JEAN-MARIE MORVAN
GEORGES ZAFINDRATAFA

Conformally flat submanifolds

Echanges Annales

Conformally flat submanifolds

JEAN-MARIE MORVAN[1] et GEORGES ZAFINDRATAFA[1]

RÉSUMÉ. — Nous étudions les propriétés locales et globales des sous-variétés conformément plates d'un espace euclidien. Nous étudions en particulier les relations entre la platitude conforme et la quasiombilicalité.

ABSTRACT. — We study local and global properties of conformally flat submanifolds in Euclidean space, and the relations between conform flatness and quasiumbilicity.

1. Introduction

A submanifold M^n of an Euclidean space E^{n+p} is conformally flat if, when it is endowed with the induced metric, each point belongs to a neighborhood which possesses coordinates $(x_1, \cdots, x_n)$ such that the metric tensor g satisfies

$$g = e^{\lambda}(dx_1 \otimes dx_1 + \cdots + dx_n \otimes dx_n),$$

where λ is a C^{∞} function. These submanifolds have been extensively studied these last 20 years, (cf Bibliography). The local structure of conformally flat hypersurfaces has been discovered by E.CARTAN [Ca1] in 1919. These hypersurfaces are generically foliated by codimension one spheres. In 1972, B.Y.CHEN and K.YANO gave a more precise description of these submanifolds [Ch-Ya1]. Finally in 1984, M.DO CARMO, M. DAJCZER and F.MERCURI, [Do-Da-Me] classified the compact conformally flat hypersurfaces. In particular, under some regularity conditions on the foliation, they proved that such an hypersurface is a topological product $S^{n-1} \times S^1$ of a sphere by a cercle. When the codimension is larger, B.Y.CHEN and K.YANO

---

[1] Faculté des Sciences d'Avignon, Département de Mathématiques 33, rue Louis Pasteur 84000 Avignon

defined in 1972 the notion of quasiumbilicity [Ch-Ya2]. This extrinsec property of the second fundamental form of an immersion is sufficient to obtain a conformally flat submanifold. In 1979, J.D.MOORE and the first author proved that this condition was also necessary for $n > 7$ and $p < 4$ [Mo-Mo]. Moreover, B.Y.CHEN and L.VERSTRALEN proved in 1978 [Ch-Ve] that a conformally flat submanifold M^n of E^{n+p}, with flat normal connexion, is quasiumbilical. Finally, in 1976, using Morse theory, J.D.MOORE gave topological restrictions of a compact conformally flat submanifold M^n of E^{n+p}, when $p < n - 3$. He proved that, in this case, M^n possesses a CW decomposition with no cells of dimension k, where $p < k < n - p$, [Moo3]. In view of these results, many problems remained open :

    i) What is the shape of the second fundamental form of a conformally flat submanifold of large codimension?

    ii) What is the extrinsec geometric structure of a conformally flat submanifold of low codimension? (Generalisation of [Do-Da-Me]).

    iii) What is the geometric meaning of quasiumbilicity?

    iv) What is the global structure of a "regular" compact conformally flat submanifold?

In this work, we give complete or partial answers to these questions :

In §3, we study the Gauss equation of flat or conformally flat submanifolds of the Euclidean space. This leads us to give an example of submanifold M^n of codimension 6 (in E^{n+6}) which is flat at some point but not quasiumbilical. However, we don't have the general solution of the Gauss equation in this case. This gives a partial answer to i).

In §4, 5, we study the local structure of a conformally flat submanifold of low codimension, and obtain a generalisation of [Do-Da-Me].

In §6, we study the notion of quasiumbilicity in terms of focal points. In particular, we obtain a new definition of quasiumbilicity which does not use a particular frame of the normal bundle.

In §7, we give the classification of compact conformally flat submanifolds of low codimension, with parallel second fundamental form.

In §8, we prove that compact regular conformally flat submanifolds are sphere-bundles.

Finally, in §9, we extend a result of [Do-Da-Me] which gives a necessary and sufficient condition for a manifold foliated by spheres to be conformally flat.

2. Notations

Let $i : M^n \hookrightarrow E^{n+p}$ be an isometric immersion of a manifold M^n of dimension n in the Euclidean space $E^{n=p}$. We shall denote by $<,>$ the scalar product in E^{n+p}, and use the same notation for $i^*(<,>)$, the metric on M^n. If $\widetilde{\nabla}$ is the canonical connexion on E^{n+p}, and ∇ the levi Civita on M^n, we put, for every vector fields X, Y belonging to TM^n, the tangent space of M^n,

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1}$$

where h is the second fundamental form associated to i. It is well known that h is a symetric bilinear tensor which takes its values in $T^\perp M^n$, the normal bundle of M^n. Let ξ a normal vector field and X a tangent vector field on M^n. We can decompose $\widetilde{\nabla}_X \xi$ in the following way : $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$, where $-A_\xi X$ and $\nabla_X^\perp \xi$ are the tangential and normal components of $\widetilde{\nabla}_X \xi$. A is called the Weingarten tensor associated to i, and is related to h by the formula :

$$< A_\xi X, Y > = < h(X,Y), \xi > \forall X, Y \in TM^n, \forall \xi \in T^\perp M^n \tag{2}$$

$\nabla^\perp$ is a metric connexion in $T^\perp M^n$, with respect to the induced metric. If $h \equiv 0$, M^n is said totally geodesic.

The mean curvature vector field of M^n is defined by $H = 1/n$ trace (h) (M^n is locally minimal for the volume if $H = 0$). Let $m \in M^n$, and $\xi_m \in T_m^\perp M^n$. ξ_m is called a quasiumbilical direction if there exists a one form ω, and two constant λ, μ such that

$$< h(X,Y), \xi >_m = \lambda \omega(X) \omega(Y) + \mu < X, Y >_m \tag{3}$$

This is equivalent to the fact that A_ξ admits an eigenvalue, with order $n-1$. If $\lambda = 0, \xi$ is called an umbilical direction. If $\mu = 0, \xi_m$ is called a cylindrical direction. M^n is totally umbilical or umbilical if there exists a normal vector field ξ such that $h(X,Y) = \mu < X, Y > \xi$, $\forall X, Y \in TM^n$. In this case, it is well known that M^n is an open set of a round sphere (of constant curvature). M^n is totally quasiumbilical (or quasiumbilical), if there exists at each point an orthonormal frame of quasiumbilical directions. M^n is totally cylindrical, or cylindrical if there exists at each point an orthonormal frame of cylindrical directions.

Let R be the curvature tensor of M^n. We denote by Ricc the Ricci tensor and r the scalar curvature of M^n. The Gauss-Codazzi-Ricci equations are given by :

$$< R(X,Y)Z,W >=< h(X,Z), h(Y,W) >$$
$$- < h(X,W), h(Y,Z) > \forall X, Y, Z, W \in TM^n \qquad \text{(Gauss) (4)}$$
$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) \; \forall X, Y, Z \in TM^n, \qquad \text{(Codazzi) (5)}$$

where $\overline{\nabla}$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla_X \big(h(Y,Z) \big) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$
$$R^\perp(X,Y)\xi, \mu >=< A_\xi(X), A_\mu(Y) >$$
$$- < A_\mu(X), A_\xi(Y) \qquad \text{(Ricci) (6)}$$
$$\forall X, Y \in TM^n, \; \forall \xi, \mu \in T^\perp M^n,$$

where $R^\perp$ the curvature of the normal bundle.

In particular, if we assume that M^n is conformally flat ($n \geq 4$), we know that the curvature tensor of M^n satisfies the following equation

$$< R(X,Y)Z, W > = \psi(X,W) < Y,Z > +\psi(Y,Z) < X,W > \atop - \psi(X,Z) < Y,W > -\psi(Y,W) < X,Z > \qquad (7)$$

where ψ is the (2,0) tensor defined by

$$\psi(X,Y) = \frac{1}{n-2} \left\{ Ricc(X,Y) - \frac{r < X,Y >}{2(n-1)} \right\}$$

$\forall X, Y, Z, W \in TM^n$. Remark that the sectional curvature of plane spanned by two orthonormal vectors X, Y is given by $K(X,Y) = \psi(X,X) + \psi(Y,Y)$.

In this case, the Gauss equation can be written

$$\psi(X,W) < Y,Z > +\psi(Y,Z) < X,W > \atop - \psi(X,Z) < Y,W > -\psi(Y,W) < X,Z > = \qquad (8)$$
$$< h(X,Z), h(Y,W) > - < h(X,W), h(Y,Z) > .$$

3. The Gauss equation of a flat submanifold

In this paragraph, we will recall well known properties of flat symetric bilinear forms. They where discoverd by E.CARTAN [Ca2]. (See also J.D.MOORE [Moo2] for an extended study of these forms).

THEOREM 3.1. — *Let $h : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^p$ be a flat symmetric bilinear form. Then*

   i) $\dim \ker h > p - 1$

   ii) If $\ker h = \{0\}$, then there exists a direction ξ in $\mathbf{E}^p$ such that $< h(\cdot, \cdot), \xi >$ is positive definite.

   iii) If ξ is a cylindrical direction, then the projection of h on $\xi^\perp$ is also a flat symmetric bilinear form.

THEOREM 3.2. — *Let $h : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^n$ be a flat symmetric bilinear form. If $\ker h = \{0\}$ then h is cylindrical.*

THEOREM 3.3. — *Let $h : \mathbf{E}^n \times \mathbf{E}^n \longrightarrow \mathbf{E}^p$, where $n \leq 3$, be a flat symmetric bilinear form. Then h is cylindrical.*

Different proofs of theorem 3.1 and 3.2 can be found in the litterature (cf. [Ca2] or [Moo2] for instance). The first author and J.D.MOORE used theorem 3.3 in [Mo-Mo]. A proof of this last theorem can be found in [Ca2]. However, we will give here an alternative proof of it.

Proof of theorem 3.3 . — Let $K = \ker h$, and factorize h through $E^n/\ker h$. Obviously, we obtain a new flat symmetric bilinear form, without kernel. Then, without restriction, we can assume that $\ker h = \{0\}$. Let us examine the three cases, $n = 1$, $n = 2$, and $n = 3$. We denote by $S_2(E^p)$ the space of symetric bilinear forms of E^p.

First case : $n = 1$. In this case, theorem 3.3 is obvious.

Second case : $n = 2$. Consider $(\xi_1, \cdots, \xi_p)$ an orthonormal frame in $\mathbf{E}^p$. Let $h^i = < h(,), \xi_i > \forall i, 1 \leq i \leq p$. Since $S_2(\mathbf{E}^2)$ has dimension 3, there are at most three linearly independant h^i, and then, the image of h lies in $\mathbf{E}^k$, $k \leq 3$. If $k = 1$, the proofs is trivial. If $k = 2$, the theorem is a direct application of theorem 3.2.

If $k = 3$, then we can assume that h^1, h^2, h^3 are linearly independant. Then it is a frame of $S_2(\mathbf{E}^2)$, and any 2-forms $\omega \otimes \omega$ ($\omega \in \mathbf{E}^{2*}$) is a linear combination of h^1, h^2, h^3. Then we can write $\omega \otimes \omega = \alpha_1 h^1 + \alpha_2 h^2 + \alpha_3 h^3$. This implies that $< h(\cdot, \cdot), \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 > = \omega \otimes \omega$. Then $\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ is a cylindrical direction. Now, let (η_1, η_2, η_3) be a new orthonormal frame of $\mathbf{E}^3$ such that $\eta_3 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$. By theorem 3.1, (iii), $\overline{h} = pr_{\eta_3}\perp h$ is flat and the dimension of Im $\overline{h} = 2$. Then we can apply theorem 3.2, to conclude that $\overline{h}$ is cylindrical, which implies that h is cylindrical.

Third case : $n = 3$. We can assume without loss of generality that $\ker h = \{0\}$, (otherwise we are in the case 1 or 2). Since $\mathcal{S}_2(\mathbf{E}^3)$ has dimension 6, $\dim[\text{Im } h] \leq 6$. If $\dim[\text{Im } h] \leq 3$, we can conclude by applying theorem 3.1 (i), 3.2, and the case 1 and 2. We will assume that $\dim[\text{Im } h] = 4, 5$ or 6.

(a) Suppose that $\dim[\text{Im } h] = 4$. Let $(\xi_1, \cdots, \xi_4)$ be an orthonormal frame of $[\text{Im } h]$. We can write

$$h = \sum_{i=1}^{4} h^i \otimes \xi_i.$$

The dimension of the vector subspace V of $\mathcal{S}_2(\mathbf{E}^3)$, spanned by $\{h^1, \cdots, h^4\}$ is 4. On the other hand, by theorem 3.1 (ii), there exists a positive definite form k which belongs to V. Let (e_1, e_2, e_3) be a frame of $\mathbf{E}^3$ which is orthonormal with respect to k. Let $\omega_1, \omega_2, \omega_3$ be the dual frame. We can remark that

$$k = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3$$

Consider the (Veronese) surface of $\mathcal{S}_2(\mathbf{E}^3)$, defined by the immersion j of the unit sphere S^2 in $\mathbf{E}^{3*}$

$$j(\omega) = \omega \otimes \omega$$

$(\widetilde{j} = j - 1/3\ Id$ is the standard immersion of the Veronese surface V in the Euclidean 5-space $\mathbf{E}^5$, as a minimal submanifold of the hypersphere of $\mathbf{E}^5$). As a consequence of the fact that two quadric of $\mathbf{E}^3$ with a null trace, have a non trivial intersection, it is easy to prove that every 3-vector subspace P of $\mathbf{E}^5$ satisfies

$$P \cap V \neq \emptyset.$$

This is equivalent to the fact that every 4-vector subspace of $\mathcal{S}_2(\mathbf{E}^3)$, containing Id, has a non trivial intersection with $j(S^2)$.

This means that there exists a form of rank one, $\omega \otimes \omega$ in V. We can write

$$\omega \otimes \omega = \sum_{i=1}^{4} \beta_i h^i$$

which implies that

$$\eta = \sum_{i=1}^{4} \beta_i \xi_i$$

is a cylindrical direction.

By theorem 3-1 (iii), $pr_{\eta^\perp} h$ is flat. Since

$$\dim\left(\eta^\perp \cap [\text{Im } h]\right) = 3$$

we conclude, using theorem 3.2, that $pr_{\eta^{\perp}} h$ and then h is cylindrical.

(b) Suppose that dim[Im h] = 5. We can apply exactly the same proof as in (a). This shows that there exists in [Im h] a section ν which is cylindrical. Then $pr_{\nu^{\perp}} h$ is also a flat, with dim[Im $pr_{\nu^{\perp}} h$] = 4. We apply (a) to conclude that $pr_{\nu^{\perp}} h$ is cylindrical, which implies that h is cylindrical.

(c) Suppose that dim[Im h] = 6. In this case, we can write $h = \Sigma_1^6 h^i \otimes \xi_i$, where $\{\xi_1, \cdots, \xi_6\}$ is an orthonormal frame of [Im h]. Then $\{h^1, \cdots, h^6\}$ is a frame of $\mathcal{S}_2(\mathbf{E}^3)$. Consequently, any form of rank 1, $\omega \otimes \omega$, can be written as a linear combinaison of $h^1, \cdots, h^6$. Let $\omega \otimes \omega = \Sigma_{i=1}^6 \gamma^i h^i$. Then $< h(\cdot, \cdot), \Sigma_{i=1}^6 \gamma^i \xi_i >$ is cylindrical, and $pr_{\xi^{\perp}} h$ is cylindrical. Since dim(Im $h \cap \xi^{\perp}$) is five, we conclude by using (b).

Remark 3.4. — In (b), we prove that any 5-vector subspace V of $\mathcal{S}_2(\mathbf{E}^3)$ which contains a positive definite form k contains also a form of rank one. Another simple proof of this can be given as follows. Let (e_1, e_2, e_3) be a frame of $\mathbf{E}^3$, orthonormal with respect to k. In this frame, the matrix of k is the Identity. Let the scalar product of $\mathcal{S}_2(\mathbf{E}^3)$ defined by :

$$< A, B >= \text{Trace } AB,$$

where the trace is taken with respect to k. Let $\psi = V^{\perp}$. We have $< \psi, k >=$ Trace $\psi = 0$ which implies that ψ is not positive definite. Then there exists a non null vector X, isotropic with respect to ψ. Let X^* be the one form dual to X with respect to k. We have $\psi(X, X) =< \psi, X^* \otimes X^* >= 0$, which implies that $X^* \otimes X^*$ lies in V.

If $n > 3$, E.CARTAN announces, without proof, in [Ca1] that theorem 3.3 is wrong. For $n = 4$, dim $\mathcal{S}_2(\mathbf{E}^4) = 10$. In this case, using the same technics than in the proof of theorem 3.3, it is easy to see that if dim[Im h] $\in$ $\{7, 8, 9, 10\}$, there exists at least in [Im h] a cylindrical direction. On the other hand, if dim[Im h] $\in \{0, 1, 2, 3, 4\}$, h is cylindrical by theorem 3.2 and 3.3. Then, the unknown cases are the cases where dim[Im h] = 5 or 6. We don't have a general method to study these two cases. However, we give here an example of a flat symmetric bilinear form from $\mathbf{E}^4 \times \mathbf{E}^4$ into $\mathbf{E}^6$ *which does not have any cylindrical direction.* Moreover, this form *is not quasiumbilical.* In fact, consider in $\mathcal{S}_2(\mathbf{E}^4)$ the space spanned by the six following symmetric matrices, which can be considered as bilinear symmetric forms, written in the canonical orthonormal frame of $\mathbf{E}^4$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Suppose that M_i representents h_i $(1 \leq i \leq 6)$, and consider

$$h : E^4 \times E^4 \longrightarrow E^6$$

defined by $\widetilde{h} = \Sigma_1^6 h^i \otimes \xi_i \circ \{\xi_1, \cdots, \xi_6\}$ is an orthonormal frame of $\mathbf{E}^6$. A long computation shows that $\widetilde{h}$ is flat, and that there does not exist any cylindrical direction with respect to $\widetilde{h}$ in $\mathbf{E}^6$. Moreover, $\widetilde{h}$ is not quasiumbilical (cf. [Za] for details).

Application 3.5. — As an obvious application of this example, it is possible to construct an immersion of an open set of $\mathbf{E}^4$ in $\mathbf{E}^{10}$, such that, for the induced metric, $0 \in \mathbf{E}^4$ is a flat point an such that the second fundamental form at 0 is not quasiumbilical (and without any cylindrical direction). For instance, let $f = (f_1, \cdots, f_{10}) : \mathbf{E}^4 \to \mathbf{E}^{10}$ be given by :

$$f_1(x, y, z, t) = \frac{1}{2}\,(x^2 + y^2 + z^2 + t^2) + zt$$
$$f_2(x, y, z, t) = xy$$
$$f_3(x, y, z, t) = yz$$
$$f_4(x, y, z, t) = \frac{1}{2}\,(y^2 - t^2) - zt$$
$$f_5(x, y, z, t) = x(z + t)$$
$$f_6(x, y, z, t) = \frac{1}{2}\,(z^2 + t^2)$$

$$f_7(x, y, z, t) = -x + y + z + t$$
$$f_8(x, y, z, t) = x - y + z + t$$
$$f_9(x, y, z, t) = x + y - z + t$$
$$f_{10}(x, y, z, t) = x + y + z - t$$

It is easy to show that f is an immersion. Its second fundamental form h at 0 satisfies $(h_{ij})_o = pr_{T_o^\perp(f(\mathbf{E}^4))} \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ which is equal to h. Then, by the Gauss equation, 0 is a flat point. On the other hand h_o does not have any cylindrical direction and is not quasi-umbilical.

4. The Gauss equation of a conformally flat submanifold of low codimension

Let us consider $i : M^n \hookrightarrow E^{n+p}$ an isometric immersion of a conformally flat manifold M^n of codimension p. Using (7), J.D.MOORE proved the following theorem, which gives a necessary condition for a submanifold of low codimension to be conformally flat.

THEOREM [Moo3] 4.1. — *Let M^n be a conformally flat submanifold of $\mathbf{E}^{n+p}$, with $1 \leq p \leq n - 3$. Then, for every $m \in M^n$, there exists an orthonormal frame in which the matrix of A_ξ has the following expression, for every $\xi \in T^\perp M^n$*

$$\begin{pmatrix} M_\xi & & & \\ & \lambda_\xi & & 0 \\ & & \ddots & \\ 0 & & & \lambda_\xi \end{pmatrix} \tag{8}$$

where M_ξ is a $k \times k$ matrix with $k \leq p$, and $\lambda_\xi \in \mathbf{R}$.

This condition is not sufficient. The simplest example is the standard immersion of $S^n \times S^n$ in $\mathbf{E}^{n+1} \times \mathbf{E}^{n+1}$. In this case, for every $\xi \in T^\perp(S^n \times S^n)$

$$A_\xi = \begin{pmatrix} \alpha_\xi & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \alpha_\xi & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \beta_\xi & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \beta_\xi \end{pmatrix}$$

where α_ξ, $\beta_\xi \in \mathbf{R}$. However $S^n \times S^n$ is not conformally flat if $n > 1$. On the other hand, if the immersion is quasiumbilical, B.Y.CHEN and K.YANO [Ch-Ya.2] proved that M^n is conformally flat. When the codimension is lower that 4, J.D.MOORE and the first author proved in [Mo-Mo], as a consequence of theorem 3.1 and 3.3, that this condition is also necessary.

THEOREM 4.2. — *Let $i : M^n \hookrightarrow \mathbf{E}^{n+p}$ be an isometric immersion of M^n into $\mathbf{E}^{n+p}$, with $n \geq 7$, $p \leq 4$. Then the immersion is quasiumbilical if and only if M^n is conformally flat.*

On the other hand, B.Y.CHEN and L.VERSTRAELEN obtained a similar result when the immersion has flat normal connexion [Ch-Ve].

THEOREM 4.3. — *Let $i : M^n \hookrightarrow \mathbf{E}^{n+p}$ be an isometric immersion of a conformally flat manifold, with $1 \leq p \leq n - 3$. If the normal bundle of M^n is flat, then the immersion is quasiumbilical.*

The example that we gave in § 3 shows that it is possible in large codimension to construct submanifolds which are conformally flat at some point but not quasiumbilical at this point. However, this notion is natural, although its definition depends on the choice of a particular normal frame.

5. Local study of conformally flat submanifold of low codimension

In this paragraph, we shall study the local shape of a conformally flat submanifold of low codimension.

(a) $\mathcal{FS}_n^p$: *The class of submanifolds generically foliated by spheres*

Let M^n be a submanifold of $\mathbf{E}^{n+p}$, $(p \leq n)$. We shall say that M^n is generically foliated by spheres if there exists a dense open set U of M^n such that $U = \cup_{r=o}^{p} U_r$, where, for every r, U_r is an open set of M^n, which is foliated by$(n - p + r)$ umbilical submanifolds of $\mathbf{E}^{n+p}$. We shall denote by $\mathcal{FS}_n^p$ this class of submanifolds. The points of U are called generical points.

Remark 5.1. — This definition implies in particular that the leaves of U_r are open set of standard spheres S^{n-p+r}. On the other hand, it is clear that the second fundamental form of M^n satisfies, in each normal direction ξ_m, $m \in M^n$, with respect to a suitable frame of M^n,

$$< h(\cdot, \cdot), \xi_m > = \begin{pmatrix} \mathcal{M}_{\xi_m} & \mathcal{M}'_{\xi_m} \\ {}^t\mathcal{M}'_{\xi_m} & < a_m, \xi_m > Id \end{pmatrix} \begin{array}{l} \} p - r \\ \} n - p + r \end{array} \qquad (9)$$

where $\mathcal{M}_\epsilon$ is a squared matrix of $(p-r)$ order, $\mathcal{M}'_\xi$ is a $(p-r) \times (n-p-r)$-matrix. Id is the identity matrix, and a_m is the projection on $T_m^\perp M$ of the mean curvature vector of the leaf. The vector field a will be called the canonical vector field of M^n. (See [Mor], [Za] for interesting geometrical and topological properties of a).

(b) $\mathcal{SFS}_n^p$: *The class of submanifolds strongly generically foliated by spheres.*

Let M^n be a submanifold of $\mathbf{E}^{n+p}$, $(p \le n)$. We shall say that M^n is strongly generically foliated by spheres if the two following conditions are satisfied :

i) $M^n \in \mathcal{FS}_n^p$

ii) The restriction of $T^\perp M$ to each leaf S of U is parallel in $T^\perp S$.

We shall denote by $\mathcal{SFS}_n^p$ this class of submanifolds. We have obviously $\mathcal{SFS}_n^p \subset \mathcal{FS}_n^p$. Moreover, it is clear that a submanifold M^n satisfies in each direction ξ_m, $m \in M^n$, with respect to a suitable frame of M^n,

$$
< h(\cdot,\cdot), \xi_m >= \begin{pmatrix} \mathcal{M}_{\xi_m} & 0 \\ 0 & < a,\xi >_m I_{n-p+r} \end{pmatrix} \begin{matrix} \} \, p-r \\ \} \, n-p+r \end{matrix} \qquad (10)
$$

(using the notation of (a)).

(c) *Local structure of conformally flat submanifolds*

Let $\mathcal{CF}_n^p$ be the class of conformally flat submanifold M^n of $\mathbf{E}^{n+p}$. We shall prove the following.

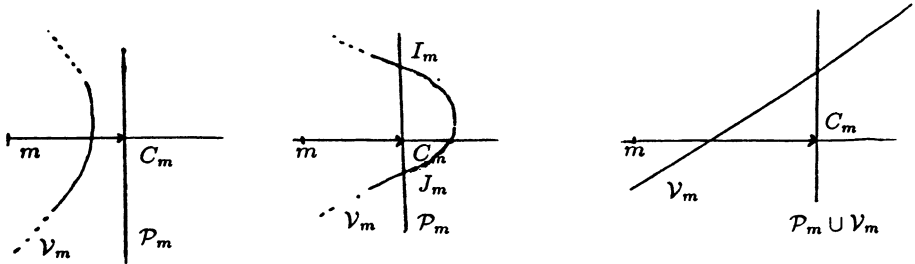THEOREM 5.2. — *If $1 \le p \le n-3$, $\mathcal{CF}_n^p \subset \mathcal{SFS}_p^n$.*

Proof of theorem 5.2. — Theorem 4.1 shows that if $1 \le p \le n-3$, the second fundamental form h of a conformally flat submanifold M^n of $\mathbf{E}^{n+p}$ satisfies, at each point $m \in M^n$, the following property : There exists a subspace $\mathbf{E}_m$ of TM^n of dimension $\ge n-p$ such that, $\forall X \in \mathbf{E}_m$, $h(X,Y) =< X,Y > a_m$ for every $Y \in TM^n$. Using a result of H.RECKZIEGEL [Re 1], we can conclude that, when m varies on M^n, $\mathbf{E}^m$ is almost every where a differentiable involutive distribution. On each open set where the dimension of E is constant, E defines a foliation, the leaves of which are umbilical in $\mathbf{E}^{n+p}$. This implies that M^n belongs to $\mathcal{FS}_n^p$. Now, it is clear, by looking at the shape of the second fundamental form given by theorem 4.1, that M^n belongs to $\mathcal{SFS}_n^p$.

6. An alternative caracterisation
of quasiumbilicity

In this paragraph, we shall give a new caracterisation of quasiumbilicity, in terms of focal set, which doesn't use particular forms on the tangent bundle of the submanifold. First of all, consider a submanifold M^n of E^{n+p}, which belongs to the class $\mathcal{SFS}_n^p$. Using the notations of §5, the focal set F_m at the point $m \in M^n$ is defined by :

$F_m = \{m + t\xi_m,$ where ξ_m is an unit normal vector such that $\det(A_{t\xi_m} - I) = \det(\mathcal{M}_{t\xi_m} - I).$ $(a_m <, t\xi_m > -1)^{n-p} = 0\}$. We deduce that $F_m = \mathcal{P}_m \cup \mathcal{V}_m$, where $\mathcal{P}_m$ is the affine subspace defined by $\mathcal{P}_m = \{m + \xi/\xi \in T_m^\perp M$ and $< a_m, \xi >= 1\}$, and $\mathcal{V}_m$ is the algebric variety defined by $\mathcal{V}_m = \{m + \xi/\xi \in T_m^\perp M\}$ and $\det(\mathcal{M}_\xi - I) = 0$. In particular the direction ξ is quasiumbilical if $\mathcal{P}_m \cap \mathcal{V}_m$ has an intersection point of multiplicity $(n-1)$ in the direction ξ. (It is easy to see that the distance from this intersection point to m is $\dfrac{1}{< C_m, \xi/\|\xi\| >}$, where $C_m \in T_m^\perp M^n$ is the point which is the projection of the center of the sphere which contains the leaf through m, of the canonical foliation; C_m is a focal point which lies in the line $m + a_m$).

Suppose, for simplicity, that the codimension is 2, and consider the generic case where $a_m \neq 0$. We have, generically, three possibilities :

Case (I) : $\mathcal{V}_m \cap \mathcal{P}_m = \emptyset$

Case (II) : $\mathcal{V}_m \cap \mathcal{P}_m$ has two points I_m and J_m.

Case (III) : $\mathcal{V}_m$ is reduced to two lines, one of which is the line $\mathcal{P}_m$. $\mathcal{V}_m \cap \mathcal{P}_m = \mathcal{P}_m$.

It is clear that I_m and J_m determines two quasiumbilical directions. More generally, any point of $\mathcal{V}_m \cap \mathcal{P}_m$ determines a quasiumbilical direction. Then,

the case (I) corresponds to the fact that no direction is quasiumbilical. The case (II) corresponds to the fact that exactly two directions are quasiumbilical. The case (III) corresponds to the fact that every direction is quasiumbilical. Consequently, using theorem, we obtain that :

   i) If M^n is locally of type (I), then M^n is not conformally flat.

   ii) If M^n is locally of type (III), then M^n is conformally flat.

   iii) If M^n is locally of type (II), then M^n is conformally flat if and only if the angle $I_m \ \widehat{m} \ J_m = \frac{\Gamma}{2}$.

It is easy to find examples of these three types of immersion :

The standard immersion of $S^2(1) \times S^n(2)$ in $\mathbf{E}^{n+4}$, $(n > 2)$ is of type (I). Consider a conformally flat hypersurface M^n of a canal hypersurface $\widetilde{M}^{n+1}$ of $\mathbf{E}^{n+2}$. Then M^n is a codimension 2 conformally flat submanifold of type (II). Consider the standard immersion i of the sphere S^n into $\mathbf{E}^{n+1}$ and a cylindrical immersion of $\mathbf{E}^{n+1}$ into $\mathbf{E}^{n+2}$. Then joi is of type (III). For a detailed study of this view point, see [Za].

7. Conformally flat submanifolds with parallel second fundamental form

In the previous paragraph, we gave a local description of the class of conformally flat submanifold of Euclidean space, by using only the Gauss equation. In this paragraph, we shall assume that the submanifold has parallel second fundamental form. Using Codazzi equation, we shall deduce important restrictions to the submanifold. Precisely, we shall prove the following.

THEOREM 7.1. — *Let $i : M^n \hookrightarrow \mathbf{E}^{n+p}$ be an isometric immersion of a conformally flat manifold M^n into $\mathbf{E}^{n+p}$, $1 \leq p \leq n - 3$, with parallel fundamental form. Then*

   i) *Either M^n is totally geodesic (and then, is an open set of a n-plane.*

   ii) *Or, M^n is properly umbilical (and then, is an open set of a round sphere S^n)*

   iii) *Or, M^n is locally a Rieamannian product $\Sigma^{n-1} \times \mathbf{R}$, where Σ^{n-1} is an open set of a $(n - 1)$-sphere and i locally a product $i_1 \times i_2$: $\Sigma^{n-1} \times \mathbf{R} \to \mathbf{E}^n \times \mathbf{E}^p$, where i_1 is an umbilical immersion and $i_2(\mathbf{R})$ is a curve in $\mathbf{E}^p$.*

   iv) *Or, M^n is locally a Riemannian product $\Sigma^{n-k} \times H^k$ $(2 \leq k \leq p)$, where Σ^{n-k} is an open set of a $(n - k)$-sphere of constant curvature*

– 343 –

$\rho > 0$, and H^k is an open set of an hyperbolic space of constant curvature $(-\rho)$.

Moreover, i is locally a product $i_1 \times i_2 : \Sigma^{n-k} \times H^k \to \mathbf{E}^{n-k+1} \times \mathbf{E}^{p+k-1}$, where i_1 is an umbilical immersion of Σ^{n-k} into $\mathbf{E}^{n-k+1}$, and i_2 is an immersion of H^k into $\mathbf{E}^{p+k-1}$.

As a consequence, we obtain the following :

COROLLARY 7.2. — Let $i : M^n \to \mathbf{E}^{n+p}$ be an isometric immersion of a compact simply connected conformally flat manifold M^n into $\mathbf{E}^{n+p}$, $(1 \le p \le n-3)$, with parallel second fundamental form. Then M^n is a standard sphere (of constant curvature).

Proof of the theorem. — We need the following lemmas.

LEMMA 7.3. — Under the assumptions of the theorem, the integral distribution which defines the foliation by spheres, is parallel. In particular, M^n is locally a product of $\Sigma^{n-p+k} \times \overline{M}^{p-k}$, where Σ^{n-p+k} is an open set of a $(n-p+k)$-sphere and $\overline{M}^{p-k}$ is a $(p-k)$ dimensional manifold.

Proof of Lemma 7.3. — Since the second fundamental form h is parallel, the dimension of the integrable distribution E defined by $E = \{X \in TM^n \,/\, (h(X,Y) = <X,Y> a\}$ has a constant dimension. This dimension is $\ge n-p$, by theorem 4.1. Then, this distribution is everywhere differentiable.

On the other hand, the Weingarten tensor A satisfies $(\widetilde{\nabla}_X A)(Y,\xi) = 0$ $\forall X,Y \in TM^n$, $\forall \xi \in T^\perp M^n$.

This implies : $\nabla_X\big(A_\xi(Y)\big) - A_\xi(\nabla_X Y) - A_{\nabla^\perp_X \xi}(Y) = \big(X(\lambda(\xi)) - \lambda(\nabla^\perp_X \xi)\big) - \big(A_\xi(\nabla_X Y) - \lambda(\xi)\nabla_X Y\big) = 0$, where $\lambda(\xi) = <a,\xi>$. From this equation, we deduce that $\forall X \in TM^n$, $\forall Y \in E$, $A_\xi(\nabla_X Y) = <a,\xi> \nabla_X Y$¿. Consequently, E is parallel. From De-Rham decomposition theorem [Ko-No], we obtain that M^n is locally the product of an integral submanifold of E and an integral submanifold of $E^\perp$. From §4 we know that an integral submanifold of E is an open set of a sphere.

The following lemma is an easy consequence of (7) (see also [La]).

LEMMA 7.4. — Let M^n be a Riemannian conformally flat manifold, endowed with a non trivial parallel distribution (i.e. $\ne \{0\}$ and TM^n). Then, for every point $m \in M^n$, there exists a neighborhood U of M^n which has one of the following expression :

i) U is an open set of $\mathbf{E}^n$

ii) $U = \Gamma \times \Sigma^{n-1}$ *is a product of a curve of* M^n *by an open set of a* $(n-1)$*-sphere.*

iii) $U = \Gamma \times H^{n-1}$ *is a product of a curve of* M^n *by an open set of a* $(n-1)$*-hyperbolic space.*

iv) $U = \Sigma^{n-k} \times H^k(2 \leq k \leq n-2)$, *is a product of an open set of a* $(n-k)$*-sphere of constant curvature* $\rho > 0$ *by an open set of a* k*-hyperbolic space of constant curvature* $-\rho$.

Using lemma 7.3 and lemma 7.4, we conclude that M^n has locally one of the previous expression i), ii), iii), iv), or is an open set of a sphere, in the case where the dimension of the integral distribution is n. We shall use, now, the fact that M^n is a submanifold of $\mathbf{E}^{n+p}$. Let us recall the following result, due to J.D.MOORE [Moo 1].

LEMMA 7.5. — *Let* M_1 *and* M_2 *be two Riemannian manifolds and* $f :$ $M_1 \times M_2 \to \mathbf{E}^N$ *be an isometric immersion of the Riemannian product* $M_1 \times M_2$. *If the second fundamental form* h *of* f *satisfies* $h(X_1, X_2) =$ $0 \, \forall X_1 \in TM_1, \, \forall X_2 \in TM_2$, *then there exists a decomposition of* $\mathbf{E}^N$, $\mathbf{E}^N =$ $\mathbf{E}^{N_1} \times \mathbf{E}^{N_2}$, $N_1 + N_2 = N$ *and two isometric immersions* $f_1 : M^1 \to E^{N_1}$, $f_2 : M^2 \to E^{N_2}$, *such that* $f = f_1 \times f_2$.

From theorem 4.1 and lemma 7.4, the condition of lemma 7.5 are satisfied. Since, by theorem 5.2 one of the two components of U is an open set of a sphere, then U is an open set of S^n or satisfies i), ii), iii), iv) of lemma 7.4. Case iv) can occurs. In fact there exists *local* isometric immersions of hyperbolic space H^k in $\mathbf{E}^{2k-1}$ (cf. [K.N.] for instance). Then $S^{n-p} \times H^p$ can be locally isometrically immersed in $\mathbf{E}^{n-p+1} \times \mathbf{E}^{2p-1} = \mathbf{E}^{n+p}$, and the theorem is proved.

Proof of the corollary. — We shall apply the global version of De Rham theorem. Since M^n is compact, the leaves of the foliation are complete [Rec 2], then they are spheres. Since M^n is simply connected, we obtain from lemma 7.4 that the only possibility for M^n is to be a sphere S^n.

8. Global structure of regular conformally flat submanifolds

We have seen in §5 that a conformally flat submanifold M^n of $\mathbf{E}^{n+p}$ ($p \leq n-3$) is "locally and generically" foliated by open sets of umbilical spheres of dimension $k \geq n-p$. Such a submanifold will be called "k-regular"

if this foliation is regular (and has dimension k everywhere). We shall prove the following theorem, which is a direct generalisation of [Do.Da.Me].

THEOREM 8.1.— *Let M^n be an oriented compact conformally flat submanifold of $\mathbf{E}^{n+p}$, ($p \leq n-3$). If M^n is k-regular, then M^n is a sphere bundle over an $(n-k)$-compact manifold.*

Proof of theorem 8.1. — Since M^n is compact, the leaves of the foliation are complete spheres [Rec 2]. Since $p \leq n-3$, and $k \geq n-p$, we have $k \geq 3$. Then the spheres are simply connected. Consequently, M^n is foliated by simply connected compact submanifolds. In particular, this foliation has no holonomy. It follows directly from [Rec] or [Ep], that M^n is a fiber-bundle over the space of leaves. The fibers of this foliation are spheres.

Remark. — i) It follows immediatly from the exact sequence of homotopy of a fiber bundle that $\Pi_1(M^n) = \Pi_1(B^{n-k})$, where B^{n-k} is the base of this fibration.

ii) Let $S^{n-2}(1)$ be the k-sphere of curvature 1, and $\overline{H}^2(-1)$ be a compact surface of constant curvature -1. Consider the standard immersion of $S^{n-2}(1)$ in $\mathbf{E}^{n-1}$ and any isometric immersion of $\overline{H}^2(-1)$ into $\mathbf{E}^{17}$ (by Nash theorem). The product $S^{n-2}(1) \times \overline{H}^2(-1)$ is then isometrically immersed in $\mathbf{E}^{n+16}$. If $n \geq 19$, we obtain an example of the situation studied in theorem 8.1. However, the codimension is not the best. In particular, we don't know examples of 2-regular compact conformally flat submanifolds M^n of $\mathbf{E}^{n+2}$.

9. Remark : Conformally flat manifolds foliated by spheres

In the previous parts of this work, we have seen that a conformally flat submanifold M^n of an Euclidean space is, at least locally, foliated by open sets of spheres of constant curvature. It is easy to see that the leaves are umbilical in M^n. In this paragraph, we shall deal with the converse problem. Precisely we shall give a necessary and sufficient condition on a manifold, which is foliated by umbilical spheres, to be conformally flat. This is a direct generalisation of [Do.Da.Me]. We obtain the following.

THEOREM .— *Let M^n be a Riemannian manifold of dimension $n \geq 4$, foliated by k-umbilical submanifolds of positive constant curvature, $3 \leq k \leq (n-1)$. Then M^n is conformally flat if and only if the sectionnal curvatures of M satisfy :*

i) $K(X,\eta) = K(Y,\eta)$ *for every* X, Y *tangent to the foliation, and for every* η *normal to the foliation.*

ii) $K(X,Y) + K(\eta,\xi) = K(X,\eta) + K(Y,\xi)$
for every orthonormal vectors (X, Y, ξ, η) *such that* (X, Y) *are tangent to the foliation, and* (ξ, η) *are normal to the foliation.*

Proof of the theorem. — Suppose that M^n is conformally flat. Let m be a point of M^n and denote by S the leaf through the point m. Let $\{e_1, \cdots, e_k, \ e_{k+1}, \cdots, e_n\}$ be an orthonormal frame at m, such that $\{e_1, \cdots, e_k\}$ is tangent to the leaf S, and such that $\{e_{k+1}, \cdots, e_n\}$ is normal to F. Let λ^2 be the curvature of S, and H be the mean curvature vector of S. Since S is umbilical, we deduce from the Gauss equation (4) that :

$$K_{ij} = \lambda^2 - \|H\|^2 \qquad \forall i,j \in \{1, \cdots, k\}, \ i \neq j.$$

This implies that $K_{ij} = K_{i'j'} \ \forall i, j, i', j' \in \{1, \cdots, k\}, \ i \neq j, \ i' \neq j'$. Let $\rho = \lambda^2 - \|H\|^2$. Using the notations of §2, we obtain that $K_{ij} = \psi(e_i, e_i) + \psi(e_j, e_j) = \rho \ \forall i, j \in \{1, \cdots, k\}$. Since $k \geq 3$, this implies that $\psi(e_i, e_i) = \psi(e_j, e_j) \ \forall i, j \in \{1, \cdots, k\}$ and $\psi(e_i, e_i) = 1/2 \ \rho \ \forall i \in \{1, \cdots, k\}$. On the other hand, $K_{i\alpha} = \psi(e_i, e_i) + \psi(e_\alpha, e_\alpha) = 1/2 \ \rho + \psi(e_\alpha, e_\alpha)$ $\forall i \in \{1, \cdots, k\}, \forall \alpha \in \{k+1, \cdots, n\}$. Then $K_{i\alpha}$ does not depend on i. Then (i) is proved. We also have $K_{\alpha\beta} + K_{ij} = \psi(e_\alpha, e_\alpha) + \psi(e_\beta, e_\beta) + \psi(e_i, e_i) + \psi(e_j, e_j) = K_{i\alpha} + K_{j\beta}$. (ii) is proved.

Conversely, assume that $K_{i\alpha} = K_{j\alpha}$ and $K_{ij} + K_{\alpha\beta} = K_{i\alpha} + K_{j\beta}$ $\forall i, j \in \{1, \cdots, k\}, \forall \alpha, \beta \in \{k+1, \cdots, n\}$. Then, by an easy computation we obtain $\psi(e_i, e_i) = 1/2 \ \rho \ \forall i \leq k$, $\psi(e_\alpha, e_\alpha) = -1/2 \ \rho + K_{i\alpha} \ \forall \alpha \geq k+1$. This implies :
$$K_{ij} = \psi(e_i, e_i) + \psi(e_j, e_j).$$
$$K_{i\alpha} = \psi(e_i, e_i) + \psi(e_\alpha, e_\alpha).$$
$$K_{\alpha\beta} = \psi(e_\alpha, e_\alpha) + \psi(e_\beta, e_\beta),$$
$$\forall i, j \in \{1, \cdots, k\}, \ \forall \alpha, \beta \in \{k+1, \cdots, n\} \ \alpha \neq \beta, \ i \neq j.$$

This implies immediately that M^n is conformally flat.

References

[Ca 1] CARTAN (E.). — La déformation des hypersurfaces dans l'espace conforme réel à $n \geq 5$ dimensions, *Bull. Soc. Math. France*, t. 45, 1917, p. 65-99.

[Ca 2] CARTAN (E.). — Sur les variétés de courbure constante d'un espace euclidien ou non-euclidien, *Bull. Soc. Math. France*, t. 47, 1919, p. 125-160.

[Ch] CHEN (B.Y.). — *Geometry of submanifolds.* — New-York, M. Dekker 1973.

[Ch-Ve] CHEN (B.Y.) and VERSTRAELEN (L.). — A characterisation of totally quasiumbilical submanifolds and its applications, *Bul. Un. Mat. Ital.*, t. (5),14.A, 1977, p. 634.

[Ch-Ya 1] CHEN (B.Y.) and YANO K. — Special quasi-umbilical hypersurfaces and locus of sphere, *Atti Ac. Naz. Lincei, serie VIII, Vol. L III*, t. **3.4**, 1972.

[Ch-Ya 2] CHEN (B.Y.) and YANO (K.). — Sous-variétés localement conformes à un espace euclidien, *C.R. Acad. Sci. Paris, série A*, t. **273**, 1972.

[Do-Da] DO CARMO (M.) and DAJCZER (M.). — *General conformally flat hypersurfaces of R^{n+1}.* — Preprint.

[Do-Da-Me] DO CARMO and DAJCZER (M.) and MERCURY (F.). — Compact conformally flat hypersurfaces, *Ann. Acad. Brasil. Ciênc.*, t. **55**, 1983 $n°2$, p. 155-157.

[Ep] EPSTEIN (D.B.A.). — Foliations with all leaves compact, *Ann. Inst. Fourier*, t. **26, 1**, 1976, p. 265-282.

[Ko-No] KOBAYASHI (S.) and NOMIZU (K.). — *Foundations of differential geometry I,II,* Wiley Interscience, 1963 and 1969, New-York.

[La] LAFONTAINE (J.). — Remarques sur les variétés conformément plates, *Math. Ann.*, t. **259**, 1982, p. 313-319.

[Moo 1] MOORE (J.D.). — Isometric immersions of riemannian products, *J. Diff. Geom.*, t. **5**, 1971, p. 159-168.

[Moo 2] MOORE (J.D.). — Isometric immersions of space forms in space forms, *Pacific Journal of Math.*, t. **40, 1**, 1972.

[Moo 3] MOORE (J.D.). — Conformally flat submanifolds of euclidian space, *Math. Ann.*, t. **225**, 1977 $J85'97$.

[Mo-Mo] MOORE (J.D.) and MORVAN (J.M.). — Sous-variétés conformément plates de codimension quatre, *C.R. Acad. Sci. Paris*, t. **287, série A**, 1978, p. 655-657.

[Mor 2] MORVAN (J.M.). — Some topological properties of totally quasi-umbilical submanifolds, *Simon Stevin*, t. **54**, 1980 $n°1$, p. 11-25.

[Ree] REEB (G.). — Sur certaines propriétés topologiques des variétés feuilletées, *Act. Sci. Ind.*, t. **1183**, 1952, p. 91-154, 157-158.

[Rec 1] RECKZIEGEL (H.). — Krümmungsflächen von isometrischen Immersionnen in Räume konstanter Krümmung, *Math. Ann.*, t. **223**, 1976, p. 169-181.

[Rec 2] RECKZIEGEL (H.). — Completeness of curvature surfaces of an isometric immersion, *J. Diff. Geom.*, t. **14**, 1979, p. 7-20.

[Za] ZAFINDRATAFA (G.). — *Sous-variétés conformément plates d'un espace Euclidien* Thèse de 3ème cycle. Université d'Avignon. France.