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Some 2-type submanifolds and applications [(1)] [(2)]

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RÉSUMÉ. — Dans cet article nous démontrons que une sous-variété de type 2 avec le champ de vecteurs de courbure moyenne parallèle dans l'espace euclidien est sphérique ou nulle. En utilisant le résultat nous donne une classification des surfaces de type 2 avec le champ de vecteurs de courbure moyenne parallèle dans l'espace euclidien. De plus nous étudions les sous-variétés de type 2 et nulle.

ABSTRACT. — In this article we prove that a 2-type submanifold M in a Euclidean space with parallel mean curvature vector is either spherical or null. By applying this result we obtain a complete classification of 2-type surfaces with parallel mean curvature vector. We also study null 2-type submanifolds.

1. Introduction

Let M be a connected (not necessary closed) n-dimensional submanifold of a Euclidean m-space E^m. Then we have the Laplacian operator Δ of M (with respect to the induced metric) acting on the space of smooth functions. By applying the Laplacian operator, we have the notion of finite type submanifolds introduced by the first author (cf. [3,4]). The Laplacian operator Δ can be extended to an operator, also denote by Δ, acting on E^m-valued smooth functions on M in a natural way. For a submanifold M in E^m, the submanifold M is said to be *of k-type* if the position vector x of M can be expressed in the following form :

$$x = c + x_{i_1} + \cdots + x_{i_k}; \ \Delta x_{i_t} = \lambda_{i_t} x_{i_t}, \ \lambda_{i_1} < \cdots < \lambda_{i_k}, \tag{1.1}$$

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for some natural number k where c is a constant map, and $x_{i_1}, \cdots, x_{i_k}$ are non-constant maps. A submanifold M is said to be *of finite type* if it is of k-type for some natural number k. Otherwise, M is said to be *of infinite type*. A k-type submanifold is said to be *null* if one of the $\lambda_{i_t}; t = 1, \cdots, k$, is null.

A submanifold M is a minimal submanifold of E^m if and only if M is of null 1-type, that is, the position vector x of M takes the following form:

$$x = c + x_o, \quad \Delta x_o = O. \tag{1.2}$$

Moreover, by a result of Takahashi, a submanifold M in E^m is a minimal submanifold of a hypersphere of E^m if and only if the submanifold M is of non-null 1-type, that, is, we have

$$x = c + x_p, \quad \Delta x_p = \lambda_p x_p, \quad \lambda_p \neq O. \tag{1.3}$$

It is well-known that 1-type submanifolds in E^m have parallel mean curvature vector.

In section 2 we prove that if a 2-type submanifold M in E^m has parallel mean curvature vector, then either (a) M is spherical or (b) M is of null 2-type, i.e., the position vector x of M takes the following form :

$$x = c + x_o + x_q, \quad \Delta x_o = 0, \quad \Delta x_q = \lambda_q x_q, \quad \lambda_q \neq 0. \tag{1.4}$$

In particular, this result shows that every closed 2-type hypersurface in a Euclidean space has non-constant mean curvature. In this section, we also show that a closed 2-type surface in E^m is the product of two plane circles with different radi if and only if it has parallel mean curverture vector. In section 3, we prove that every null 2-type submanifold in E^m with parallel mean curvature vector is an a-submanifold. By applying this result we obtain a complete classification of 2-type surfaces with parallel mean curvature vector. In the last section, we will apply our previous results to show that a surface in E^3 is an open portion of a circular cylinder if and only if it is of flat null 2-type.

2. 2-type submanifold in E^m

For an n-dimensional submanifold M in E^m, we denote by h, A, H, ∇ and D, the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the submanifold M, respectively. A submanifold M is said to have parallel mean curvature vector if $DH = 0$ identically. For a hypersuface M, the parallelism of mean curvature vector equivalent to the constancy of mean curvature $\alpha = \|H\|$ If the submanifold M is closed (i.e.,M is compact and without boundary), then every eigenvalue λ_t of Δ is ≥ 0 and the only harmonic functions on M are constant functions. In this case, the constant vector c in the spectral decomposion (1.1) is nothing but the center of mass of M in E^m. A submanifold M of a hyperphere S^{m-1} of E^m is said to be *mass-symmetric* if the center of mass of M in E^m is the center of the hyperphere S^{m-1} in E^m. In this section we study 2-type submanifolds in a Euclidean space with parallel mean curvature vector.

THEOREM 1.— *Let M be a 2-type submanifold of E^m. If M has parallel mean curvature vector, then one of the following two cases occurs :*

(a) M is spherical;

(b) M is of null 2-type.

In particular, if M is closed, then M is spherical and mass-symmetric.

Proof.— Let X, Y be two vector fields tangent to M. Then, for any fixed vector a in E^m, we have

$$\begin{aligned} YX < H, a > = < D_Y D_X H, a > - < \nabla_Y(A_H X), a > - < A_{D_X H} Y, a > \\ - < h(Y, A_H X), a >, \end{aligned} \tag{2.1}$$

where $<,>$ denotes the inner product of E^m. Let $e_1, \cdots, e_n$ be an orthonormal local frame field tangent to M. Then equation (2.1) implies

$$\Delta H = \Delta^D H + \sum \{h(e_i, A_H e_i) + A_{D_{e_i} H} e_i + (\nabla_{e_i} A_H) e_i\} \tag{2.2}$$

where

$$\Delta^D H = \sum \{D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H\} \tag{2.3}$$

is the Laplacian of H with respect to the normal connection D. Regard ∇A_H and A_{DH} as (1,2)-tensors on M and we set

$$\overline{\nabla} A_H = \nabla A_H + A_{DH}.$$

Then we have

$$\overline{\nabla} A_H = \sum \{(\nabla_{e_i} A_H) e_i + A_{D_{e_i} H} e_i\}. \tag{2.4}$$

Let $e_{n+1},...,e_m$ be an orthonormal normal basis of M such that e_{n+1} is parallel to H. Then we have

$$\sum h(e_i, A_H e_i) = \|A_{n+1}\|^2 H + a(H) \tag{2.5}$$

where

$$A_r = A_{e_r}, \quad \|A_{n+1}\|^2 = tr(A_{n+1})^2,$$

and

$$a(H) = \sum_{r=n+2}^{m} (tr(A_H A_r))e_r \tag{2.6}$$

is called the *allied mean curvature vector* of M in E^m. Combining (2.2), (2.4), (2.5) and (2.6), we have the following useful formula [3, p.271] :

$$\Delta H = \Delta^D H + \|A_{n+1}\|^2 H + a(H) + tr(\overline{\nabla} A_H). \tag{2.7}$$

Moroever, we also have the following [4,5]

$$tr(\overline{\nabla} A_H) = (n/2)\operatorname{grad}\alpha^2 + 2trA_{DH}, \quad \alpha^2 =< H, H > . \tag{2.8}$$

Therefore, if $DH = O$, then we have $\Delta^D H = tr(\overline{\nabla} A_H) = 0$ which implies

$$\Delta H = \|A_{n+1}\|^2 H + a(H). \tag{2.9}$$

Now, assume that M is of 2-type in E^m. Then the position vector x of M in E^m has the following spectral decomposition:

$$x - c = x_p + x_q, \quad \Delta x_p = \lambda_p x_p, \quad \Delta x_q = \lambda_q x_q. \tag{2.10}$$

From (2.10) we have

$$\Delta^2 x = (\lambda_p + \lambda_q)\Delta x - \lambda_p \lambda_q (x - c). \tag{2.11}$$

On the other hand, we also have

$$\Delta x = -nH. \tag{2.12}$$

Therefore, by using (2.9),(2,11),(2.12), we obtain

$$\|A_{n+1}\|^2 H + a(H) = (\lambda_p + \lambda_q)H + (\lambda_p \lambda_q / n)(x - c). \tag{2.13}$$

From (2.13) we have either $\lambda_p\lambda_q = 0$ or $x - c$ is normal to M at every point in M. If $\lambda_p\lambda_q = 0$, then M is of null 2-type . If $x - c$ is normal to M, then $< x - c, x - c >$ is a positive constant. In this case, M is contained in a hypersphere S^{m-1} centered at c. In particular, if M is closed, then because λ_p and λ_q are positive, M cannot be null. Moreover, in this case, because c is the center of mass of M in E^m, M is mass-symmetric in S^{m-1}.

(Q.E.D.)

Remark 1. — For a closed n-dimensional 2-type submanifold M in E^m with $x = c + x_p + x_q, \lambda_p < \lambda_q$, the first author had shown that if M lies in a unit hypersphere of E^m, then the mean curvature function satisfies $\alpha^2 \geq \lambda_p/n$, moreover, if $\alpha^2 = \lambda_p/n$ at a point $u \in M$, then $DH = O$ at u and M is speudo-umbilical at u (i.e., the Weingarten map with respect to H is proportional to the identity map).

COROLLARY 2. — *Every 2-type closed hypersurface of E^{n+1} has non-constant mean curvature.*

This corollary follows immediately from Theorem 1, since for a hypersuface the constancy of mean curvature is the same as the parallelism of mean curvature vector.

Remark 2. — This corollary was also obtained independently by Garay.

THEOREM 3. — *Let M be a closed 2-type surface in E^m.Then M has parallel mean curvature vector if and only if M is the product of two plane circles with different radii.*

Proof. — If M has parallel mean curvature vector, then M is one the following surfaces (cf.[2,p.106]): (i) a minimal surface of E^m,(ii) a minimal surface of a hypersphere of E^m, (iii) a surface in a 3-dimensional linear subspace E^3 or (iv) a surface in a 3-sphere S^3 in a 4-dimensional linear subspace. For the first two cases, M is of 1-type which contradicts to the hypothesis. If M lies in a 3-dimensional linear subspace, then, by Theorem 1, M is a 2-sphere which is of 1-type again. If M lies in a 3-sphere, then by parallelism of H, we see that the mean curvature is constant and so M is mass-symmetric. Consequetly,according to Theorem 4.5 of [3,p.279], we know that M is the product surface of two plane circles with different radii.

(Q.E.D.)

3. Null 2-type submanifolds in E^m

Let M be an n-dimensional null 2-type submanifold of E^m. Then we have the following spectral decomposition of the position vector x of M in E^m :

$$x = c + x_o + x_q, \quad \Delta x_o = 0, \quad \Delta x_q = \lambda_q x_q, \tag{3.1}$$

where c is a constant vector, and x_o and x_q are non-constant maps from M into E^m.

LEMMA 4.— *If M is a null 2-type submanifold in E^m, then we have*
(1) $tr(\overline{\nabla} A_H) = 0$,
(2) $\Delta^D H = (\lambda_q - \|A_{n+1}\|^2)H + a(H)$.

Proof.— Since M is of null 2-type, equation (3.1) implies

$$\Delta H = \lambda_q H. \tag{3.2}$$

Therefore, by applying formula (2.7), we obtain

$$\Delta^D H + \|A_{n+1}\|^2 H + a(H) + tr(\overline{\nabla} A_H) = \lambda_q H. \tag{3.3}$$

Because $tr(\nabla D_H)$ is tangent to M and all other terms in (3.3) are normal to M, formula (3.3) implies the lemma.

From Theorem 1 and Lemma 4 we have the following.

THEOREM 5.— *Let M be a 2-type submanifold in E^m with parallel mean curvature vector. Then either (a) M is spherical and non-null or (b) M is a 2-type a-submanifold with $\|A_{n+1}\|^2 = \lambda_q$ which is a nonzero constant.*

Proof.— This lemma follows form Theorem 1 and statement (2) of lemma 4 , since the parallelism of H implies $\Delta^D H = 0$.

Now, we apply Theorem 1 and Lemma 5 to obtain the following generalization of Theorem 3 which gives a complete classification of 2-type surfaces with parallel mean curvature vector.

THEOREM 6.— *Let M be a surface in E^m with parallel mean curvature vector. Then M is of 2-type if and only if M is one of the following two surfaces:*

(a) an open portion of the product surface of two plane circles with different radii;

(b) an open portion of a circular cylinder.

Proof. — Let M be a 2-type surface in E^m with parallel mean curvature vector. Then M must lies either in a 3-dimensional linear subspace with constant mean curvature or in a hyperphere S^3 in a 4-dimensional linear subspace of E^m with constant mean curvature (cf.[2,p.106]). According to Theorem 1, M is either spherical or null. We consider these two cases separately.

Case (1): M is null. In this case, Theorem 5 implies that $\|A_3\|$ is a nonzero constant. Since M either lies in a 3-dimensional linear subspace or lies in a 3-sphere S^3, the constancy of $\|A_3\|$ is equivalent to the constancy of the length of the second fundamental form $\|h\|$. Because the mean curvature is also constant, the equation of Gauss implies that M has constant Gaussian curvature. Consequetly, by applying Proposition 3.2 of [2,p.118], we conclude that M is either an open portion of the product of two plane circles or an open portion of a circular cylinder. In the first case, the radii of the two plane circles must be different, since M is of 2-type.

Case (2): M is non-null and M lies in a 3-sphere S^3. Without loss of generality, we may assume that S^3 is of radius one and centered at the origin. Since the mean curvature vector H of M in E^m and the mean curvature vector H' of M in S^3 are related by $H = H' - x$, formula (2.13) gives

$$\|h\|^2 H' - 2 < H, H > x = (\lambda_p + \lambda_q)(H' - x) + (\lambda_p \lambda_q / 2)(x - c). \quad (3.4)$$

.

Because M is non-null, this implies that $< x, c >$ is a constant function on M. If $c \neq 0$, then M is a small hyperphere of S^3 which is a contradiction. So we obtain $c = 0$. Consequently, by (3.4), we find

$$\|h\|^2 = \lambda_p + \lambda_q. \quad (3.5)$$

.

Thus, by the constancy of the mean curvature and the equation of Gauss, the Gaussian curvature of M is also constant. Hence, by applying Proposition 3.1 of [2, p.116], we know that the surface M is flat. From these we may conclude that M is an open portion of the product of two plane circles with different radii.

The converse follows from Theorem 4.5 of [3,p.279] and Lemma 2 of [6].

From Theorem 6, we obtain the following new characterization of circular cylinders.

LEMMA 7.— *Let M be a surface in E^3. Then M is an open portion of a circular cylinder if and only if M has constant mean curvature and is of 2-type.*

4. Null 2-type surfaces.

In this section we study null 2-type surfaces in E^3. Let M be a null 2-type surface in E^3. According to Lemma 4 we know that for such surfaces we have

$$tr(\overline{\nabla} A_H) = 0 \text{ and } \Delta^D H = (\lambda_q - \|h\|^2)H. \tag{4.1}$$

Assume that the mean curvature of M is non-constant. We put $U = \{u \in M : \operatorname{grad} \alpha^2 \neq 0 \text{ at } u\}$. Then U is non-empty open subset of M. By formula (2.8) and condition (4.1), we have

$$\nabla \alpha^2 + 2trA_D H = 0 \tag{4.2}$$

where ∇f denotes the gradient of f. Let $H = \alpha e$, where $\alpha = \|H\|$ is the mean curvature and e a unit normal vector of M in E^3. Then, by the definition, we have $tr\ A_{DH} = A(\nabla\alpha), A = A_e$ Therefore, (4.2) implies the following.

LEMMA 8.— *If M is a null 2-type surface in E^3, then, on U, $\nabla\alpha$ is an eigenvector of the Weingarten map A with eigenvalue add α.*

From Lemma 8, we see that A has eigenvalues add α and 3α on U. Let e_1, e_2 be an orthonormal local frame field on U such that e_1 is parallel to $\nabla\alpha$. Then we have

$$e_2\alpha = 0. \tag{4.3}$$

Since we have $Ae_1 = -\alpha e_1$ and $Ae_2 = 3\alpha e_2$, (4.3) and Codazzi's equation imply

$$\omega_1^2(e_1) = 0 \quad \text{and} \quad 3(e_1\alpha) = -4\alpha\omega_1^2(e_2) \tag{4.4}$$

where we put

$$\nabla e_i = \sum \omega_i^j e_j \quad i, j = 1, 2. \tag{4.5}$$

On the other hand , Lemma 4 implies

$$\Delta\alpha = (\lambda_q - 10\alpha^2)\alpha. \tag{4.6}$$

By using (4.4) we may obtain

$$e_1e_1\alpha = (4/3)(e_1\alpha)\omega_2^1(e_2) + (4\alpha/3)(e_1(\omega_2^1(e_2))), \tag{4.7}$$

$$e_2e_2\alpha = 0 \qquad (\nabla_{e_2}e_2)(\alpha) = \omega_2^1(e_2)(e_1\alpha). \tag{4.8}$$

Thus, by applying (4.7) and (4.8), we may obtain

$$-9\ \alpha\Delta\alpha = 4\alpha^2\big(\omega_2^1(e^2)\big)^2 + 4\alpha^2\big(e_1\big(\omega_2^1(e_2)\big)\big). \tag{4.9}$$

From (4.4) we also have

$$9\|\nabla\alpha\|^2 = 16\alpha^2\big(\omega_2^1(e^2)\big)^2. \tag{4.10}$$

Combining (4.9) and (4,10), we find

$$-12\alpha\Delta\alpha = 3\|\nabla\alpha\|^2 + 14\alpha^2\big(e_1(\omega_2^1(e_2))\big). \tag{4.11}$$

Now, we need another expression of the last term of (4.11). By the definition of the curvature tensor R of M and the connection form, we may obtain

$$\begin{aligned} e_1\omega_2^1(e_2) =&< R(e_1,e_2)e_2, e_1 > + \\ &< \nabla_{[e_1,e_2]}e_2, e_1 > \end{aligned} \tag{4.12}$$

where we have used formula (4.4). By using the definition of Lie braket and (4.4), we may show that the last term of (4.12) is equal to $\big(\omega_2^1(e_2)\big)^2$. Therefore, from (4.12), we may get

$$e_1\omega_2^1(e_2) = -3\alpha^2 + (9/16\alpha^2)\|\nabla\alpha\|^2. \tag{4.13}$$

Combining (4.11) and (4.13) we find

$$\alpha\Delta\alpha = 4\alpha^4 - \|\nabla\alpha\|^2. \tag{4.14}$$

From (4.6) and (4.14) we obtain

$$\|\nabla\alpha\|^2 = (14\alpha^2 - \lambda_q)\alpha^2 \text{ on } U. \tag{4.15}$$

If the closure of U is a proper subset of M, then, according to Corollary 7, a connected component, say V, of $M\backslash$closure (U) is an open portion of a circular cylinder. And hence the mean curvature α on V is nonzero constant. Moreover, according to Lemma 5, $\alpha^2 = \lambda_q$ on V. On the other hand, when a point in U is approaching to V, the mean curvature function α is approaching to λ_q which is nonzero. Therefore, we also have $\alpha^2 = \lambda_q/14$ on $M\backslash$closure (U) by virtue of (4.15). This is impossible unless either U is empty or closure $(U) = M$. Consequently, by applying Theorem 6, we have obtain the following.

LEMMA 9.— *Let M be a null 2-type surface in E^3. Then either (a) M is an open portion of a circular cylinder or (b) α^2 is non-constant almost everywhere, i.e., closure $(U) = M$.*

By applying Lemma 9 we way obtain the following.

LEMMA 10.— *Let M be a null 2-type surface in E^3. If M is not an open portion of a circular cylinder, then the Gaussian curvature $G = -3\alpha^2 \leq 0$ and $G \neq 0$ almost everywhere.*

Proof.— Let M be a null 2-type surface in E^3 and M is not an open portion of a circular cylinder. Then, by Lemma 9, the mean curvature function is non-constant almost everywhere. Moreover, from Lemma 8, we know that the Gaussian curvature G of M is equal to $-3 < H, H >$ on U. So, by continuity, we obtain $G = -3\alpha^2 \leq 0$ on the whole surface M and $G \neq 0$ almost everywhere.

By applying Lemma 10 we have the following characterizations of circular cylinder.

THEOREM 11.— *Let M be a complete null 2-type surface in E^3. Then M is a circular cylinder if and only if one of the following conditions holds :*

(a) $\lambda_q > 0$

(b) $\alpha^2 \geq$ *const* > 0;

(c) α^2 *has a relative maximum;*

(d) G *has a relative minimum;*

Proof.— If M is a circular cylinder. Then it is easy to verify that M is a null 2-type surface with zero Gaussian curvature G, non-zero constant mean curvature, and positive λ_q (cf.[6]). Now, we prove the converse.

(a) Let M be a complete null 2-type surface in E^3 with $\lambda_q > 0$. If M is not a circular cylinder, then, according to Lemma 10, the Gaussian curvature G of M satisfies

$$G = -3\alpha^2 \leq 0 \text{ on } M. \tag{4.16}$$

Since α is non-constant almost everywhere, formula (4.15) implies

$$\lambda_q \leq 14\alpha^2. \tag{4.17}$$

From (4.16) and (4.17) we obtain $G \leq -3\lambda_q/14 < 0$. On the other hand, a well-know result of Efimov [7] (see also [8]) says that no surface can be immersed in a Euclidean 3-space so as to be complete in the induced metric with Gaussian curvature $G \leq$ const. < 0. And hence we conlude that this is impossible. Consequently, M must be a circular cylinder in E^3.

(b) Assume that M is a complete null 2-type surface with $\alpha^2 \geq$ const. > 0. Then, from Lemma 10, we see that either M is a circular cylinder or $G \leq$ const. < 0. Hence, by applying Efimov's theorem, we see that the later case is impossible.

(c) Let M be a complete null 2-type surface such that α^2 has a relative maximum at a point u in M. If $\alpha \neq 0$ *at* u, then formula (4.15) implies that $14\alpha^2 = \lambda_q > 0$. Thus M is a circular cylinder by part (a). The remaining part of this theorem follows from Lemma 10. (Q.E.D.)

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