the form $u(t)=\sum_{j=1}^{\infty} u_j(t)w_j$.

Since $u(t)$ is Lipschitz continuous from $[0,T]$ in $\mathcal{V}_m$, as we have proved above, we have :

$$\left(v, \frac{u(t+h)-u(t)}{h}\right)_r \to \big(v,u'(t)\big)_r \text{ as } h\to 0$$

for a.e. t in $[0,T[$ and $r\leq m$ for $v \in V_m$.

Therefore, we can see that for a.e. $t,s\in[0,T[$, we obtain :

$$\left(\frac{\partial u(t)}{\partial t}, \frac{\partial u(s)}{\partial s}\right)_r + v\left(u(t), \frac{\partial u(s)}{\partial s}\right)_{r+1} + \\ + \left((u(t)\cdot\nabla)u(t), \frac{\partial u(s)}{\partial s}\right)_r = \left(f(t), \frac{\partial u(s)}{\partial s}\right)_r,$$

for $r\leq m-1$.

Now, we show for $k=m-2$ and a.e. $t\in[0,T[$:

$$|u'(t)-u'(s)|_k^2 + \big((u(s)\cdot\nabla)u(s),u'(s)\big)_k \\ \big((u(t)\cdot\nabla)u(t),u'(t)\big)_k - \big(f(s),u'(s)-u'(t)\big)_k \\ \leq C\{|s-t| + \|f(t)-f(s)\|_k\}. \tag{1.5}$$

We get for a.e. $t, s \in [0, T[$,

$$\begin{aligned}
&|u'(s) - u'(t)|_k^2 = |u'(s)|_k^2 - 2\big(u'(s), u'(t)\big)_k + |u'(t)|_k^2 \\
&= -v\big(u(s), u'(s)\big)_{k+1} - \big((u(s) \cdot \nabla)u(s), u'(s)\big)_k + f(s), u'(s)\big)_k \\
&\quad + 2v\big(u(s), u'(t)\big)_{k+1} + 2\big((u(s) \cdot \nabla)u(s), u'(t)\big)_k - 2\big(f(s), u'(t)\big)_k \\
&\quad - \big(u(t), u'(t)\big)_{k+1} - \big((u(t) \cdot \nabla)u(t), u'(t)\big)_k + \big(f(t), u'(t)\big)_k \\
&= v\big(u(s) - u(t), u'(t)\big)_{k+1} + v\big(u(s), u'(t) - u'(s)\big)_{k+1} \\
&\quad + \big((u(s) \cdot \nabla)u(s) - \big(u(t) \cdot \nabla\big)u(t), u'(t)\big)_k \\
&\quad + \big((u(s) \cdot \nabla)u(s), u'(t) - u'(s)\big)_k + \big(f(s), u'(s) - u'(t)\big)_k \\
&\quad + \big(f(t) - f(s), u'(t)\big)_k.
\end{aligned}$$

Here we know that :

$$\begin{aligned}
&\big|v\big(u(s) - u(t), u'(t)\big)_{k+1}\big| \le C\big|u(s) - u(t)\big|_m |u'(t)|_{m-2} \le C|t - s|, \\
&v\big(u(s), u'(t) - u'(s)\big)_{k+1} = v\big\{\big(u(s), u'(t)\big)_{k+1} - \big(u(s), u'(s)\big)_{k+1}\big\} \\
&= v\Big\{-v\big(u(t), u(s)\big)_{k+2} - \big((u(t) \cdot \nabla)u(t), u(s)\big)_{k+1} + \big(f(s), u(s)\big)_{k+1} \\
&\quad v\big(u(s), u(s)\big)_{k+2} + \big((u(s) \cdot \nabla)u(s), u(s)\big)_{k+1} - \big(f(s), u(s)\big)_{k+1}\Big\},
\end{aligned}$$

therefore

$$\begin{aligned}
\Big|v\big(u(s), u'(t) - u'(s)\big)_{k+1}\Big| &\le v^2\big|\big(u(s) - u(t), u(s)\big)\big|_{k+2} \\
&\quad + v\Big|\big((u(t) \cdot \nabla)u(t) - \big(u(s) \cdot \nabla\big)u(s), u(s)\big)_{k+1}\Big| \\
&\quad + v\big|\big(f(t) - f(s), u(s)\big)_{k+1}\big| \\
&\le C\big\{|t - s| + \|f(s) - f(t)\|_{m-2}\big\}
\end{aligned}$$

and

$$\big|\big(f(t) - f(s), u'(t)\big)_k\big| \le C\|f(t) - f(s)\|_{m-2}.$$

Further, we get :

$$\begin{aligned}
&\big((u(s) \cdot \nabla)u(s), u'(t)\big)_k - \big((u(s) \cdot \nabla)u(s), u'(s)\big)_k \\
&= \big((u(s) - u(t) \cdot \nabla)u(s), u'(t)\big)_k + \big((u(t) \cdot \nabla)\big(u(s) - u(t)\big), u'(t)\big)_k \\
&+ \big((u(t) \cdot \nabla)u(t), u'(t)\big)_k - \big((u(s) \cdot \nabla)u(s), u'(s)\big)_k.
\end{aligned}$$

Therefore, from these facts, we get :

$$\begin{aligned}
&\big|u'(s) - u'(t)\big|_k^2 + \big((u(s) \cdot \nabla)u(s), u'(s)\big)_k - \big((u(t) \cdot \nabla)u(t), u'(t)\big)_k \\
&\quad - \big(f(s), u'(s) - u'(t)\big)_k \le C\big\{|s - t| + \|f(s) - f(t)\|_k\big\}
\end{aligned}$$

which proves the inequality (1.5).

Thus, if we change the situation of s and t, we should have :

$$\begin{aligned}&\left|u'(t)-u'(s)\right|_k^2+\big((u(t)\cdot\nabla)u(t),u'(t)\big)_k\\&\quad-\big((u(s)\cdot\nabla)u(s),u'(s)\big)_k-\big(f(t),u'(t)-u'(s)\big)_k\\&\leq C\big\{|t-s|+\|f(t)-f(s)\|_{m-2}\big\}.\end{aligned}$$

Consequently, adding these two last inequalities, we obtain :

$$\begin{aligned}2\left|u'(t)-u'(s)\right|_k^2+\big(f(t)-f(s),u'(s)-u'(t)\big)_k\\\leq 2C\big\{|t-s|+\|f(t)-f(s)\|_{m-2}\big\},\end{aligned}$$

and then,

$$|u'(t)-u'(s)|_{m-2}^2\leq C\big\{|t-s|+\|f(t)-f(s)\|_{m-2}\big\}, \tag{1.6}$$

for a.e. $t,s\in[0,T[$.

Now, for any $t_o\in[0,T[$, there exists a sequence (t_μ) in $[0,T[$ such that $\lim_{\mu\to\infty}t_\mu=t_o$, $u'(t_\mu)\in\mathcal{V}_{m-2}$. From the inequality (1.5) it follows that $\big(u'(t_\mu)\big)$ is a Cauchy sequence in $\mathcal{V}_{m-2}$. So we have the limit $\big(u'(t_o)\big)=\lim_{\mu\to\infty}u'(t_\mu)\in\mathcal{V}_{m-2}$, and the limit dose not depend on the choice of the sequence (t_μ). Therefore, we can say $u'(t)\in C(0,T;\mathcal{V}_{m-2})$.

Then, from the continuity of $u(t)$, $f(t)$, we have :

$$\big(u'(t),v\big)+v\big(u(t),v\big)_1+\big((u(t)\cdot\nabla),v\big)=\big(f(t),v\big)$$

for all $t\in[0,T[$ and $v\in V$. From this we obtain the condition of the Proposition 1.2.

Q.E.D.

PROPOSITION 1.3.— *Under the same assumptions as in Proposition 1.1, we can not have two different* (m)*-solutions for (1.1) in* $[0,T[$.

Proof.— If there exist two (m)-solutions $u(t)$, $\widehat{u}(t)$, then $w(t)=u(t)-\widehat{u}(t)$ should satisfy a.e. in $[0,T[$:

$$\Big(\frac{1}{2}\,|w(t)|_o^2\Big)'+v|w(t)|_1^2+\big((u\cdot\nabla)u-(\widehat{u}\cdot\nabla)\widehat{u},w(t)\big)=0.$$

Here we know :

$$\begin{aligned}|(u\cdot\nabla)u-(\widehat{u}\cdot\nabla)\widehat{u},w)|&=\big|\big((w\cdot\nabla)u-(\widehat{u}\cdot\nabla)w,w\big)\big|\\&\leq|\nabla u|_\infty\,|w|_o^2+|\widehat{u}|_\infty|\nabla w|_o|w|_o\leq C\big(|w|_o^2+|w|_1\cdot|w|_o\big)\end{aligned}$$

because $u, \widehat{u} \in L^\infty(0,T;\mathcal{V}_{m+1})$.

We obtain

$$\Big(\frac{1}{2}\ |w(t)|_o^2\Big)' + \nu|w(t)|_1^2 \leq C\big(|w|_o^2 + |w|_1|w|_o\big).$$

Thus,

$$\big(|w(t)|_o^2\big)' \leq c|w|_o^2 \text{ a.e. in } t \in [0,T[.$$

Since $w(0) = 0$, we get $w(t) = 0$ in $[0,T[$.

Q.E.D.

Remark 1.2. — In the case of $n = 2$ we have uniqueness for (1.1) when the solution u belongs to the weak class $L^2(0,T;\mathcal{V}_1) \cap L^\infty(0,T;\mathcal{V}_o)$ as proved by LIONS-PRODI [6]. In LIONS [7] p.84, he proved that if $n \geq 3$ and the weak solutions u belong to :

$$L^2(0,T;\mathcal{V}_1) \cap L^\infty(0,T;\mathcal{V}_o) \cap L^r(0,T;[L^s(\Omega)]^n),$$

for $2/s + n/r \leq 1$, we have also uniqueness for the weak solutions for the Navier-Stokes equations (1.1). It is important to observe that the assumptions of the Proposition 1.3 implies that our (m)-solutions belong to the above Lions class. So, the proof of Proposition 1.3 follows also from this Lions result.

Now we state our assertions.

THEOREM 1. — *Let $m \geq [n/2] + 2$, $u_o \in \mathcal{V}_{m+2}$, $f(t) \in C(0,\infty;W_m)$, $f'(t) \in C(0,\infty;W_{m-1})$. Then, there exists a positive number δ and a unique function $u(x,t)$ which is an (m)-solutions of (1.1) in $[0,\delta[$.*

Corollary of Theorem 1. — If $m \geq [n/2] + 3$, the function u in Theorem 1 satisfies (1.1) for all (x,t) in Q.

THEOREM 2. — *Under the same assumptions as in Theorem 1, we assume $f(t) \in L^1(0,\infty;W_m)$. Then, if*

$$|u_o|_m + \int_0^\infty \|f(t)\|_m\, dt$$

is small enough, the function $u(t)$ exists in $[0,\infty[$.

THEOREM 3. — *Under the same assumptions as in Theorem 1, we assume*

$$\|f(t)\|_m < \frac{C}{(l+t)^\alpha},$$

for some constants $C > 0$, $\alpha > 0$. Then we have a function $u(x,t)$ and positive real numbers $0 < T_1 \leq T_2 < +\infty$ such that :

$$u(x,t) \in L^{\infty}_{loc}(0,\infty;\mathcal{V}_{m+1})$$
$$u'(x,t) \in L^{\infty}_{loc}(0,\infty;\mathcal{V}_m),$$

and $u(x,t)$ is an (m)-solution for (1,1) in $[0,T_1[$, and $u(x,t)$ solves (ii) of Definition 1.1 in the interval $[T_2,+\infty[$. Moreover :

$$|u(t)|_m < \frac{\widehat{C}}{(1+t)^{\alpha}}$$

holds, for some $\widehat{C} > 0$ and large t.

THEOREM 4.— *Let $m \geq [n/2]+2$, $f(x,t) = \nabla g(x)$ for $g \in \overset{\circ}{H}{}^m(\Omega)$. Then, we have a set $W \subset \mathcal{V}_{m+1}$ such that if $\widetilde{u}_o \in W$ there exists an (m)-solution $\widetilde{u}(x,t)$ for (1.1) in $[0,\infty[$ with initial condition $\widetilde{u}(x,0) = \widetilde{u}_o(x)$. The set W is not bounded in $\mathcal{V}_{m+1}$.*

Remark 1.3.— Note that $u \in L^{\infty}_{\text{loc}}(0,\infty;H)$ means $u \in L^{\infty}(0,T;H)$ for any $0 < T < \infty$.

2. Galerkin approximation scheme

In this section we prove the Theorems 1 and 2. We get for $u_o \in \mathcal{V}_{m+2}$ a sequence $(a_j)_{j\in\mathbf{N}}$ of real numbers such that

$$u_{oj} = \sum_{i=1}^{j} a_i w_i \text{ converges to } u_o \text{ strongly in } \mathcal{V}_{m+2} \tag{2.1}$$

For each $j \in \mathbf{N}$, we define $u_j(t) = \sum_{i=1}^{j} \chi_{ji}(t) w_i$, which belongs to the space V_j $[w_1, w_2, \cdots, w_j]$. The approximated Galerkin scheme consists in determining χ_{ji} such that u_j is a solution of :

$$\begin{cases} \big(u_j'(t),v\big) + v\big(u_j(t),v\big)_1 + \big((u_j(t)\cdot\nabla)u_j(t),v\big) \\ \qquad = \big(f(t),v\big) \text{ for each } v \in V_j \\ u_j(0) = u_{oj} \end{cases} \tag{2.2}$$

We know from the standard theory of ordinary differential equations that $u_j(t)$, given as a solution of the ordinary differential system, exists in some interval $[0,\delta_j[$.

To obtain a priori estimates, we put :

$$C_o = \sup_{\substack{u \in \mathcal{V}_m \\ u \neq 0}} \frac{\|(u \cdot \nabla)u\|_{m-1}}{|u|_m^2}; C_T = \sup_{0 \leq t \leq T} \|f(t)\|_{m-1}.$$

Then $C_o > 0$ is well defined because $m \geq [n/2] + 2$. In the following, $u_j(t)$ means the solution of (2.2).

LEMMA 2.1. — *It holds that :*

$$|u_j(t)|_m^2 \leq \frac{|u_{oj}|_m^2 + \dfrac{C_T}{C_o}}{1 - C_*\Big(|u_{oj}|_m^2 + \dfrac{C_T}{C_o}\Big)t} - \frac{C_T}{C_o},$$

for $t \in [0, \overline{\delta}_j[$, *where*

$$C_* = \frac{\big(\gamma(m-1)C_o\big)^2}{4v}, \overline{\delta}_j = \min\Big\{C_*\Big(|u_{oj}|_m^2 + \frac{C_T}{C_o}\Big)^{-1}, T\Big\}.$$

Proof. — In $(2.2)_1$ we take $v = \sum_{i=1}^{j} \chi_{ji}(t)\lambda_i^m w_i$ in $[0, \delta_j[$. So we get :

$$\Big(\frac{1}{2}\ |u_j(t)|_m^2\Big)' + v|u_j(t)|_{m+1}^2 + \Big(\big(u_j(t) \cdot \nabla\big)u_j(t),\ u_j(t)\Big)_m = \big(f(t),\ u_j(t)\big)_m. \tag{2.3}$$

We know that from Remark 1.1,

$$\begin{aligned}\Big|\Big(\big(u_j(t) \cdot \nabla\big)u_j(t), u_j(t)\Big)_m\Big| &\leq \gamma(m-1)|u_j(t)|_{m+1}\big\|\big(u_j(t) \cdot \nabla\big)u_j(t)\big\|_{m-1} \\ &\leq \gamma(m-1)C_o|u_j(t)|_{m+1}|u_j(t)|_m^2. \\ \big|\big(f(t), u_j(t)\big)_m\big| &\leq \gamma(m-1)|u_j(t)|_{m+1}\|f(t)\|_{m-1}.\end{aligned}$$

Then, substituting in (2.3) we obtain :

$$\big(|u_j(t)|_m^2\big)' \leq C_*\big(C_o|u_j(t)|_m^2 + C_T\big)^2,$$

for each $t \in [0, \delta_j[$.

Solving this differential inequality, we get the assertion of Lemma 2.1.

Q.E.D.

From Lemma 2.1 and definition (2.1), choosing $T > 0$ large enough, we know that for small $\mu > 0$, there exists $j_o \in \mathbf{N}$ such that :

$$\delta_\mu = \frac{1}{C_*\Big(|u_o|_m^2 + \mu + \dfrac{C_T}{C_0}\Big)} < \bar{\delta}_j,\ j \geq j_o.$$

Thus we have proved :

$$\sup_{j\geq j_o}\ \sup_{t\in[0,\delta_\mu]} |u_j(t)|_m^2 < C < \infty. \tag{2.4}$$

LEMMA 2.2.— *It holds that :*

$$\sup_{j\geq j_o}\ \sup_{t\in[0,\delta_\mu]} |u_j(t)|_{m+1}^2 < C < \infty \tag{2.5}$$

$$\sup_{j\geq j_o}\ \sup_{t\in[0,\delta_\mu]} |u_j'(t)|_m^2 < C < \infty \tag{2.6}$$

Proof.— To obtain (2.5) we take $v = \sum_{i=1}^{j} \chi_{ji}'(t)\lambda_i^m w_i$ in $(2.2)_1$. Then we have :

$$|u_j'(t)|_m^2 + \Big(\frac{v}{2}|u_j(t)|_{m+1}^2\Big)' + \big((u_j(t)\cdot\nabla)u_j(t), u_j'(t)\big)_m$$
$$= \big(f(t), u_j'(t)\big)_m,$$

for $0 \leq t \leq \delta_\mu$.

So by the estimate (2.4), we obtain :

$$\Big(\frac{v}{2}|u_j(t)|_{m+1}^2\Big)' \leq \big(\|(u_j(t)\cdot\nabla)u_j(t)\|_m^2 + \|f(t)\|_m^2\big)$$
$$\leq C|u_j(t)|_m^2|u_j(t)|_{m+1}^2 + C\|f(t)\|_m^2$$

or

$$\Big(\frac{v}{2}\ |u_j(t)|_{m+1}^2\Big)' \leq C|u_j(t)|_{m+1}^2 + C\|f(t)\|_m^2. \tag{2.7}$$

Applying Gronwall inequality to (2.7) we obtain (2.5).

To get (2.6) we differentiate the equation $(2.2)_1$ with respect to t, and do $v = \sum_{i=1}^{j} \chi_{ji}'(t)\lambda_i^m w_i$. We have then :

$$\Big(\frac{1}{2}|u_j'(t)|_m^2\Big)' + v|u_j'(t)|_{m+1}^2$$
$$+ \big((u_j'(t)\cdot\nabla)u_j(t) + (u_j(t)\cdot\nabla)u_j'(t), u_j'(t)\big)_m$$
$$= \big(f'(t), u_j'(t)\big)_m, \tag{2.8}$$

for $0 \leq t \leq \delta_\mu$.

We have :

$$\Big|\big((u'_j(t)\cdot\nabla)u_j(t),u'_j(t)\big)_m\Big| \leq C|u'_j(t)|_m^2\ |u_j(t)|_{m+1}$$
$$\leq C|u'_j(t)|_m^2 \tag{2.9}$$
$$\Big|\big((u_j(t)\cdot\nabla)u'_j(t),\ u'_j(t)\big)_m\Big| \leq C|u_j(t)|_m|u'_j(t)|_m|u_j(t)|_{m+1}$$
$$\leq C|u'_j(t)|_m|u'_j(t)|_{m+1}. \tag{2.10}$$

By (2.9), (2.10), the inequality (2.8) can be transformed in the following :

$$\big(|u'_j(t)|_m^2\big)' \leq C + C|u'_j(t)|_m^2\ ,\ 0\leq t\leq \delta_\mu. \tag{2.11}$$

From (2.11) we get :

$$|u'_j(t)|_m^2 \leq C + C|u'_j(0)|_m^2\ ,\ 0\leq t\leq \delta_\mu. \tag{2.12}$$

If we have boundedness of $\{|u'_j(0)|_m\}$ independent of $j\in\mathbf{N}$, then we obtain the assertion (2.6).

In fact, let t goes to zero in $(2.2)_1$. We get :

$$\big(u'_j(0),v\big)+v(u_{oj},v)_1+\big((u_{oj}\cdot\nabla)u_{oj},v\big)=\big(f(0),v\big),$$

for all $v\in V_j$. Doing $v=\sum_{i=1}^{j}\chi'_{ji}(0)\lambda_i^m w_i$ in this equation, we obtain :

$$|u'_j(0)|_m^2+v\big(u_{oj},u'_j(0)\big)_{m+1}+\big((u_{oj}\cdot\nabla)u_{oj},u'_j(0)\big)_m$$
$$=\big(f(t),u'_j(0)\big)_m.$$

So, we get :

$$|u'_j(0)|_m^2 \leq v|u_{oj}|_{m+2}|u'_j(0)|_m + C\|(u_{oj}\cdot\nabla)u_{oj}\|_m|u'_j(0)|_m +$$
$$+C\|f(0)\|_m|u'(0)|_m$$

or

$$|u'_j(0)|_m \leq v|u_{oj}|_{m+2}+C|u_{oj}|_m|u_{oj}|_{m+1}+C\|f(0)\|_m < C < \infty. \tag{2.13}$$

From (2.12) and (2.13) we obtain the estimates (2.6) of Lemma 2.2.

Q.E.D.

By the condition (2.6) of Lemma 2.2, we get :

$$|u_j(t) - u_j(s)|_m \leq C|t-s|, 0 \leq t,\ s \leq \delta_\mu. \tag{2.14}$$

Consequently, applying compactness argument (cf. [1,7]) to the results in Lemmas 2.1 and 2.2, we have a subsequence of $\left(u_j(t)\right)_{j \in \mathbf{N}}$ such that :

$$\begin{aligned} &u_j(t) \to u(t) \text{ weak star in } L^\infty(0,\delta_\mu;\mathcal{V}_{m+1}) \\ &u_j(t) \to u(t) \text{ strongly in } \mathcal{V}_m \text{ for every } t \in [0,\delta_\mu] \\ &u_j'(t) \to u'(t) \text{ weak star in } L^\infty(0,\delta_\mu : \mathcal{V}_m) \end{aligned}$$

as $j \to \infty$, and from (2.14), the limit $u(t)$ satisfies :

$$|u(t) - u(s)|_m \leq C|t-s|,\ 0 \leq t,\ s \leq \delta_\mu,$$

that is, $u(t)$ is Lipschitz continuous in $[0,\delta_\mu]$, with values in $\mathcal{V}_m$.

From this, $u(t)$ satisfies, for a.e. $t \in [0,\delta_\mu]$ the equation :

$$\left(u'(t),v\right) + v\left(u(t),v\right)_1 + \left((u(t)\cdot\nabla)u(t),v\right) = \left(f(t),v\right) \tag{2.15}$$

for all $v \in V$.

Therefore, the function $u(t)$ in an (m)-solution for (1.1) in $[0,\delta_\mu]$. Thus we have proved Theorem 1.

For the proof of Theorem 2, it is sufficient to prove the following :

LEMMA 2.3.— *If u_o, f satisfy :*

$$|u_o|_m^2 + 2\gamma(m)\left(\int_0^\infty \|f(t)\|_m dt\right)\frac{v}{C_1} < \left(\frac{v}{C_1}\right)^2, \tag{2.16}$$

where

$$C_1 = \sup_{\substack{u \in \mathcal{V}_{m+1} \\ u \neq 0}} \frac{\gamma(m-1)\|(u\cdot\nabla)u\|_{m-1}}{|u|_m|u|_{m+1}},$$

then, for large j, we have :

$$|u_j(t)|_m < \frac{v}{C_1},\ t \in [0,\infty[.$$

Proof.— In fact, if $u_o,\ f$ satisfy (2.16), we know that :

$$|u_{oj}|_m^2 + 2\gamma(m)\left(\int_0^\infty \|f(t)\|_m dt\right)\frac{v}{C_1} < \left(\frac{v}{C_1}\right)^2$$

holds for large j, by (2.1). Therefore, in the existence interval $[0,\delta_j[$ we get :

$$\left(\frac{1}{2}\,|u_j(t)|_m^2\right)' + v|u_j(t)|_{m+1}^2 + \big((u_j(t)\cdot\nabla)u_j(t), u_j(t)\big)_m = \big(f(t), u_j(t)\big)_m$$

hence

$$\left(|u_j(t)|_m^2\right)' \le 2|u_j(t)|_{m+1}^2\{-v + C_1|u_j(t)|_m\} + 2\gamma(m)\|f(t)\|_m\,|u_j(t)|_m.$$

Integrating from 0 to t this inequality, in the existence interval, we have :

$$|u_j(t)|_m^2 \le 2\int_0^t |u_j(s)|_{m+1}^2\{-v + C_1|u_j(s)|_m\}ds$$
$$+ 2\gamma(m)\int_0^t \|f(s)\|_m |u_j(s)|_m ds + |u_{oj}|_m^2.$$

Here, if we have a time $t^* \in (0,\delta_j)$ such that

$$|u_j(t^*)|_m = \frac{v}{C_1}\ ,\ |u_j(t)|_m < \frac{v}{C_1}\ ,\ t \in [0,t^*[,$$

then at $t = t^*$ we should have :

$$\left(\frac{v}{C_1}\right)^2 \le 2\int_0^{t^*} |u_j(t)|_{m+1}^2\{-v + C_1|u_j(t)|_m\}dt$$
$$+ 2\gamma(m)\int_0^{t^*} \|f(t)\|_m|u_j(t)|_m dt + |u_{oj}|_m^2$$
$$\le |u_{oj}|_m^2 + 2\gamma(m)\left(\int_0^\infty \|f(t)\|_m^2 dt\right)\frac{v}{C_1},$$

which is a contradiction. This proves the Lemma 2.3.

Q.E.D.

From this estimate we get as in previous argument :

$$\sup_{j\ge j_o}\ \sup_{0\le t\le T}\left\{|u_j(t)|_{m+1}\ ,\ |u_j'(t)|_m\right\} < \infty,$$

for every fixed $T > 0$.

Then, applying Proposition 1.3, uniqueness, we have the proof of Theorem 2.

3. Galerkin approximation scheme with penalty term

In this section we prove the Theorems 3, 4. For this purpose, putting $\{\lambda_i, w_i\}_{i\in\mathbf{N}}$ and $(u_{oi})_{i\in\mathbf{N}}$ the same sequences as in Section 2, we set, for $\epsilon > 0$,

$$u_j^\epsilon(t) = \sum_{i=1}^{j} \chi_{ji}^\epsilon(t) w_i,$$

as a solution of the system :

$$\begin{cases} \big((u_j^\epsilon(t))',v\big)+\nu\big(u_j^\epsilon(t),v\big)_1+\big((u_j^\epsilon(t)\cdot\nabla)u_j^\epsilon(t),v\big) \\ +\epsilon F\Big(\dfrac{K-|u_j^\epsilon(t)|_m^2}{\epsilon}\Big)\big(u_j^\epsilon(t),v\big)=\big(f(t),v\big) \\ \text{for each } v\in V_j \\ u_j^\epsilon(0)=u_{oj} \end{cases} \tag{3.1}$$

In (3.1) the function $F(\xi)$ satisfies the conditions :

$$\begin{cases} F(\xi)\in C^1(0,\infty) \\ F(\xi)\geq \dfrac{A}{\xi^\beta},\ \xi\in]0,\delta_o[\text{ for some } A>0,\ \beta\geq 1,\ 0<\delta_o\leq 1 \\ F'(\xi)\leq 0,\ \text{for } \xi>0 \\ F(\xi)=1,\ \text{for } \xi\geq 1 \end{cases} \tag{3.2}$$

The constant K in (3.1) is chosen for u_o such that

$$|u_o|_m^2<K. \tag{3.3}$$

We have :

LEMMA 3.1.— *For large j, ($j\geq j_o$), it holds that :*

$$|u_j^\epsilon(t)|_m^2<K,\ \textit{for}\ \geq 0.$$

Proof.— For large j we have that $u_j^\epsilon(t)$ exists in some interval $[0,\delta_j^\epsilon]$, from (2.1) and (3.3), since it holds that $|u_{oj}|_m^2<K$ for large j, and therefore the system (3.1) has no singularities near $t=0$.

Now, doing $v=\sum_{i=1}^j \xi_{ji}^\epsilon(t)\lambda_i^m w_i$ in (3.1), we get :

$$\Big(\frac12\ |u_j^\epsilon(t)|_m^2\Big)'+\nu|u_j^\epsilon(t)|_{m+1}^2+\big((u_j^\epsilon(t)\cdot\nabla)u_j^\epsilon(t),u_j^\epsilon(t)\big)_m$$
$$+\epsilon F\Big(\frac{K-|u_j^\epsilon(t)|_m^2}{\epsilon}\Big)\ |u_j^\epsilon(t)|_m^2=\big(f(t),u_j^\epsilon(t)\big)_m.$$

Hence, as in a previous manner,

$$\Big(\frac12\ |u_j^\epsilon(t)|_m^2\Big)'+\nu|u_j^\epsilon(t)|_{m+1}^2+\epsilon F\Big(\frac{K-|u_j^\epsilon(t)|_m^2}{\epsilon}\Big)\ |u_j^\epsilon(t)|_m^2$$
$$\leq C|u_j^\epsilon(t)|_{m+1}\|(u_j^\epsilon(t)\cdot\nabla)u_j^\epsilon(t)\|_{m+1}+C\|f(t)\|_m|u_j^\epsilon(t)|_m$$
$$C|u_j^\epsilon(t)|_{m+1}|u_j^\epsilon(t)|_m^2+C\|f(t)\|_m|u_j^\epsilon(t)|_m.$$

Therefore,

$$\begin{aligned}&\left(|u_j^\epsilon(t)|_m^2\right)' + \epsilon F\Big(\frac{K - |u_j^\epsilon(t)|_m^2}{\epsilon}\Big)\, |u_j^\epsilon(t)|_m^2\\ &\le C|u_j^\epsilon(t)|_m^4 + C\|f(t)\|_m |u_j^\epsilon(t)|_m.\end{aligned}$$

Here, if we have a time t_1, $(0 < t_1 \le \delta_j^\epsilon)$, such that

$$|u_j^\epsilon(t_1)|_m^2 = K,\ |u_j^\epsilon(t)|_m^2 < K,\ 0 < t < t_1,$$

then we see that

$$\left(|u_j^\epsilon(t)|_m^2\right)'\Bigg|_{t\,=\,t_1} \ge 0.$$

It follows, from the above inequality, that

$$\begin{aligned}&\lim_{t\to t_1}\left[\left(|u_j^\epsilon(t)|_m^2\right)' + \epsilon F\Big(\frac{K - |u_j^\epsilon(t)|_m^2}{\epsilon}\Big)\, |u_j^\epsilon(t)|_m^2\right]\\ &\le \lim_{t\to t_1}\left\{C|u_j^\epsilon(t)|_m^4 + C\|f(t)\|_m |u_j^\epsilon(t)|_m\right\}\\ &= CK^2 + C\|f(t)\|_m\sqrt{K} < +\infty.\end{aligned}$$

However, the left hand side of (3.4) should be $+\infty$ from $(3.2)_2$. Thus, we cannot have such t_1. This implies our assertion of Lemma 3.1.

Q.E.D.

From Lemma 3.1, we know that $u_j^\epsilon(t)$ exists in $[0,\infty[$ for each $\epsilon > 0$, $j \ge j_o$.

Remark 3.1. — This type of argument was used by the first time by Ebihara [2].

LEMMA 3.2. — *For every fixed $T > 0$, we have :*

$$\sup_{j\ge j_o}\ \sup_{0\le t\le T} |u_j^\epsilon(t)|_{m+1} < C(t) \tag{3.5}$$

$$\sup_{\substack{j\ge j_o\\ \epsilon\le\epsilon_o}}\ \sup_{0\le t\le T} \left|\left(u_j^\epsilon(t)\right)'\right|_m < C(T) \tag{3.6}$$

for some $\epsilon_o > 0$.

Proof. — We may omit the subindex ϵ, j of $u_j^\epsilon(t)$ representing the function by $u(t)$. This procedure makes the notation better. Now, to prove (3.5) we do $v = \sum_{i=1}^{j} (\chi_{ji}^\epsilon(t))\lambda_i^{m+1} w_i$ in $(3.1)_1$ and we get :

$$\Big(\frac{1}{2}\,|u(t)|_{m+1}^2\Big)' + v|u(t)|_{m+2}^2 + \big((u(t)\cdot\nabla)u(t), u(t)\big)_{m+1} + \epsilon F\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\,\big(u(t),u(t)\big)_{m+1} = \big(f(t),u(t)\big)_{m+1}.$$

Then

$$\Big(\frac{1}{2}\,|u(t)|_{m+1}^2\Big)' + v|u(t)|_{m+2}^2 + \epsilon F\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\,|u(t)|_{m+1}^2 \leq C|u(t)|_{m+2}|u(t)|_{m+1}|u(t)|_m + C|u(t)|_{m+2}\|f(t)\|_m.$$

Therefore, we get

$$\big(|u(t)|_{m+1}^2\big)' \leq C(K)|u(t)|_{m+1}^2 + C\|f(t)\|_m^2$$

Thus, we have (3.5) of Lemma 3.2.

For the proof of (3.6), we differentiate both sides of $(3.1)_1$ and after we do $v = \sum_{i=1}^{j} (\chi_{ji}^\epsilon(t))'\lambda_i^m w_i$. Then we obtain :

$$\Big(\frac{1}{2}\,|u'(t)|_m^2\Big)' + v|u'(t)|_{m+1}^2 + \big((u'(t)\cdot\nabla)u(t) + (u(t)\cdot\nabla)u'(t), u'(t)\big)_m + \epsilon F\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\big(u'(t),u'(t)\big)_m - 2F'\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\big(\big(u(t),u'(t)\big)_m\big)^2 = \big(f'(t),u'(t)\big)_m.$$

Here, from (3.2), we know that

$$\epsilon F\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\,|u'(t)|_m^2 - 2F'\Big(\frac{K-|u(t)|_m^2}{\epsilon}\Big)\,\big(\big(u(t),u'(t)\big)_m\big)^2 \geq 0.$$

Therefore, we have :

$$\Big(\frac{1}{2}\,|u'(t)|_m^2\Big)' + v|u'(t)|_{m+1}^2 \leq C|u'(t)|_m^2|u(t)|_{m+1} + C|u(t)|_m|u'(t)|_m|u'(t)|_{m+1} + C\|f'(t)\|_m|u'(t)|_m$$

then,

$$\Big(\frac{1}{2}\,|u'(t)|_m^2\Big)' \leq C|u'(t)|_m^2 + C\|f'(t)\|_m\ ,\ 0 < t \leq T.$$

Thus, if we prove the boundedness of $\{|(u^\epsilon j(0))'|_m\}$ we have the proof of the estimate (3.6).

In fact, let $t \to 0$ in $(3.1)_{1'}$. We get :

$$((u_j^\epsilon(0))', v) + \nu(u_{oj}, v)_1 + ((u_{oj} \cdot \nabla)u_{oj}, v) \\ + \epsilon F\left(\frac{K - |u_{oj}|_m^2}{\epsilon}\right)(u_{oj}, v) = (f(0), v).$$

From (3.2), we can see that we have :

$$F\left(\frac{K - |u_{oj}|_m^2}{\epsilon}\right) = 1 \ , \ (0 < \epsilon \le \epsilon_o \ , \ j \ge j_o).$$

Therefore, if we take $v = \sum_{i=1}^{j} (\chi_{ji}^\epsilon(0))' \lambda_i^m w_i$ we obtain :

$$\begin{aligned} |(u_j^\epsilon(0))'|_m^2 &= (f(0), (u_j^\epsilon(0))')_m - \nu(u_{oj}, (u_j^\epsilon(0))')_{m+1} \\ &\quad - ((u_{oj}, (u_j^\epsilon(0))')_m - \epsilon(u_{oj}, (u_j^\epsilon(0))')_m \\ &\le C(\|f(0)\|_m + \nu|u_{oj}|_{m+2} + |u_{oj}|_m |u_{oj}|_{m+1} \\ &\quad + \epsilon|u_{oj}|_m)\ |(u_j^\epsilon(0))'|_m. \end{aligned}$$

This implies the boundedeness of $\{|(u_j^\epsilon(0))'|_m\}$, which proves (3.6) of Lemma 3.2.

Q.E.D.

From the estimation in Lemmas 3.1, we get, for each fixed ϵ, $(0 < \epsilon < \epsilon_o)$, a function $u^\epsilon(t) = \sum_{i=1}^\infty (u^\epsilon(t), w_i) w_i$ which satisfies :

$$\begin{cases} u^\epsilon(t) \in L^\infty_{\text{loc}}(0, \infty, \mathcal{V}_{m+1}); (u^\epsilon(t))' \in L^\infty_{\text{loc}}(0, \infty; \mathcal{V}_m) \\ ((u^\epsilon(t))', v) + \nu(u^\epsilon(t), v)_1 + ((u^\epsilon(t) \cdot \nabla)u^\epsilon(t), v) \\ \quad + \epsilon F\left(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\right)(u^\epsilon(t), v) = (f(t), v) \\ \quad \text{a.e., } t > 0, \text{ for every } v \in V \\ u^\epsilon(0) = u_o \end{cases} \tag{3.7}$$

At the same time, we know that :

$$|u^\epsilon(t)|_m^2 < K, \text{ for every } t > 0 \tag{3.8}$$

$$\sup_{0<\epsilon<\epsilon_o} |u^\epsilon(t)|_{m+1} \le C(T) < \infty \tag{3.9}$$

$$\sup_{0<\epsilon<\epsilon_o} |(u^\epsilon(t))'|_m \le C(T) < \infty \tag{3.10}$$

$$|u^\epsilon(t) - u^\epsilon(s)|_m \le C(T)\ |t - s|, \ 0 \le t, \ s \le T, \tag{3.11}$$

for each $T > 0$.

LEMMA 3.3.—

$$\sup_{0<\epsilon<\epsilon_0} \epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big) \le C(T) < \infty. \tag{3.12}$$

Proof.— In (3.7) we take $v = \sum_{i=1}^{\infty} \lambda_i^m (u^\epsilon(t), w_i) w_i$ and we obtain

$$\begin{aligned}
&\big((u^\epsilon(t))', u^\epsilon(t)\big)_m + \nu |u^\epsilon(t)|_{m+1}^2 + \big((u^\epsilon(t)\cdot\nabla)u^\epsilon(t), u^\epsilon(t)\big)_m \\
&\quad + \epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big)|u^\epsilon(t)|_m^2 = \big(f(t), u^\epsilon(t)\big)_m \ , \ \text{a.e. } t > 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big)|u^\epsilon(t)|_m^2 \\
&\qquad \le |(u^\epsilon(t))'|_m |u^\epsilon(t)|_m + \nu |u^\epsilon(t)|_{m+1}^2 + C|u^\epsilon(t)|_m^2 |u^\epsilon(t)|_{m+1} \\
&\qquad\quad + C\|f(t)\| |u^\epsilon(t)|_m \\
&\qquad \le C(T) \ , \ 0 \le t \le T.
\end{aligned}$$

Hence, if $|u^\epsilon(t)|_m \le K/2$, then,

$$\epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big) = \epsilon,$$

and if $|u^\epsilon(t)|_m^2 > K/2$, then

$$\epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big) \le \frac{2}{K}\, C(T) \ , \ 0 \le t \le T,$$

which proves Lemma 3.3.

Q.E.D.

Therefore, it follows from (3.8) to (3.12), that we obtain $u(x,t), \chi(t)$ such that :

$$u^\epsilon(\cdot,t) \to u(\cdot,t) \text{ weak star in } L^\infty(0,T;\mathcal{V}_{m+1}), \tag{3.13}$$

$$u_t^\epsilon(\cdot,t) \to u_t(\cdot,t) \text{ weak star in } L^\infty(0,t;\mathcal{V}_m), \tag{3.14}$$

$$u^\epsilon(\cdot,t) \to u(\cdot,t) \text{ strongly in } C(0,T;\mathcal{V}_m), \tag{3.15}$$

$$\epsilon F\Big(\frac{K - |u^\epsilon(t)|_m^2}{\epsilon}\Big) \to \chi(t) \text{ weak star in } L^\infty(0,T), \tag{3.16}$$

and $\chi(t) \geq 0$ a.e. in $[0,T]$ for every $T > 0$,

$$\Big(\frac{\partial u(t)}{\partial t}, v\Big) + \nu\big(u(t), v\big)_1 + \big((u(t)\cdot\nabla)u(t), v\big) \\ + \chi(t)\big(u(t), v\big) = \big(f(t), v\big)$$

a.e. for $t \geq 0$ and for each $v \in V$,

$$u(0) = u_o \tag{3.18}$$

Here we know that :

$$|u(t)|_m^2 \leq K, \text{ for all } t \text{ in } [0, \infty[.$$

We have : if $|u(t_o)|_m^2 < K - \theta$ $(0 < \theta < K)$, then $\chi(t) = 0$ at $t = t_o$. In fact, by the strong convergence $u^\epsilon(t) \to u(t)$ in $C(0,T; \mathcal{V}_m)$ and

$$|u^\epsilon(t_o)|_m^2 \leq K - \frac{\theta}{2} \text{ for small } \epsilon \ (\epsilon \leq \epsilon_o),$$

then

$$\epsilon F\Big(\frac{K - |u^\epsilon(t_o)|_m^2}{\epsilon}\Big) \leq \epsilon F\Big(\frac{\theta}{2\epsilon}\Big) \to 0 \text{ as } \epsilon \to 0.$$

Now, we prove Theorem 3.

From (3.17) we get for a.e. t in $[0, \infty[$:

$$\Big(\frac{1}{2}\ |u(t)|_m^2\Big)' + \nu|u(t)|_{m+1}^2 + \big((u(t)\cdot\nabla)u(t), u(t)\big)_m + \chi|u(t)|_m^2 \\ = \big(f(t), u(t)\big)_m$$

or

$$\Big(\frac{1}{2}\ |u(t)|_m^2\Big)' + \nu|u(t)|_{m+1}^2 \leq C\|f(t)\|_m|u(t)|_m + C|u(t)|_m^2|u(t)|_{m+1},$$

that is,

$$\Big(\frac{1}{2}\ |u(t)|_m^2\Big)' \leq C\|f(t)\|_m\sqrt{K} + CK^2.$$

Therefore,

$$|u(t)|_m^2 \leq |u_o|_m^2 + C\sqrt{K}\int_0^t \|f(s)\|_m ds + CK^2 t.$$

From this, we can find $T_1 > 0$ such that :

$$|u_o|_m^2 + C\sqrt{K}\int_0^t \|f(s)\|_m ds + CK^2 t < K \ , \ 0 \le t \le T_1.$$

This implies$|u(t)|_m^2 < K$, $0 \le t \le T_1$, therefore it follows that $\chi(t) \equiv 0$ a.e. in $t \in [0, T_1]$.

In the next step, we do $v = u(t)$ in (3.17) and we get for a.e. $t \in [0, T]$:

$$\Big(\frac{1}{2}\ |u(t)|_o^2\Big)' + v|u(t)|_1^2 + \chi(t)|u(t)|_o^2 = \big(f(t), u(t)\big).$$

Therefore,

$$\frac{1}{2}\big(|u(t)|_o^2\big)' + v_o|u(t)|_o^2 \le |f(t)|_o|u(t)|_o,$$

where v_o is a positive constant such that $v_o|u(t)|_o^2 \le v|u(t)|_1^2$. Thus

$$\big(|u(t)|_o\big)' + v_o|u(t)|_o \le |f(t)|_o \le \frac{C_o}{(1+t)^\alpha}.$$

From this inequality we obtain :

$$|u(t)|_o \le e^{-v_o t}\left\{|u_o|_o + C_o\int_0^t \frac{e^{v_o s}}{(1+t)^\alpha}\ ds\right\}$$

or

$$|u(t)|_o \le \frac{\widetilde{C}}{(1+t)^\alpha}\ ,\ t \ge 0,$$

holds for some positive constant $\widetilde{C}$, applying l'Hospital Theorem. Next we get

$$\Big(\frac{1}{2}\ |u(t)|_m^2\Big)' + v|u(t)|_{m+1}^2 + \big((u(t)\cdot\nabla)u(t), u(t)\big)_m + \chi(t)|u(t)|_m^2$$
$$= \big(f(t), u(t)\big)_m.$$

Then

$$\Big(\frac{1}{2}\ |u(t)|_m^2\Big)'$$
$$\le -v|u(t)|_{m+1}^2 + \big|\big((u(t)\cdot\nabla)u(t), u(t)\big)_m\big| + C\|f(t)\|_m|u(t)|_m$$
$$\le -v|u(t)|_{m+1}^2 + C\|(u(t)\cdot\nabla)u(t)\|_{m+1}|u(t)|_{m+1} + C\|f(t)\|_m|u(t)|_m$$
$$\le -v|u(t)|_{m+1}^2 + C_2|u(t)|_{m+1}|u(t)|_m|u(t)|_{m+1} + C_3\|f(t)\|_m|(u(t)|_m$$

for some $C_2 > 0$.

Here, we know that there exists ρ_o, $(0 < \rho_o < 1)$, such that

$$|u|_{m-1}|u|_m \leq |u|_m^{-\rho_o}|u|_{m+1},$$

for all $u \in \mathcal{V}_{m+1}$, by using generalized Schwarz inequality :

$$(u,u)_k \leq |u|_{k-1}|u|_{k+1}\ ,\ k = 1, 2, \cdots .$$

Therefore, we have :

$$\begin{aligned}
&\left(|u(t)|_m^2\right)' \\
&\leq 2|u(t)|_{m+1}^2\Big\{-v + C_2|u(t)|_o^{\rho_o}|u(t)|_m^{1-\rho_o}\Big\} + 2C_3\|f(t)\|_m|u(t)|_m \\
&\leq 2|u(t)|_{m+1}^2\Big\{-v + C_2\frac{\widetilde{C}^{\rho_o}}{(1+t)^{\alpha\rho_o}}K^{\frac{1-\rho_o}{2}}\Big\} + 2C_3\|f(t)\|_m|u(t)|_m.
\end{aligned}$$

Thus, we get $\widetilde{T} > 0$ such that :

$$C_2\frac{\widetilde{C}^{\rho_o}}{(1+t)^{\alpha\rho p_o}}\ K \leq \frac{v}{2}\ ,\ \widetilde{T} \leq \forall t.$$

Therefore, if $t \geq \widetilde{T}$ we get :

$$\left(|u(t)|_m^2\right)' \leq -v|u(t)|_{m+1}^2 + 2C_3\|f(t)\|_m|u(t)|_m\ ,$$

and we have :

$$\left(|u(t)|_m\right)' + \frac{v_o}{2}\ |u(t)|_m \leq 2C_3\|f(t)\|_m \leq \frac{C}{(1+t)^\alpha}\ ,\ t \geq \widetilde{T}$$

by assumption.

Thus

$$|u(t)|_m \leq e^{-\frac{v_o}{2}(t-\widetilde{T})}\left\{|u(\widetilde{T})|_m + C\int_{\widetilde{T}}^t \frac{e^{-\frac{v_o}{2}}(s-\widetilde{T})}{(1+)^\alpha}ds\right\}$$

or

$$|u(t)|_m \leq \frac{\widehat{C}}{(1+t)^\alpha}\ ,\ t \geq \widetilde{T},$$

for some positive constant $\widehat{C}$, applying l'Hopital Theorem.

This implies that there exists $T_2 > 0$ such that :

$$|u(T_2)|_m = |u_o|_m \ , \ |u(t)|_m^2 < |u|_m^2 < K \ , \ t > T_2. \tag{3.19}$$

Since $|u(t)|_m^2 < K$, $t \in [T_2, \infty[$, we get $\chi(t) = 0$ for $t \in [T_2, +\infty[$. This shows that $u(t)$ solves (i) of the Definition 1.1 in $[T_2, \infty[$.

We finally prove Theorem 4.

Since $f(x,t) = \nabla g(x)$, $g \in \overset{\circ}{H}{}^1(\Omega) \cap H^m(\Omega)$, for any $v \in V$, $(f, v)_k = 0$, $k = 0, 1, 2, \cdots, m$, holds. Therefore, in this case, the function $u(x,t)$ constructed by the system (3.1) and (3.7) should satisfy all the properties from (3.31) to (3.16) and solves :

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + v\big(u(t), v\big)_1 + \big((u(t)\cdot\nabla)(t), v\big) \\ \quad + \chi(t)\big(u(t), v\big) = 0 \\ v \in V, \text{ a.e. } t \geq 0 \\ u(0) = u_o \end{cases} \tag{3.20}$$

In particular as in (3.19), we can find $T > 0$ such that $u(T) \in \mathcal{V}_{m+1}$ and

$$|u(T)|_m = |u_o|_m \ , \ |u(t)|_m^2 < |u_o|_m^2 < K \ , \ t > T. \tag{3.21}$$

So, obtaining $\chi(t) \equiv 0$ for $t > T$, $u(t)$ solves :

$$\left(\frac{\partial u}{\partial t}, v\right) + \big(u(t), v\big)_1 + \big((u(t)\cdot\nabla)u(t), v\big) = 0$$

for each $v \in V$, a.e. in $t \geq T$.

Therefore, if we put :

$$W \equiv \big\{u(T) \in \mathcal{V}_{m+1}; u_o \in \mathcal{V}_{m+2}\big\},$$

where $u(t)$ is the function constructed by our scheme and $u(T)$ is the datum given by (3.21), we can see that W is the desired set for the proof of Theorem 4. In fact, W is not bounded in $\mathcal{V}_{m+1}$ because $u_o \in \mathcal{V}_{m+2}$ has no restriction on the size of its norm and $U(t) = u(t+T)$ solves, in our sense, the system :

$$\begin{cases} \left(\dfrac{\partial U}{\partial t}, v\right) + v\big(U(t), v\big)_1 + \big((U(t)\cdot\nabla)U(t), v\big) = 0 \\ \text{a.e. in } t \geq 0 \\ U(0) = u(T) \end{cases} \tag{3.22}$$

which proves Theorem 4.

Q.E.D.

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(Manuscrit reçu le 29 octobre 1985)