GIOVANNI MANCINI
ENZO MITIDIERI

Positive solutions of some coercive-anticoercive elliptic systems

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GIOVANNI MANCINI[1] AND ENZO MITIDIERI[1]

RÉSUMÉ. — Dans ce travail nous traitons des résultats d'existence ou non-existence pour une classe de systèmes elliptiques semi-linéaires, avec des nonlinéarités critiques, en connexion avec un principe de maximum pour un système elliptique associé.

ABSTRACT. — In this paper we discuss existence-nonexistence results for a class of semilinear elliptic systems with critical nonlinearities in connection with a maximum principle for a related elliptic system.

0. Introduction

In dealing with elliptic systems,

$$
\begin{aligned}
-\Delta u &= f(u,v) & &\text{in } \Omega \subset \mathbf{R}^N \\
-\Delta v &= g(u,v) & &\text{in } \Omega \subset \mathbf{R}^N \\
u &= v = 0 & &\text{on } \partial\Omega
\end{aligned}
\qquad (0.1)
$$

where Ω is a smooth bounded domain and $f,g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are given mappings, we must face [2,7-11,15,17,18,21,24] all the typical problems associated with the scalar case, like nonuniqueness, nonexistence, breaking of symmetries, lack of compactness related to critical growth of the nonlinearities f and g, as well as some specific features, mainly the lack of a general maximum principle.

We will investigate in this paper existence and nonexistence results for a special class of systems of the type (0.1) in connection with a kind of

[1] Dipartimento di Scienze Matematiche, Università degli Studi di Trieste Piazzale Europa, 1 - 34100 Trieste (Italia)

"maximum principle" and critical nonlinearities. Namely we concentrate our attention on the problem

$$-\Delta u = \lambda u - \delta v + f(u) \quad \text{in } \Omega \subset \mathbf{R}^N \qquad (0.2)$$

$$-\Delta v = \delta u + \gamma v + h(v) \quad \text{in } \Omega \subset \mathbf{R}^N \qquad (0.3)$$

$$u = v = 0 \qquad \qquad \text{on } \partial\Omega$$

(here λ, γ, δ are real constants).

Special cases of (0.2)-(0.3) (e.g. $h \equiv 0$) were considered previously in [8,13,15-17,21] where those authors treat essentially sublinear nonlinearities (i.e. f sublinear).

Here we are concerned with the problem of existence and non-existence results for (0.2)-(0.3) for nonlinearities of the type,

$$(f) \qquad \qquad f(t) = |t|^{2^*-2}t, \ 2^* = \frac{2N}{N-2}$$

$$(h_1) \qquad \qquad h \in C^1(\mathbf{R}), \ h(0) = h'(0) = 0, h'(t) \le 0, \ t \in \mathbf{R}$$

i.e. for systems which are *sublinear* in v, but *superlinear* with limiting growth in u. The main motivation of our study is the following nonexistence result :

Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet b.c.

PROPOSITION 0.1.— *Assume* (f) *and* $(h)_1$, *and let* Ω *be a smooth starshaped domain contained in* $\mathbf{R}^N$ $(N \ge 3)$.

If,

$$(\Lambda_1) \qquad \qquad 0 < \delta < \lambda_1 - \gamma$$

$$(h_2) \qquad \qquad h(t)t \le 2H(t) = 2\int_0^t h(s)ds, \quad t \in \mathbf{R}$$

then (0.2)-(0.3) has no positive solutions $(u,v) \not\equiv (0,0)$ *provided*

$$\gamma + 2\delta \le \lambda \le 0$$

or

$$\lambda_1 + \frac{\delta^2}{\lambda_1 - \gamma} \le \lambda.$$

In contrast, we can prove the following existence result.

Let λ_j be j-th eigenvalue of $-\Delta$ with Dirichlet boundary condition.

THEOREM 1. — *Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain $(N \geq 4)$. Assume that (f) $(h_1) - (h_2)$ hold and*

$$0 < \lambda < \min_j \left\{ \lambda_j + \frac{\delta^2}{\lambda_j - \gamma} \right\}$$

Then there exists a non trivial solution (u,v) to (0.1)-(0.2).

If, furthermore

$$\lambda \geq \gamma + 2\delta$$

holds, then (0.2)-(0.3) has a solution $(u,v) \in \left(H_0^1 \cap L^\infty \right)^2$, with $u \geq v > 0$.

As far as the nonexistence result is concerned, we have to point out that while the inequality $\lambda \geq \lambda_1 + \frac{\delta^2}{\lambda_1 - \gamma}$ is sharp, we don't know wether the inequality $\gamma + 2\delta \leq \lambda \leq 0$ can be improved. This is because the argument a la "Pohozaev" we use, requires a kind of maximum principle which in general does not hold without some restrictions on the parameters λ, γ and δ.

Because of this we obtain different bounds on λ, in Theorem 1, as far as existence and, respectively, non existence of positive solutions are concerned.

The paper is organized as follows :

In Section 1. we recall some kind of maximum principle for an integro-differential operator related to (0.2)-(0.3) and we prove radial properties of positive solutions in case Ω is a ball.

In Section 2. we prove some non existence results, first analyzing the case of a general starshaped domain, secondly concentrating our attention to the special case $\Omega = B_1(0)$ and $N = 3$.

In Section 3. following Brezis-Kato [5], we prove L^∞-regularity and obtain L^∞-bounds for solutions of an approximate problem related to (0.2)-(0.3).

In Section 4. we prove some existence results in the case of $\Omega \subset \mathbf{R}^N$ with $N \geq 4$.

Finally **Section 5.** deals with the special case of $h \equiv 0$, $\gamma \equiv 0$ and $N = 3$.

Notations. — Throughout the paper we will use standard notations for Sobolev spaces H_0^1 and corresponding norms

$$\|u\|^2 = \int_\Omega | \nabla u |^2 dx \qquad |u|_P^P = \int_\Omega |u|^P dx$$

$< \cdot, \cdot >$ will denote the scalar product in H_0^1, while $(\cdot, \cdot)$ will be the scalar product in $\mathbf{R}^N$.

1. A maximum principle and radial properties of solutions

Let us first consider the linear system,

$$-\Delta u = \lambda u - \delta v + f(x) \quad \text{in } \Omega \qquad u = 0 \text{ on } \partial\Omega \qquad (1.1)$$

$$-\Delta v = \delta u + \gamma v + h(x) \quad \text{in } \Omega \qquad v = 0 \text{ on } \partial\Omega \qquad (1.2)$$

Throughout this Section we will always assume

(Λ_1) $$0 < \delta < \lambda_1 - \gamma$$

(Λ_2) $$\lambda < \lambda_1 + \frac{\delta^2}{\lambda_1 - \gamma}$$

For completeness, we present here some results contained in [13].

PROPOSITION 1.1. — *Assume* $(\Lambda_1) - (\Lambda_2)$ *and*

(Λ_3) $$\gamma + 2\delta \leq \lambda$$

If $h(x) \leq 0 \leq f(x)$, $f, h \in L^{2N/N+2}$ *and* $(u, v) \in H_0^1 \times H_0^1$, *solves (1.1)-(1.2), then u is non negative.*

Proof. — Let us denote by G the Green operator

$$\forall u \in L^{\frac{2N}{N+2}} :< G(u), \varphi >:= \int_\Omega u\varphi dx, \qquad \varphi \in H_O^1$$

If $(u, v) \in H_0^1 \times H_0^1$ solves (1.1)-(1.2), then

$$u - \gamma G(u) = (\lambda - \gamma)G(u) - \delta G(v) + G(f)$$
$$v - \gamma G(v) = \delta G(u) + G(h)$$

Setting $B = (I - \gamma G)^{-1} G$ the system rewrites

$$\left(I - (\lambda - \gamma)B + \delta^2 B^2\right)u = Bf - \delta B^2 h \geq 0.$$

So it is enough to prove that the operator

$$L := \left(I - (\lambda - \gamma)B + \delta^2 B^2\right)^{-1}$$

is order preserving. But $\lambda \geq \gamma + 2\delta$ implies $L = \alpha \cdot \beta(I - \alpha B)^{-1}(I - \beta B)^{-1}$ where α^{-1}, β^{-1} are the two positive solutions of $\delta^2 x^2 + (\gamma - \lambda)x + 1 = 0$. So that $\alpha\beta = \delta^2$, $\alpha + \beta = \lambda - \gamma$ and hence $\alpha + \delta^2\alpha^{-1} = \beta + \delta^2\beta^{-1} = \lambda - \gamma < \lambda_1 - \gamma + \frac{\delta^2}{\lambda_1 - \gamma}$. This implies $\alpha < \lambda_1 - \gamma$, $\beta < \lambda_1 - \gamma$, because the function $t \to t + \delta^2 t^{-1}$ is convex, with minimum value at $t = \sigma$ given by 2σ, and $\lambda_1 - \gamma > \delta$. In turns, this implies $\|\alpha\beta\| = \alpha(\lambda_1 - \gamma)^{-1} < 1$, $\|\beta B\| < 1$ and hence $(I - \alpha B)^{-1}$, $(I - \beta B)^{-1}$ are order preserving.

Remark. — (Λ_1) is a necessary condition for the validity of Prop. 1.1, because it is necessary for L, B to be order preserving. In fact (Λ_1) just means that the positive eigenfunction φ_1, for LB corresponds to the largest eigenvalue of LB, since the spectrum of LB is given by $(\lambda_j - \gamma + \delta^2(\lambda_j - \gamma)^{-1} - (\lambda_j - \gamma)^{-1}$, $(-\Delta\varphi_j = \lambda_j \varphi_j, \varphi_j \in C_0^\infty(\Omega))$ and if $\lambda_1 - \gamma > \delta$, it is no longer true, in general, that $\lambda_1 - \gamma + \delta^2(\lambda_1 - \gamma)^{-1} \leq \lambda_j - \gamma + \delta^2(\lambda_j - \gamma)^{-1}$ $j > 1$.

PROPOSITION 1.2. — *Assume $f, h \in C^1$ satisfy the following assumption :*

$$(f_1) \quad 0 \leq f(t) \leq a + b|t|^{2^* - 1} t, \qquad a, b \in \mathbf{R}^+$$

$$(h_1) \quad h(0) = h^+(0) = 0, \quad -M \leq \frac{h(s)}{s} \leq 0, \ s > 0$$

$$h(s) = 0 \quad s \leq 0.$$

Then, if $(\delta, \gamma, \lambda)$ satisfy $(\Lambda_1)-(\Lambda_2)-(\Lambda_3)$, every H_0^1-solution (u, v) of (0.1)-(0.2) satisfies

$$u \geq \frac{\lambda - \gamma}{\delta} v > 0 \quad in \ \Omega \tag{1.3}$$

$$\frac{\partial u}{\partial n} < \frac{\partial v}{\partial n} < 0 \ on \ \partial\Omega \quad (n. \ exterior \ normal \ to \ \partial\Omega) \tag{1.4}$$

Proof. — Let (u, v) be a solution of (0.1)-(0.2).

Using Prop. 1.1 with $f(x) := f(u(x)), h(x) := h(v(x))$ we easily see that $u \geq 0$.

Setting

$$a(x) = \begin{cases} 0 & \text{if } v(x) \leq 0 \\ \dfrac{h(v(x))}{v(x)} & \text{if } v(x) > 0 \end{cases}$$

equation (0.2) gives

$$-\Delta v - (\gamma - a(x))v = \delta u \geq 0$$

and since $a \in L^\infty$, $a \geq 0$, the weak maximum principle implies $v \geq 0$.

Finally, if ϵ_i is a solution of $(i = 1.2)$

$$\delta\epsilon^2 + (\gamma - \lambda)\epsilon + \delta = 0$$

we have

$$-\Delta(u - \epsilon v) \geq (\lambda - \delta\epsilon)u - (\delta + \gamma\epsilon)v = (\lambda - \delta\epsilon)(u - \epsilon v)$$

which implies in particular

$$u \geq \frac{\lambda - \gamma}{2\delta}v \geq v \quad \text{and} \quad \frac{\partial u}{\partial n} < \frac{\partial v}{\partial n} < 0.$$

We end this Section with a result concerning symmetry properties of solutions, which seems to be new and of interest in itself.

PROPOSITION 1.3. — *If in addition to the assumptions in Prop. 1.2 we assume*

$$(*) \quad f'(t) + \lambda - \gamma - 2\delta \geq 0 \quad h'(t) \leq 0 \quad t \geq 0$$

an $\Omega = B_R(0)$, then the solutions of (0.1)-(0.2) are radially symmetric.

Proof. — After introducing the new variable $w := u - v$ we see that (u, v, w) is a *positive* solution of the system

$$
\begin{aligned}
-\Delta u &= f_1(u, v, w) := f(u) + (\lambda - \delta)u + \delta w \\
-\Delta v &= f_2(u, v, w) := \delta u + \gamma v + h(v) \qquad\qquad\qquad \text{in } \Omega \\
-\Delta w &= f_3(u, v, w) := f(u) + (\lambda - 2\delta - \gamma)u - h(v) + (\delta + \gamma)w \\
u &= v = w = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega
\end{aligned}
$$

Since $\dfrac{\partial f_i}{\partial x_j} \geq 0$ (far $i \neq j$) we can apply a result by Troy [24], to infer that (u, v, w) is radial and $w'(r) = u'(r) - v'(r) < 0$, $r \in (0, R)$.

Remark 1.4. — As one easily sees the above result can be used to obtain L^∞-bounds of positive solutions to (0.2)-(0.3) in the same spirit as Cosner [9].

2. Some non existence results

In this Section we first give a general result concerning non existence of positive solutions for (0.1)-(0.2) in a starshaped domain contained in $\mathbf{R}^N$.

In case $N = 3$ we will sharpen our result, following BREZIS-NIRENBERG [6] in case Ω is a ball. In both cases our technique heavily relies on the maximum principle for (0.1)-(0.2) given in Section 1, and consequently we restrict our attention to parameters $(\lambda, \delta, \gamma)$ satisfying $(\Lambda_1) - (\Lambda_2) - (\Lambda_3)$.

Nevertheless, we don't know whether our limitations an $(\lambda, \delta, \gamma)$ are sharp or not.

POHOZAEV *identity for (0.1)-(0.2).* (see [13]). *Let (u, v) be a smooth solution of (0.1)-(0.2), then*

$$(P) \qquad \int_{\partial\Omega} < x, \nu > \left[(\frac{\partial u}{\partial \nu})^2 - (\frac{\partial v}{\partial \nu})^2 \right] d\sigma =$$

$$= 2\lambda \int_{\Omega} u^2 - 2\gamma \int_{\Omega} v^2 - 4\delta \int_{\Omega} uv + (2 - N) \int_{\Omega} u f(u) +$$

$$+ 2N \int_{\Omega} F(u) + (N - 2) \int_{h} h(v)v - 2N \int_{\Omega} H(v)$$

PROPOSITION 2.1. — *Let $\Omega \subset \mathbf{R}^N$ $N \geq 3$) be a smooth bounded domain. Assume that $(\Lambda_1) - (\Lambda_2) - (\Lambda_3)$ and (f), $(h_1) - (h_2)$ hold. Then (0.1)-(0.2) has no positive solutions provided either*

$$(i) \qquad \lambda \geq \lambda_1 + \frac{\delta^2}{\lambda_1 - \gamma}$$

or

$$(ii) \qquad 2\delta + \gamma \leq \lambda \leq 0 \text{ and } \Omega \text{ is starshaped.}$$

Proof. — The proof of case (i) is straighforward and will be omitted. To deal with the case (ii), we use (P) together with the assumptions on $f(\cdot)$ and $h(\cdot)$.

It is easy to see that $(\Lambda_1) - (\Lambda_2) - (\Lambda_3)$ imply that the right hand side of (P) is non positive, while by Prop. 1.2, the left hand side is positive in view of (1.4) and the starshapeness of Ω.

Remark 2.2. — It is not difficult to check that the above Proposition holds under the more general assumptions on the non linearities :

$$s f(s) \geq 2^* \int_0^s f(t)dt$$
$$s h(s) \leq 2^* \int_0^s h(t)dt \qquad , \quad s \in \mathbf{R}$$

We now show, as in the case of a single equation considered by BREZIS-NIRENBERG [6], we have a more delicate situation in case $N = 3$ and Ω is a

ball. For simplicity, we will limit ourselves to the case $\gamma = 0$, $h(\cdot) \equiv 0$ and $\Omega = B_1(0)$.

PROPOSITION 2.3. — *Let* $\Omega = \{x \in \mathbf{R}^3 : |x| < 1\}$, *and let* (u,v) *be a* C^2-*solution of*

$$-\Delta u = \lambda u + (u^+)^5 - \delta v \quad on \ \Omega \qquad (2.1)$$

$$-\Delta v = \delta u \qquad\qquad on \ \Omega \qquad (2.2)$$

$$u = v = 0 \qquad\qquad on \ \partial\Omega$$

Assume that $0 < \delta < \pi^2/16$, *and let*

$$\sigma(\delta) \in \left(2\sqrt{\delta}, \pi\right)$$

be the lowest solution of

$$L(s) := s \ sen \ s - \frac{4\delta}{s} \ sen \ \frac{4\delta}{s} = 0$$

in such an interval. Then

$$\lambda \in \left(2\delta, \frac{\sigma^2(\delta)}{4} + \frac{4\delta^2}{\sigma^2(\delta)}\right) \implies u \equiv v \equiv 0.$$

Remark 2.4. — Let $0 < \delta < \dfrac{\pi^2}{16}$ the function $L(s)$ is positive in $\left(2\sqrt{\delta}, \pi/2\right)$ and becomes negative for $s = \pi$, hence there is a *lowest* solution of $L(s) = 0$ in $(2\sqrt{\delta}, \pi)$. If we denote by $\sigma(\delta)$ such a solution, clearly $\lim_{\delta \to 0+} \sigma(\delta) = \pi$; thus, "in the limit" we get a non existence interval for values of λ, given by $\left(0, \pi^2/4\right)$, which is the one obtained by BREZIS-NIRENBERG in the case of a single equation.

Nevertheless we don't know if $\dfrac{\sigma^2(\delta)}{4} + \dfrac{4\delta^2}{\sigma^2(\delta)}$ is a sharp bound for non-existence, as well as our method, relying on a maximum principle, doesn't apply (in the case $\gamma = 0$) to negative $\lambda's$, and hence it is an open question to know whether non-existence results hold true in this case.

Proof. — Thanks to Prop. (1.3), we are reduced to prove that our system has no positive *radial* solutions, for $\lambda \in \left(2\delta, \dfrac{\sigma^2(\delta)}{4} + \dfrac{4\delta^2}{\sigma^2(\delta)}\right)$. Again, our proof relies on a appropriate :

POHOZAEV *identity for (radial) solution of (2.1)-(2.2).*

Let $(u, v) \in C^2(0, 1)$ be a radial solution to (2.1)-(2.2). Let $\psi \in C^\infty(\mathbf{R})$ be such that $\psi(0) = 0$ and $\psi(1) \geq 0$. Then,

$$(P_R) \qquad \frac{1}{2}\psi(1)\big[u'(1)^2 - v'(1)^2\big] = \frac{2}{3}\int_0^1 u^6(r^2\psi' - r\psi)dr +$$

$$+ \int_0^1 r^2\Big[(r\frac{1}{4}\psi''' + \lambda\psi')u^2 - \frac{1}{4}\psi'''v^2 - 2uv\psi'\Big]dr.$$

Proof of (P_R). — The radial solution (u, v) of (2.1)-(2.2) satisfies

$$-u'' - \frac{2}{r}u' = \lambda u + u^5 - \delta v$$

$$-v'' - \frac{2}{r}v' = \delta u$$

$$u'(0) = v'(0) = u(1) = v(1) = 0$$

Following BREZIS-NIRENBERG [6], we first multiply the first equation (respectively the second equation) by $r^2\psi'u'$ $(r^2\psi'v')$, and by $(1/2r^2\psi' - r\psi)u$ $((1/2r^2\psi' - r\psi)v)$. After integration by parts we obtain

$$\frac{1}{2}|u'(1)|^2\psi(1) = \int_0^1 (\frac{1}{4}\psi''' + \lambda\psi')r^2u^2dr +$$

$$+ \frac{2}{3}\int_0^1 (r^2\psi' - r\psi)u^6dr + \delta\int_0^1 u'vr^2\psi dr - \delta\int_0^1 uv(\frac{1}{2}r^2\psi' - r\psi)dr$$

$$(2.3)$$

and

$$\frac{1}{2}|v'(1)|^2\psi(1) = \int_0^1 \frac{1}{4}\psi'''v^2r^2 - \delta\int_0^1 uv'r^2\psi dr +$$

$$+ \delta\int_0^1 uv(\frac{1}{2}r^2\psi - r\psi)dr.$$

$$(2.4)$$

Taking (2.3)-(2.4), after integration by parts we get (P_R).

Now using Prop. 1.2 we get,

$$A = \int_0^1 r^2\Big[(\frac{1}{4}\psi''' + \lambda\psi')u^2 - \frac{1}{4}\psi'''v^2 - 2\delta uv\psi'\Big]dr +$$

$$+ \frac{2}{3}\int_0^1 u^6(r^2\psi' - r\psi)dr \geq 0$$

with strict inequality if (u, v) is not identically zero.

Our claim is that if $\lambda \in \left(2\delta, \dfrac{\sigma^2(\delta)}{4} + \dfrac{4\delta^2}{\sigma^2(\delta)}\right)$ we can choose $\psi(\cdot)$ to get $A < 0$, which will imply $u \equiv v \equiv 0$. We choose $\psi(\cdot)$, among the solutions (ψ, φ) of the O.D.E. system :

$$
\begin{aligned}
&\frac{1}{4}\psi'' + \lambda\psi = 2\delta\varphi \\
&\varphi'' = -2\delta\psi \\
&\varphi' \leq 0 \qquad\quad \text{in } (0,1) \\
&\psi(0) = 0
\end{aligned}
\tag{2.5}
$$

with such a choice of (ψ, φ) we have the following :

LEMMA 2.5. — *Let* $\lambda > 2\delta$, *and set* $\Delta = \lambda^2 - 4\delta^2$, $c = \dfrac{\lambda - \sqrt{\Delta}}{2}$. *Then*

$$
A \leq \int_2^1 v(\lambda v - 2\delta u)\left(\psi' - \frac{2}{c}\,\varphi'\right)r^2\,dr + \frac{2}{3}\int_0^1 u^6(r^2\psi' - r\psi)dr.
$$

Proof. — Since $c(\lambda - c) = \delta^2$, we have

$$
\begin{aligned}
(u^2 - v^2) + \frac{1}{c}\,(\lambda v^2 - 2\delta uv) &= \\
&= \frac{1}{c}\left[cu^2 + (\lambda - c)v^2 - 2\sqrt{c}\,\sqrt{\lambda - c}\,uv\right] \geq 0.
\end{aligned}
$$

Thus

$$
2\delta(u^2 - v^2)\varphi' \leq -\frac{2\delta}{c}(\lambda v^2 - 2\delta uv)\varphi'
$$

because $\varphi' \leq 0$. Hence using (2.5);

$$
\begin{aligned}
\left(\frac{1}{4}\psi''' + \lambda\psi'\right)u^2 - \frac{1}{4}\,\psi'''v^2 - 2\delta uv\psi' &= \\
&= 2\delta\varphi'(u^2 - v^2) + (\lambda v^2 - 2\delta uv)\psi' \leq (\lambda v^2 - 2\delta uv)\left(\psi' - \frac{2\delta}{c}\varphi'\right).
\end{aligned}
$$

Remark 2.6. — Let (ψ, φ) be a solution to (2.5); then

$$
(x) \qquad\qquad \varphi'''' + 4\lambda\,\varphi'' + 16\delta^2\,\varphi = 0.
$$

Conversely if φ solves (x), the pair $(\psi, \varphi|$ with

$$\psi = -\frac{1}{2\delta}\varphi''$$

solves (2.5).

Assuming $\lambda > 2\delta$ and denoted by $\pm i\rho$, $\pm i\sigma$ the roots of the characteristic equation of (x), we have the following relations

$$\rho^2 = 2(\lambda - \sqrt{\Delta}), \qquad \sigma^2 = 2(\lambda + \sqrt{\Delta}) \qquad (2.6)$$
$$\rho^2 = 4c, \ \rho^2 + \sigma^2 = 4\lambda, \ \rho^2\sigma^2 = 16\delta^2. \qquad (2.7)$$

Since $\sigma^2 > 4\delta$, we also have $\rho^2 < 4\delta < \sigma^2$. Now assuming as above $\lambda > 2\delta$, let us choose the following solution (ψ, φ) to (2.5)

$$\psi(t) = \frac{\sigma^2}{2\delta} \sin \sigma t - 2 \sin \frac{4\delta}{\sigma} t$$
$$\varphi(t) = \sin \sigma t - \frac{\sigma^2}{4\delta} \sin \frac{4\delta}{\sigma} t \qquad (2.8)$$

$\sigma = \sigma(\lambda, \delta)$, $\rho = \rho(\lambda, \delta)$, being given as in (2.6).

Remark 2.7. — It is clear that φ chosen as above, satisfies $\varphi' \le 0$ in $(0,1)$, provided $\sigma \le \pi$, because

$$\varphi'(t) = \sigma\left(\cos \sigma t - \cos \frac{4\delta}{\sigma} t\right)$$

and $\dfrac{4\delta}{\sigma} < \sigma$.

We want also to emphasize that, in view of the existence result; which we will give later, and which holds true if

$$\frac{\pi^2}{4} + \frac{4\delta^2}{\pi^2} \le \lambda < \pi^2 + \frac{\delta^2}{\pi^2}$$

(here obviously $\lambda_1 = \pi^2$), we have to assume $\sigma < \pi$. In fact by (2.6)

$$\frac{\sigma^2}{4} + \frac{4\delta^2}{\sigma^2} = \lambda$$

and the necessary assumption (in order to get non existence)

$$\lambda < \frac{\pi^2}{4} + \frac{4\delta^2}{\pi^2}$$

yields $\sigma < \pi$.

To complete the Proof of the Proposition 2.3, we need some simple estimates which we collect in a lemma whose proof can be obtained by direct calculation.

LEMMA 2.8. — *Let $\sigma(\delta)$ be the lowest solution of*

$$\sigma \, \sin \, \sigma - \frac{4\delta}{\sigma} \, \sin \, \frac{4\delta}{\sigma} = 0$$

in $(2\sqrt{\delta}, \pi)$. Let (ψ, φ) be given by (2.8). Then

(i) $\psi' - \dfrac{8\delta}{\pi^2} \varphi' \geq 0$ *in (0,1)*

(ii) $\psi - t\psi' \geq 0$ *in (0,1)*

(iii) $\psi(1) > 0$

Completiton of the Proof of Proposition 2.2.

Now the result follows, using Lemma 2.5 and Lemma 2.8, taking into account that by (1.3) and (1.4) we have $u > \lambda/2\delta \ v$ and $u'(r) - v'(r) < 0$, $r \in (0,1)$.

3. Regularity and L^∞-estimates

The main purpose of this section is to get some uniform L^∞-estimates for weak solutions of families of elliptic systems of the type

$$-\Delta u = \lambda u - \delta v + f(u) \quad \text{in } \Omega \subset \mathbf{R}^N \tag{0.1}$$

$$-\Delta v = \delta u + \gamma v + h_R(v) \quad \text{in } \Omega \subset \mathbf{R}^N \tag{0.2}_R$$

$$u = v = 0 \qquad \text{on } \partial\Omega$$

where $(\lambda, \gamma, \delta) \in \mathbf{R}^3$, $\gamma < \lambda_1$ (the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions) and $f(\cdot)$, $h_R(\cdot)$ are allowed to have "limiting growth". More precisely we will assume that :

There exist positive constants a, b, a_R, b_R, such that for every $t \in \mathbf{R}$ we have

$$|f(t)| \leq a|t|^{2^*-1} + b, \quad |h_R(t)| \leq a_R|t|^{2^*-1} + b_R \tag{3.1}$$

Our argument follow closely the work of BREZIS-KATO [5] devoted to a single equation. In what follows we deal with a given system, so that we drop subscript R.

PROPOSITION 3.1. — *Let $(u,v) \in H_0^1$ be a solution of*

$$- \Delta u = a\, u + b\, v + f(u) \qquad (3.1)$$

$$- \Delta v = c\, u + d\, v + h(v) \qquad (3.2)$$

where $(a,b,c,d) \in \mathbf{R}^4$, $f, h \in C^1$ satisfy (3.1) and $f(0) = f'(0) = h(0) = h'(0)$. Then $(u,v) \in L^\infty$.

For convenience of the reader we state here some facts which are essentially contained in [5].

LEMMA 3.2. — *Let $a_j \in L^{N/2}$, $(j \in N)$ be such that :*

$$\forall \epsilon > 0 \; \exists\, K_\epsilon > 0 : \sup_j \int_{|a_j| \geq K_\epsilon} |a_j|^{N/2} \leq \epsilon \qquad (3.4)$$

Then there exists $\mu \in \mathbf{R}$ such that for every $f \in L^{2N/N+2}$ each equation

$$-\Delta u + a_j(x)u + \mu u = f \qquad (3.5)$$

has a unique solution $u_j \in H_0^1$. Furthermore $\|u_j\| \leq C|f|_{2N/N+2}$ and $f \geq 0 \Longrightarrow u_j \geq 0$.

Proof. — Using (3.4) we see that,

$$\int_\Omega |a_j|^2 u^2 \leq \int_{|a_j| \geq K_\epsilon} a_j u^2 + K_\epsilon \int_{|a_j| < K} u^2 \leq$$

$$\leq K_\epsilon |u|_2^2 + |u|_{2^*}^{N-2/N} \Big(\int_{|a_j| \geq K_\epsilon} |a_j|^{N/2} \Big)^{N/2} \leq$$

$$\leq K_\epsilon |u|_2^2 + \epsilon |u|_{2^*}^{N-2/N}$$

which implies

$$\int_\Omega |a_j|^2 u^2 \leq K_\epsilon |v|_2^2 + \epsilon \|u\|_2^2 \qquad (3.6)$$

Thus, the bounded selfadjoint linear operators in H_O^1 given by

$$< L_j u, \; \varphi > := \int_\Omega \nabla u \nabla \varphi + a_j(x)u\varphi + K_{1/2}u\varphi, \quad \varphi \in H_0^1$$

are (uniformly) positive definite : taking $\epsilon = 1/2$, we find K, independent of j, such that

$$< L_j u, u > \geq \frac{1}{2}\, \|u\|^2 + K|u|_2^2 - K|u|_2^2 = \frac{1}{2}\, \|u\|^2.$$

Hence, for such K, and any given $f \in L^{2N/N+2}$ the equations

$$< L_j u_j, \varphi > = \int_\Omega f\varphi \quad , \quad \varphi \in H_0^1$$

are uniquely solvable for every $j \in \mathbf{N}$ and

$$\frac{1}{2} \|u_j\|_2^2 \leq < L_j u_j, u_j > = \int_\Omega f u_j \leq |f|_{2N/N+2} |u_j|_{2N/N-2}$$

implies

$$\|u_j\| \leq \cos t |f|_{2N/N+2}$$

Finally, if $f \geq 0$, multiplying (3.5) by $\min(u,0)$ (dropping subscript j) and using again (3.6), we get $\min(u,0) = 0$.

LEMMA 3.3. — Let $a \in L^{N/2}$ and $f \in L^p$ with $p \geq 2^*$. Let $u \in H_0^1$ be a weak solution of

$$-\Delta u + a(x)u = f.$$

Then $u \in L^p \Longrightarrow u \in L^{\tau p}$ (with $\tau = N/N - 2$), and

$$|u|_{\tau p} \leq \cos t(|f|_p + |u|_p).$$

Proof. — Set

$$a_j(x) = \begin{cases} a(x) & \text{if } |a(x)| \leq j \\ j \text{ sign } a(x) & \text{otherwise} \end{cases}$$

Lemma (3.2) applies, to find $K \in \mathbf{R}$ such that the equations (omitting subscript j)

$$\begin{aligned} -\Delta z_j + a_j(x)z_j + kz_j &= f + ku \\ -\Delta z + a(x)z + kz &= f + ku \end{aligned} \tag{3.7}$$

have a unique solution z_j, $z = u$ respectively. Furthermore,

$$\|z_j\| \leq \cos t \ |f + Ku|_{2^*},$$

so that we can assume $z_j \xrightarrow{H_0^1} \overline{z}$, for some $\overline{z}$. Clearly $\overline{z} = u$, because $a_j \xrightarrow{L^{N/2}} a$ and $z_j\varphi \xrightarrow{L^{N/N-2}} \overline{z}\varphi$ for every $\varphi \in H_0^1$, and then we can pass to the limit in

$$\int_\Omega \left(\nabla z_j \nabla\varphi + a_j z_j + kz_j\varphi\right) = \int_\Omega (f + ku)\varphi, \ \varphi \in H_0^1.$$

Now, let us fix $q \leq p$. To estimate $|z_j|_{\tau q}$, first notice that taking eventually the positive and negative part of $f + Ku$, it is enough to estimate $|z_j|_{\tau q}$ assuming $z_j \geq 0$. Dropping subscript j, let us consider $z_n = \min(z, n)$. Since $|z_n|^{q-1} \in H_0^1$, denoting $\widehat{f} = f + Ku$, we get from (3.7) :

$$(q-1)\int_\Omega |\nabla z_n|^2 z_n^{q-2} = \int_\Omega \nabla z \nabla z_n^{q-1} \leq \int_\Omega |a_j| z\, z_n^{q-1} + \widehat{f} z_n^{q-1} \leq$$

$$|\widehat{f}|_q |z_n|_q^{q-1} + \int_{z \leq n} |a_j| z_n^q + \int_{z \geq n} |a_j| z\, z_n^{q-1}$$

and hence,

$$\frac{4(q-1)}{q^2} \int_\Omega |\nabla z_n^{q/2}|^2 \leq |\widehat{f}|_q |z_n|_q^{q-1} + \int_\Omega |a_j| z_n^q + j \int_{z \geq n} z^q.$$

Using (3.6), with $u = z_n^{q/2}$ we finally get

$$\frac{4(q-1)}{q^2} \int_\Omega |\nabla z_n^{q/2}|^2 \leq |\widehat{f}|_q |z_n|_q^{q-1} + \epsilon \int_\Omega |\nabla z_n^{q/2}|^2 + k_\epsilon \int_\Omega |z_n|^q\; j \int_{z \geq n} z^q$$

Taking $\epsilon = \dfrac{2(q-1)}{q^2}$, and using Sobolev inequality we obtain,

$$C_q |z_n|^q_{\frac{qN}{N-2}} \leq |\widehat{f}|_q |z_n|_q^{q-1} + k_\epsilon |z_n|_q^q + j \int_{z \geq n} z^q$$

sending n to infinity and using Fatou lemma we get

$$C_q |z|_q^q \leq k_\epsilon \left(|\widehat{f}|_q + |z|_q \right)^q \quad (z = z_j) \tag{3.8}$$

with C_q and k independent on j.

Using lemma 3.2 and taking $q = 2^*$ in (3.8) we get

$$C_{2^*} |z_j|_{\tau 2^*} \leq k_1 |\widehat{f}|_{2^*}$$

Iterating this procedure, we get, up to $\tau^{l-1} 2^* \leq p < \tau^l 2^*$

$$C_l |z_j|_{\tau^l 2^*} \leq k_1 |\widehat{f}|_{\tau^{l-1} 2^*}$$

and then

$$c |z_j|_{\tau p} \leq K |\widehat{f}|_p \tag{3.9}$$

with c and K independent on j. Passing to the limit (as $j \to +\infty$) in (3.9) we get

$$c|u|_{\tau p} \le K|f + Ku|_p$$

Combining Lemma (3.3) and elliptic regularity applied to both equations (3.1)-(3.2), we obtain the claim of Proposition 3.1.

We are now in position to give L^∞-estimates for solutions of (0.1)-$(0.2)_R$.

PROPOSITION 3.4. — *Let f, h_R satisfy (3.1) and*

$$f(0) = f'(0) = h'_R(0) = h'_R(0) = 0, \quad h'_R(t) \le 0, \quad t \in \mathbf{R} \qquad (3.10)$$

Let (u_R, v_R) be solutions of (0.1)-$(0.2)_R$ such that,
(i) $\{u_r\}$ is precompact in L^{2^*}, and $\sup_R \|v_R\| < +\infty$. Then,

$$\sup_R |u_R|_{L^\infty} < +\infty, \quad \sup_R |v_R|_{L^\infty} < +\infty.$$

Proof. — According to Proposition 3.1, we know that $(u_R, v_R) \in L^\infty$. Set $a_R(x) := \lambda + \widehat{a}_R(x)$, where

$$\widehat{a}_R(x) = \begin{cases} \dfrac{f(u_R(x))}{u_R(x)} & \text{if } u_R(x) \ne 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see, using the compactness assuptions, that

$$\forall \epsilon > 0 : \exists K_\epsilon > 0 : \sup_R \int_{a_R \ge K_\epsilon} |a_R|^{N/2} \le \epsilon. \qquad (3.11)$$

So that, by Lemma 3.2, there exists $\mu \in \mathbf{R}$, independent on R, such that $u_R(\cdot)$ is the unique H_0^1-solution of

$$-\Delta z - a_R(x)z + \mu z = \mu u_R - \delta v_R. \qquad (3.12)$$

To get a uniform bound for $\{|u_R|\}$, we first prove L^p-estimates. To this extent, let us introduce z_R, w_R, solutions of (3.12) corresponding to right hand side $(\mu u_R - \delta v_R)^+$ and $(\mu u_R - \delta v_R)^-$ respectively : they exists and are unique because of Lemma 3.2. We claim that, setting $\tau = N/N - 2$, we have

$$\forall p \ge 1 \; \exists \; C_p : \begin{cases} |w_R|_{\tau p} \le C_p(|u_R|_p + |w_R|_p) \\ |z_R|_{\tau p} \le C_p(|u_R|_p + |z_R|_p) \end{cases} \qquad (3.13)$$

where C_p is independent on R.

Since for $p = 2$, the right hand side in (3.13) are bounded with respect to R, we will get

$$\sup_R |u_R|_{2r} \leq \sup_R |w_R|_{rp} + \sup_R |z_R| < +\infty$$

because obviously $u_R = w_R + z_R$. In turn this implies $\sup_R |u_R|_{2r}2 < +\infty$, and thus, by iteration, we will get

$$\forall p > 1, \sup_R |u_R|_p < +\infty \tag{3.14}$$

To prove (3.13) for, say z_R (the same procedure works as well for w_R) we first need

$$\forall p > 1 : \exists\, D_p > 0 : |v_R|_p \leq D_p |u_R|_p \tag{3.15}$$

where D_p does not depend on R. Let us prove (3.15) with v_R replaced by $\overline{v}_R$, solution of

$$
\begin{aligned}
-\Delta v - \gamma v - h_R(v) &= \delta u_R^+ && \text{in } \Omega \\
v &= 0 && \text{on } \partial\Omega
\end{aligned}
\tag{3.16}
$$

A similar argument, taking instead u_R^-, will give (3.15). If $\overline{v}_R \geq 0$ is the solution of (3.16), multiplying the equation by $\overline{v}_R^{p-1}$, we get (observe that $h_R(\overline{v}_R(x)) \leq 0$).

$$-\gamma \int_\Omega \overline{v}_R^p + \frac{4(p-1)}{p^2} \int_\Omega |\nabla \overline{v}_R^{p/2}|^2 = (p-1) \int_\Omega |\nabla \overline{v}_R|^2 \overline{v}_R^{p-2} - \gamma \int_\Omega \overline{v}_R^p \leq$$

$$\leq \delta \int_\Omega u_R^+ \overline{v}_R^{p-1},$$

and using Poincaré, inequality,

$$\left(\frac{4(p-1)}{p^2} \lambda_1 - \gamma \right) \int_\Omega \overline{v}_R^p \leq \delta |u_R^+|_p \, |\overline{v}_R|_p^{p-1}.$$

Since $\gamma < \lambda_1$, if $p > 2$ is sufficiently close to 2, we get

$$|\overline{v}_R|_p \leq C_p |u_R^+|_p \text{ (uniformly in } R)$$

Starting with such a p, and using Sobolev inequality, we have

$$\alpha_p \left(\int_\Omega \left(\overline{v}_R^{p/2} \right)^{2^*} \right)^{\frac{N-2}{N}} \leq \gamma |\overline{v}_R|_p^p + \delta |u_R^+|_p |\overline{v}_R|_p^{p-1} \leq \beta_p \left(|\overline{v}_R|_p + |u_R^+|_p \right)^p$$

and hence

$$|\bar{v}_R|_{\tau p}^p \leq \beta_p' |u_R^+|_p^p$$

which implies $|\bar{v}_R|_{\tau p} \leq C_p |u_R^+|_{\tau p}$. By iterating the above procedure, we thus get $\forall p > 2, \ \exists \ C_p > 0 : |\bar{v}_R| \leq C_p |u_R^+|_p$ (uniformly in R).

We now go back to the proof of (3.13). Multiplying the equation

$$-\Delta z_R - a_R(x) z_R + \mu z_R = (\mu u_R - \delta v_R)^+$$

by z_R^{p-1}, we get as above

$$\frac{4(p-1)}{p^2} \int_\Omega |\nabla z_r^{p/2}|^2 \leq \int_\Omega a_R(x) z_R^p + \int_\Omega (\mu u_R - \delta v_R)^+ z_R^{p-1}.$$

Now, in view of (3.11) we can apply (3.6) to get, with a suitable choice of ϵ and using Sobolev's and Holder's inequalities :

$$C_p |z_R|_{\tau p}^p \leq k_\epsilon \int_\Omega z_R^p + C_p \big(|u_R|_p + |v_R|_p\big)|z_R|_p^{p-1}$$

for some positive constants $C_p, \ k_\epsilon$ independent on R.

Using (3.15), we finally get

$$|z_R|_{\tau p}^p \leq C_p \big(|z_R|_p + |u_R|_p\big)^p$$

Now, elliptic regularity, Sobolev imbedding and (3.14)-(3.15) applied to $(0.1)_R$ give uniform (w.r. to R) L^∞-bound for u_R.

Finally, if $\sup_R |u_R|_{L^\infty} \leq M$, setting

$$-\Delta\psi - \gamma\psi = \delta M \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \partial\Omega$$

we get,

$$-\Delta(v_R - \psi) - \gamma(v_R - \psi) = \delta u_R + h_R(v_R) - \delta M \leq h_R(v_R),$$

and multiplying the above inequality by $\mathcal{V}_R = \max\{v_R - \psi, 0\}$, we obtain

$$\int_\Omega |\nabla(v_R - \psi)^+|^2 - \gamma \int_\Omega (v_R - \psi)^{+2} \leq \int_\Omega h_R(v_R)(v_r - \psi)^+.$$

Thus $(v_R - \psi)^+ = 0$, i.e. $v_R(x) \leq \psi(x) \quad x \in \Omega, \ \forall R$.

Similarly we get a uniform lower bound for v_R.

4. The variational principle and existence results

In this section we first consider system (0.1) (0.2) for some classes of superlinear-sublinear nonlinearities, reducing our system to an integrodifferential equation of variational type. The early stage, when we consider $f(\cdot)$ of the type (f) and $h(\cdot)$ satisfying $(h_1) - (h_2)$ and asymptotically linear we already deal with the main difficulty, i.e. a "mountain pass" situation with the lack of $P.S.$, which we over come in the spirit of [6].

Nevertheless, while the existence technique is quite the same as in [19] (see also [3]), due to the lack of a general maximum principle, we are able to produce two quite different existence results, as far as we are concerned with existence and with existence of positive solutions respectively.

We believe that this is a peculiar difference between anticoercive-coercive systems considered by us and elliptic equations.

Our first result is the following :

THEOREM 4.1. — *Let* $\Omega \subset \mathbf{R}^N$ $(N \geq 4)$ *be a smooth bounded domain. Assume*

$$f(t) = |t|^{2^* - 2} t$$

and

(h_1) \qquad $h \in C^1(\mathbf{R}),\ h(0) = h'(0) = 0,\ h'(t) \leq 0,\ t \in \mathbf{R}$

(h_2) \qquad $h(t)t \leq 2H(t) \quad t \in \mathbf{R}$

(h_3) \qquad $\lim_{t \to +\infty} h'(t)$ *exists and is finite*

Then, there exists a non-trivial solution $(u, v) \in H_0^1 \times H_0^1$ *to*

$$-\Delta u = \lambda u - \delta v + f(u) \qquad in\ \Omega \qquad (1.1)$$
$$-\Delta v = \delta u + \gamma v + h(v) \qquad in\ \Omega \qquad (1.2)$$
$$u = v = 0 \qquad\qquad on\ \partial\Omega$$

provided,

(Λ) \qquad $\gamma < \lambda_1,\ 0 < \lambda < \lambda_j + \dfrac{\delta^2}{\lambda_j - \gamma} \quad \forall j$

If, in addition, we assume

(Λ_1) $\qquad\qquad\qquad\qquad 0 < \delta < \lambda_1 - \gamma$

(Λ_2) $\qquad\qquad\qquad\qquad 2\delta + \gamma \leq \lambda$

then (1.1)-(1.2) has a solution (u,v) satisfying $u \geq v > 0$.

In what follows (see Th. 4.9 below), we will improve our result, as far as growth conditions on $h(\cdot)$ are concerned, making use of the a priori estimates and the results of Section 3.

Finally, we will see that a more delicate situation occours if $N = 3$ (see Section 5 and [6] for the scalar case).

In order to prove th. 4.1 we first state a variational principle and we analyse its geometric features.

Setting,

$$< Uv, \varphi >:= \int_\Omega \nabla v \nabla \varphi - \gamma v \varphi - h(v)\varphi, \quad v, \varphi \in H_0^1$$

one can easily see, using assumptions $(h_1) - (h_2)$ and $\gamma < \lambda_1$, that $U(\cdot)$ is a strongly monotone operator. In addition using standard tools one can easily find that the operator $T = U^{-1}G$ with

$$< Gu, \varphi >:= \int_\Omega u\varphi \quad, \quad u, \varphi \in H_0^1$$

satisfies the following properties; which we collect in

LEMMA 4.2.—

(i) $\qquad (u_n) \in H_0^1 : u_n \xrightarrow{H_0^1} u \implies Tu_n \xrightarrow{H_0^1} Tu$

(ii) $\qquad \exists\, C > 0 : \|Tu\| \leq C|u|_{2N/N+2}, \ u \in H_0^1$

(iii) $\qquad \exists\, C > 0 : 0 < \int_\Omega Tu\, u \leq C|u|^2_{2N/N+2}, \ u \in H_0^1 - \{0\}$

Now, let us denote

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\gamma}{2}\int_\Omega v^2 - \int_\Omega H(v), \qquad v \in H_0^1$$

$$\Gamma(u) := < GT(u), u > -J(Tu) = \int_\Omega U^{-1}G(u)u - J(Tu), u \in H_0^1.$$

Notice that

$$< \nabla Jv, v >=< Uv, v >$$

and hence

$$\Gamma(u) = <Tu, UT(u)> -J(T(u)) = <\nabla J(Tu), T(u)> -J(T(u)) =$$
$$= J(T(u)) + 2\int_\Omega H(Tu) - \int_\Omega h(T(u))T(u) \geq \frac{1}{2}(1 - \frac{\gamma}{\lambda_1})\|Tu\|^2$$

because of $(h_1) - (h_2)$.

Using the above facts, we easily get the following :

A variational principle

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \frac{1}{2*}\int_\Omega |u|^{2^*} + \Gamma(\delta u), \quad u \in H_0^1$$

is a C^2-functional on H_0^1, with

$$<\nabla E(u), \varphi> = \int_\Omega \nabla u \nabla \varphi - \lambda u\varphi - |u|^{2^*-2}u\varphi + \delta\int_\Omega U^{-1}G(\delta u)\varphi, \quad \varphi \in H_0^1$$

Henceforth, if $\nabla E(u) = 0$, the pair (u, v) with $v := T(\delta u)$ is a solution to (0.1)-(0.2). This easily follows from the relation $\nabla J(v) = U(v)$, so that

$$<\nabla \Gamma(u), \varphi> = <GT(u), \varphi> + <GT'(u)(\varphi), u> -$$
$$- <G(u), T'(u)\varphi> = <GT(u), \varphi>, \quad \varphi \in H_0^1.$$

Thus if $v = T(\delta u)$ and $\nabla E(u) = 0$, we have

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \lambda u\varphi + |u|^{2^*-2}u\varphi - \delta v\varphi, \quad \varphi \in H_0^1$$
$$\int_\Omega \nabla v \nabla \varphi = \int_\Omega \delta u\varphi + \gamma v\varphi + h(v)\varphi, \quad \varphi \in H_0^1$$

i.e. (u, v) is a solution to (1.1)-(1.2).

Remark 4.3. —

(i) If (u, v) is an H_0^1-solution to (0.1)-(0.2) then

$$E(u) = \frac{1}{N}\int_\Omega |u|^{2^*} - \int_\Omega \left(\frac{1}{2}h(v)v - H(v)\right)$$

(ii) If $v = T(u)$ then for $u \in H_0^1$ it follows that

$$\Gamma(u) - \frac{1}{2}<\nabla \Gamma(u), u> = -\frac{1}{2}\int_\Omega h(v)v + \int_\Omega H(v) \geq 0$$

In the following lemma we describe the geometric properties of "mountain-pass" type of the energy $E(\cdot)$.

LEMMA 4.4. —

(i) $\quad E(0) = 0,\ \nabla E(0) = 0$ *and*

$$\exists\ C > 0 :\ < E''(0)\varphi, \varphi > \ \geq\ C\|\varphi\|, \varphi \in H_0^1$$

(ii) $\quad E(tu) \xrightarrow{t \to +\infty} -\infty\ ,\ u \in H_0^1 - \{0\}.$

Proof. — Clearly $E(0) = 0,\ \nabla E(0) = 0$ because $U(0) = 0$. Furthermore $E''(0) = I - \lambda G + \delta^2 G(I - \gamma G)^{-1}G$ has positive eigenvalues given by $\lambda_j^{-2}(\lambda_j + \delta^2(\lambda_j - \delta)^{-1} - \lambda)$, which are bounded away from zero because of (Λ)

(ii) Since, in view of Lemma 4.2 and the positivity of J

$$\Gamma(u) = \int_\Omega T(u)u - J\big(T(u)\big) \leq C|u|^2_{2N/N+2}$$

we immediately get $E(tu) \to -\infty$ as $t \to +\infty$. To make use of the geometric informations given by Lemma 4.4 we need the following crucial compactness property of $E(\cdot)$.

Denote by

$$S := \inf_{u \neq 0} \frac{\|u\|^2}{|u|^2_{\frac{2N}{N+2}}}$$

the best Sobolev constant, we have :

LEMMA 4.5. — *Let* $(u_n) \in H_0^1$ *be such that* $\nabla E(u_n) \to 0$ *as* $n \to +\infty$. *Then* (u_n) *has an* H_0^1-*convergent subsequence provided* $\sup_n E(u_n) < \dfrac{1}{N} S^{N/2}$.

The proof of such (P.S.) condition, which makes use of arguments from [18] (see also [23]), will be postponed, as well as the following estimate of the "mountain pass level" : Setting

$$\sum := \Big\{p \in C\big([0,1] : H_0^1\big) : p(0) = 0,\ E(p(1)) < 0\Big\}$$

we have :

LEMMA 4.6. — *Let $N \geq 4$ and $\lambda > 0$. Then*

$$C = \inf_{p \in \Sigma} \sup_{t \in [0n,1]} E(p(t)) < \frac{1}{N} S^{N/2}$$

Proof of Th. 4.1. — In view of Lemmas 4.4-4.6, the "mountain-pass lemma" (see [1]) applies to get a critical point of $E(\cdot)$ at positive energy C. Our variational principle implies the result.

In order to get existence of *positive* solution under the additional assumption $(\Lambda_1) - (\Lambda_2)$, we have to slight modify our arguments, first replacing $|u|^{2^*-2}u$ in (0.1) and $h(v)$ in (0.2) respectively with

$$f(t) := |t^+|^{2^*-1} \text{ and } \widehat{h}(t) = \begin{cases} h(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The new system

$$-\Delta u = \lambda u + |u^+|^{2^*-1} - \delta v \qquad \text{in } \Omega \qquad (0.1)'$$

$$-\Delta v = \delta u + \gamma v + \widehat{h}(v) \qquad \text{in } \Omega \qquad (0.2)'$$

$$u = v = 0 \qquad \text{on } \partial\Omega$$

has only L^∞-positive solutions by Prop. 1.4. In turns, solutions to (0.1)'-(0.2)' can be obtained as critical points of the modified functional

$$\widehat{E}(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |u|^2 - \frac{1}{2^*} \int_\Omega |u^+|^{2^*} + \widehat{\Gamma}(\delta u)$$

where $\widehat{\Gamma}$ is defined in the obvious way. Slight modifications of our arguments, show that Lemma 4.4 (with (ii) only for positive $u's$) - 4.5 and 4.6 hold with E replaced by $\widehat{E}$, and hence the result follows.

Now we go back to the proofs of Lemma 4.5 and 4.6.

Proof of Lemma 4.5. — It will require several steps :

Step 1. — $\sup_n \|u_n\| < +\infty$, so that, w.l.o.g. : $u_n \rightharpoonup u$, $u_n \to u$ a.e; and $u_n \to u$ in L^α for every $\alpha \in (1, 2^*)$.

Furthermore there exists (see [19]) positive measures μ and ν and numbers $a_j \geq 0$ such that for every $\varphi \in L^\infty(\Omega)$, it results

$$\int_\Omega |\nabla u_n|^2 \varphi \, dx \to \int_\Omega \varphi \, d\mu$$

$$\int_\Omega |u_n|^{2^*} \varphi \, dx \to \int_\Omega \varphi \, d\nu \qquad (4.1)$$

$$\int_\Omega d\nu = \int_\Omega |u|^{2^*} dx + \sum a_j \qquad (4.2)$$

$$\int_\Omega d\mu \geq \int_\Omega |\nabla u|^2 \varphi \, dx + S \sum a_j^{\frac{N-2}{N}} \qquad (4.3)$$

Step 2.— $\quad a_j \not\equiv 0 \Longrightarrow \sum a_j \geq S^{N/2}$

Step 3.— $\quad \lim_n \inf E(u_n) \leq \dfrac{1}{N} \sum a_j.$

The lemma will follow noticing that assumption $\sup E(u_n) < 1/N\ S^{N/2}$ and Steps 2,3 imply $a_j \equiv 0$, (i.e. $\int_\Omega |u_n|^{2^*} \to \int_\Omega |u|^{2^*}$ in view of (4.1)-(4.2)) which joint with $u_n \rightharpoonup u$ in L^{2^*} imply $u_n \to u$ in L^{2^*}. Finally, setting

$$< Nu, \varphi > := \int_\Omega |u|^{2^*-2} u\varphi, \ \varphi \in H_0^1$$

we easily see from

$$\nabla E(u_n) = u_n - \lambda G(u_n) - N(u_n) + \delta GT(\delta u_n) \xrightarrow{n\to\infty} 0$$

that $u_n \to u$ in H_0^1.

Proof of Step 1. — By Assumption we have

$$0(1) + 0(1)\|u_n\| = E(u_n) - \frac{1}{2} < \nabla E(u_n), \ u_n > = \frac{1}{N} \int_\Omega |u_n|^{2^*} + \Gamma(\delta u_n) -$$

$$- \frac{1}{2} < \nabla\Gamma(\delta u_n), \ \delta u_n > \geq \frac{1}{N} \int_\Omega |u_n|^{2^*}$$

and hence

$$E(u_n) \geq \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2} |u_n|^2 - \frac{1}{2^*} \int_\Omega |u_n|^{2^*}$$

implies $\sup_n \|u_n\| < +\infty$.

Proof of Step 2. — Passing to the limit in

$$< \nabla E(u_n), \varphi > = \int_\Omega \nabla u_n \nabla\varphi - \lambda u_n\varphi - \int_\Omega |u_n|^{2^*-2} u_n\varphi +$$

$$+ \delta \int_\Omega T(\delta u_n)\varphi \ , \ \varphi \in H_0^1$$

we see that, taking in the limit $\varphi = u$,

$$\int_\Omega \left(|\nabla u|^2 - \lambda u^2 - |u|^{2^*} \right) + \delta T(\delta u)u = 0 \tag{4.4}$$

On the other hand, since T is completely continuous and

$$< \nabla E(u_n), u_n > = \|u_n\|^2 - \lambda|u_n|^2 - |u_n|_{2^*}^{2^*} + \delta \int_\Omega T(\delta u_n)u_n \xrightarrow{n\to\infty} 0$$

we get

$$\mu(\Omega) = \lambda \int_\Omega u^2 dx + \nu(\overline{\Omega}) - \delta \int_\Omega T(\delta u)u \ dx \qquad (4.5)$$

But $\mu(\overline{\Omega}) \geq \int_\Omega |\nabla u|^2 + S\Sigma a_j^{N-2/N}$ by (4.3), so that using (4.4) and (4.5) we conclude

$$S\left(\sum a_j\right)^{N-2/N} \leq S \sum a_j^{N-2/N} \leq - \int_\Omega u^{2^*} + \nu(\overline{\Omega}) = \sum a_j.$$

Proof of Step 3. — From (4.2)-(4.3) and (4.5) we obtain

$$\lim_n E(u_n) = \frac{1}{2} \mu(\overline{\Omega}) - \frac{\lambda}{2} \int_\Omega u^2 - \frac{1}{2*} \nu(\overline{\Omega}) + \Gamma(\delta u) =$$

$$= \frac{1}{N} \nu(\overline{\Omega}) + \Gamma(\delta u) - \frac{1}{2} < \nabla\Gamma(\delta u), \delta u > \geq \frac{1}{N} \sum_j a_j$$

Proof of Lemma 4.6. — It is enough to find a point $\overline{u}$ such that $\sup_{t\geq 0} E(t\overline{u}) < \frac{1}{N}S^{N/2}$. For this purpose, we will use "instanton" like functions employed by BREZIS-NIRENBERG [6].

For a given cut-off function $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ in a neighbourhood of zero, we set

$$u_\epsilon(x) = K \frac{\epsilon^{\frac{N-2}{N}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}} \varphi(x) \quad x \in \Omega, \ \epsilon > 0.$$

It is well know that (see BREZIS-NIRENBERG [6]) for a suitable choice of the normalization constant K, one has the following estimates

$$\|u_\epsilon\|^2 = S^{N/2} + 0(\epsilon^{N-2}) \qquad (4.6)$$

$$|u_\epsilon|_{2^*}^{2^*} = S^{N/2} + 0(\epsilon^N) \qquad (4.7)$$

$$|u_\epsilon|_p^2 = \begin{cases} 0(\epsilon^{N-2}) & \text{if } 1 \leq p < \dfrac{N}{N-2} \\[2mm] 0(\epsilon^{\frac{2N}{p}-(N-2)}) & \text{if } p > \dfrac{N}{N-2} \\[2mm] 0(\epsilon^{N-2})|\log \epsilon|^{\frac{2(N-2)}{N}} & \text{if } p = \dfrac{N}{N-2} \end{cases} \qquad (4.8)$$

(here $0(\epsilon^\alpha)$ mean $0(\epsilon^\alpha)\epsilon^{-\alpha}$ bounded, and bounded away from zero).

In particular

$$|u_\epsilon|_2^2 = \begin{cases} 0(\epsilon^2) & \text{if } N > 4 \\ 0(\epsilon^2) \, |\log \epsilon| & \text{if } N = 4 \end{cases} \tag{4.9}$$

$$|u_\epsilon|_p^2 = o\big(|u_\epsilon|_2^2\big) \qquad \text{if } 1 < p < 2 \tag{4.10}$$

$$\|u_\epsilon\|^2 = S^{N/2} + o\big(|u_\epsilon|_2^2\big)$$

$$|u_\epsilon|_{2^*}^{2^*} = S^{N/2} + o\big(|u_\epsilon|_2^2\big) \tag{4.11}$$

Now, if $E(t_\epsilon u_\epsilon) = \max_{t \geq 0} E(t u_\epsilon)$, then $d/dt \, E(t u_\epsilon)_{t=t_\epsilon} = 0$, i.e. $t_\epsilon u_\epsilon$ satisfies

$$t_\epsilon^2 \big(\|u_\epsilon\|^2 - \lambda |u_\epsilon|_2^2\big) + \int_\Omega T(\delta t_\epsilon u_\epsilon) \delta t_\epsilon u_\epsilon = t_\epsilon^{2^*} \, |u_\epsilon|_{2^*}^{2^*}.$$

Using (4.11) and Lemma 4.2-(iii) we get

$$t_3^2 \big(S^{N/2} + o\big(|u_\epsilon|_2^2\big)\big) - \lambda |u_\epsilon|_2^2 \leq t_\epsilon^{2^*} \big(S^{N/2} + o\big(\|u_\epsilon\|_2^2\big)\big) \leq$$
$$\leq t_\epsilon^2 \big(S^{N/2} + o_1\big(|u_\epsilon|_2^2\big) - \lambda |u_\epsilon|_2^2 + o_2\big(|u_\epsilon|_2^2\big)\big)$$

and hence,

$$t_\epsilon^{2^*-2} = 1 - \lambda S^{-N/2} |u_\epsilon|_2^2 + o\big(|u_\epsilon|_2^2\big) \tag{4.12}$$

Thus, writing $w_\epsilon = t_\epsilon u_\epsilon \; v_\epsilon = T(\delta w_\epsilon)$ from

$$E(w_\epsilon) = \frac{1}{N} \int_\Omega |w_\epsilon|^{2^*} dx - \int_\Omega \Big(\frac{1}{2} \, h(v_\epsilon) v_\epsilon - H(v_\epsilon)\Big) dx,$$

and since $-h(s)s \leq \text{const.} S^2$ by (h_3), we obtain

$$E(t_\epsilon u_\epsilon) \leq \frac{1}{N} \Big(S^{N/2} - \frac{N}{2} \, \lambda |u_\epsilon|_2^2\Big) + o\big(|u_\epsilon|_2^2\big)$$

Using (4.12), inequality $\|Tu\| \leq C|u|_{2N/N+2}$ and (4.10) we finally obtain

$$E(t_\epsilon u_\epsilon) \leq \frac{1}{N} \Big(S^{N/2} - \frac{1}{2} \, \lambda |u_\epsilon|_2^2\Big) + o\big(|u_\epsilon|_2^2\big) < \frac{1}{N} \, S^{N/2}$$

provided $|u_\epsilon|_2^2$ is sufficiently small, i.e. if ϵ is small.

Remark 4.7. — In dealing with the modified functional $\widehat{E}$, we need a few modification to get Lemma 4.5. As far as Step 1 is concerned, we get instead of (4.7) :

$$\int_\Omega (u_n^+)^{2^*} \leq C(1 + \|u_n\|) \tag{4.13}$$

$$\|u_n^+\| \geq < \nabla E(u_n), u_n^+ > \geq \|u_n^+\|^2 - \lambda \left(\int_\Omega |u_n^+|^{2^*} \right)^{\frac{N-2}{N}} - \int_\Omega |u_n^+|^{2^*} -$$

$$- C|u_n|_{\frac{2N}{N+2}} |u_n^+|_{\frac{2N}{N+2}}$$

and then

$$\frac{1}{2} \|u_n^+\|^2 \leq \lambda \|u_n\|^{\frac{N-2}{N}} + C_1 \|u_n\| + C_2 \|u_n\|^{1+\frac{N-2}{2N}} + C_3 \qquad (4.14)$$

Assuming, by contradiction, $\|u_n\| \to +\infty$, we get from (4.14) $\widehat{u}_n := \dfrac{u_n^+}{\|u_n\|} \to 0$ in H_0^1, and consequently, if $\overline{u} := w - \lim \dfrac{u_n}{\|u_n\|}$, it results $\overline{u} \leq 0$. Furthermore, from

$$\nabla E(u_n) = u_n - \lambda G u_n - N u_n + \delta G T(\delta u_n) \to 0 \qquad (4.15)$$

here

$$< Nu, \varphi > := \int_\Omega (u^+)^{2^*} \overline{\varphi}^1 \ , \ \|Nu\| \leq |u^+|_{2_*}^{2_*-1} \ , \quad \varphi \in H_0^1$$

we get

$$\overline{u} = \lim_n \left(\lambda G\left(\frac{u_n}{\|u_n\|} \right) + \frac{\delta G T(\delta u_n)}{\|u_n\|} \right) \qquad (4.16)$$

strongly in H_0^1, because $\|T(\delta u_n)\| \leq \cos t |u_n|$: thus, setting

$$\overline{v} := w - \lim \frac{T(\delta u_n)}{\|u_n\|},$$

$(\overline{u}, \overline{v})$ satisfy

$$\overline{u} = \lambda G(\overline{u}) + \delta G(\overline{v}), \quad \overline{u} \neq 0 \qquad (4.17)$$

$(i.e)$

$$-\Delta \overline{u} = \lambda \overline{u} - \delta \overline{v}$$

On the other hand, since

$$\int_\Omega \nabla v_n \nabla \varphi = \delta \int_\Omega u_n \varphi + \gamma \int_\Omega v_n \varphi + \int_\Omega \widehat{h}(v_n) \varphi, \quad \varphi \in H_0^1$$

by definition of $v_n := T(\delta u_n)$, we also have, dividing by $\|u_n\|$ and passing to the limit (using (h_3)) :

$$\int_\Omega \nabla \overline{v} \nabla \varphi = \delta \int_\Omega \overline{u} \varphi + \gamma \int_\Omega \overline{v} \varphi + \int_\Omega \widehat{h}'(+\infty) \overline{v}^+ \varphi, \quad \varphi \in H_0^1$$

But since $\overline{u} \leq 0$, by the maximum principle $\overline{v} \leq 0$ and hence

$$-\Delta \overline{v} = \delta \overline{u} + \gamma \overline{v} \qquad (4.18)$$

Using assumption $(\Lambda)_1 - (\Lambda)_2$, we see that (4.17)-(4.18) ha only the trivial solution, i.e. $\overline{u} = \overline{v} = 0$, a contradiction. This proves step 1 for the energy of the modified problem. As for step 2, it is enough to observe that if $u_n \to u$, $v_n := T(\delta u_n) \to v = T(\delta u)$, (u,v) solve the system (0.1)'-(0.2)' and hence they are positive functions in view of assumptions $(\Lambda_1) - (\Lambda_2)$ and Prop. 1.2. But then we still have (4.4), while (4.5) follows observing that $\lim_n \int_\Omega |u_n^+|^{2^*} = \lim_n \int_\Omega |u_n|^{2^*}$. In fact,

$$< \nabla e(u_n), u_n^- > = \int_\Omega |\nabla u_n^n|^2 - \lambda |u_n^-|^2 + \delta \int_\Omega T(\delta u_n) u_n^- \to 0$$

implies $\int_\Omega |\nabla u_n^-|^2 \to 0$ because $u_n^- \to \overline{u} = 0$ in H_0^1. The same remark allows to work out Step 3, and thus the proof of Lemma 4.5, for the modified problem, is complete.

Remark 4.8. — Step 1 in the proof of Lemma 4.5 can be obtained replacing $(h)_3$ with the more general assumption

(h) $\qquad \exists\, a,b : |h(s)| \leq a + b|s|^\alpha$, $1 \leq \alpha < 2^* - 1$

Similarly, assumption (h_1) could be replaced by

$(h)_\alpha$ $\qquad$ There exists $\alpha \in (1, 2^*) : \sup_s s^{-\alpha} \left(\dfrac{1}{2}\, sh(s) - H(s) \right) < +\infty$

We want to show now how to drop the growth assumption on $h(\cdot)$, making use of Theorem 4.1 and of the a priori bounds given by Proposition 3.4.

THEOREM 4.9. — *Let $N \geq 4$, and assume (Λ_1) and (Λ_2). Then if*

(h_1) $\qquad\qquad h(0) = 0 = h'(0)$, $h'(s) \leq 0$, $s \in \mathbf{R}$

(h_5) $\qquad\qquad \dfrac{h(s)}{s} \geq h'(s)$, $s \geq 0$

are satisfied, the problem

$$- \Delta u = \lambda u - \delta v + |u|^{2^* - 2} u$$
$$- \Delta v = \delta u + \gamma v + h(v)$$

has a non trivial solution $(u, v) \in H_0^1(\Omega)^2$. *If in addition*

(Λ_3) $\qquad\qquad\qquad\qquad \gamma + 2\delta \leq \lambda,$

holds, then the above system has a solution (u, v) *satisfying*

$$u \geq \frac{\lambda}{2\delta}\ v > v > 0.$$

Remark 4.10. — $(h_5) \Longrightarrow (h_2)$.

Proof of Theorem 4/9. — Given $M > 0$, let us consider the truncated nonlinearity :

$$h_M(s) = \begin{cases} h(s) & \text{if } |s| \leq M \\ h(M) + (s - M)h'(M) & \text{if } s \geq M \\ h(-M) + (s + M)h'(-M) & \text{if } s \geq -M \end{cases}$$

Notice that h_M satisfies (h_5), and hence (h_2) too (h_3). Thus Theorem 4.1 applies to get solution (u_M, v_M) of the modified system :

$$- \Delta u = \lambda u - \delta v + |u|^{2^* - 2} u$$
$$- \Delta v = \delta u + \gamma v + h_M(v)$$

The proof will be now performed in several steps. Indicating with E_M the energy corresponding to h_M, (u_M, v_M) can be chosen such that

Step 1. $\qquad\qquad\qquad \sup_M \big(\|u_M\| + \|v_M\| \big) < +\infty$

Step 2. $\qquad\qquad\qquad \sup_M E_M(u_M) < \dfrac{S^{N/2}}{N}$

Step 3. $\qquad\qquad\qquad (u_M)$ is precompact in L^{2^*}

Step 4. $\qquad\qquad\qquad \sup_M |v_M|_\infty < +\infty$

Clearly step 4 implies the result.

We will chosse (u_M, v_M) as given by Theorem 4.1, so that they satisfy the inequality $E_M(u_M) < 1/N\ S^{N/2}$.

Proof of Step 1.— Here, and in what follows we will use the obvious notation :

$$H_M(s) = \int_0^s h_M(\sigma)d\sigma, \ u_M(v) = v - \gamma G(v) - N_M(v), \qquad v \in H_0^1$$

$$< N_M(v), \varphi > := \int_\Omega h_M(v)\varphi, \ T_M = U_M^{-1}G, \qquad v \in H_0^1$$

$$J_M(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\gamma}{2}\int_\Omega v^2 - \int_\Omega H_M(v), \qquad v \in H_0^1$$

$$\Gamma_M(u) = \int_\Omega T_M(u)u - J_M(T_M(u)), \qquad u \in H_0^1$$

Notice that $v_M = T_M(\delta u_M)$ so that $\int_\Omega v_M u_M = \int_\Omega T_M(\delta u_M)u_M$.

Now, given u_M, v_M as above, we have

$$\frac{S^{N/2}}{N} \geq E_M(u_M) \geq \frac{1}{N}\int_\Omega |u_M|^{2^*}$$

and also,

$$\|u_M\|^2 = \lambda|u_M|^2 + \int_\Omega |u_M|^{2^*} - \delta\int_\Omega v_M u_M$$

and thus, since $\int_T u(u)u > 0$ ($u \neq 0$,

$$\|u_M\|^2 \leq S^{N/2} + c\lambda s^{\frac{N-2}{N}}.$$

Proof of Step 2.— It is enough to remark that (4.19) rewrites :

$$t_\epsilon^{2^*-2} = 1 - \lambda|u_\epsilon|_2^2 + 0_M(|u_\epsilon|_2^2) \text{ with } \frac{0_M(|u_\epsilon|_2^2)}{|u_\epsilon|_2^2} \xrightarrow{\epsilon \to 0} 0$$

uniformly with respect to M, because the inequality $\int_\Omega T_M(\delta t_\epsilon u_\epsilon)\delta t_\epsilon u_\epsilon \leq ct_\epsilon^2|u_\epsilon|_p^2$, $\left(p \in \left[\frac{2N}{N+2}, 2^*\right]\right)$ holds true with c independent on M. As a consequence, following the proof of Lemma 4.6, we get

$$E_M(t_\epsilon u_\epsilon) \leq \frac{S^{N/2}}{N} - \frac{\lambda}{2}S^{N/2}|u_\epsilon|_2^2 + 0_M(|u_\epsilon|_2^2)$$

which implies

$$\sup_M E_M(t_\epsilon u_\epsilon) < \frac{1}{N}S^{N/2}$$

if ϵ is sufficiently small.

Proof of Step 3. — This is a slight modification of the arguments used in the proof of Lemma 4.5. We already know that, given M_j, passing eventually to a subsequence, we have $u_j := u_{M_j} \rightharpoonup u$, $v_j := v_{M_j} \rightharpoonup v$ for some u and v, because $\|v_{M_j}\| \le c\|u_{M_j}\|$ by lemma 4.2-ii, with some c independent on M_j. Following the proof of Lemma 4.5, we have just to prove that

$$\int_\Omega \left(|\nabla u|^2 - \lambda|u|^2 - |u|^{2^*} + \delta vu = 0 \right) \tag{4.8}$$

$$\mu(\overline{\Omega}) = \lambda \int_\Omega u^2 + \nu(\overline{\Omega}) - \delta \int_\Omega uv \tag{4.9}$$

(where $\mu(\cdot)$ an $\nu(\cdot)$ are limits, in the sense of measures, respectively of $|\nabla u_j|^2$ and $|u_j|^{2^*}$, and $\nu = |u|^{2^*} + \Sigma\, a_j \delta x_j$)

$$\liminf_j E_{M_j}(u_j) \ge \frac{\Sigma a_j}{N}. \tag{4.10}$$

Now, (4.8) and (4.9) easily follow from

$$\int_\Omega \nabla u_j \nabla u = \lambda \int_\Omega u_j u + \int_\Omega |u_j|^{2^*-2} u_j u - \delta \int_\Omega v_j u$$

and

$$\int_\Omega |\nabla u_j|^2 = \lambda \int_\Omega u_j^2 + \int_\Omega |u_j|^{2^*} - \delta \int_{\Omega_1} u_j v_h$$

and the fact that $v_j \rightharpoonup v$, $u_j \rightharpoonup u$ in H_0^1. As far (4.24), using Remark 4.3 and (h_2), we see that

$$E_j(u_j) \ge \frac{1}{N} \int_\Omega |u_j|^{2^*} \to \frac{1}{N} \left(\int_\Omega |u|^{2^*} + \Sigma\, a_j\right)$$

Proof of Step 4. — This is just a consequence of the precompactness of $\{u_M\}$ in H_0^1 and $\sup_M \|v_M\| < +\infty$ in view of Proposition 3.4.

5. The case N = 3

In this section we concentrate our attention on the existence of solutions to the very special system

$$\begin{aligned} -\Delta u + \lambda u + u^5 - \delta v \qquad & \text{in } \Omega \subset \mathbf{R}^3 \\ -\Delta v = \delta u \qquad & \text{in } \Omega \subset \mathbf{R}^3 \\ u = v = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

or equivalently to the problem

$$-\Delta u - \lambda u + \delta^2 B(u) = u^5 \quad \text{in } \Omega$$
$$u = 0 \qquad\qquad\qquad \text{on } \partial\Omega$$

(5.2)

obtained from (4.1) by inverting the second equation (i.e. $< Bu, \varphi > = \int_\Omega u\varphi$, $\varphi \in H_0^1$).

We will follow closely the approach of Bahri-Coron (see also BREZIS [4], SCHOEN [22], MC LEOD [20]) which describes the important role played by the regular part of the Green function of the linear operator involved in the given nonlinear problem, in estimating

$$\widehat{S}_\lambda := \inf_{\substack{u \in H_0^1 \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2 + \delta^2 \int_\Omega B(u)u}{\left(\int_\Omega u^6\right)^2}$$

For definitness, let $G(\cdot, \cdot)$ be the Green function relative to

$$-\Delta z - \lambda z + \delta^2 B(z) = h \quad \text{in } \Omega$$
$$z = 0 \qquad\qquad\qquad \text{on } \partial\Omega$$

(5.3)

(here $h \in L^2$, $0 < \lambda < \lambda_1 + \dfrac{\delta^2}{\lambda_1}$ and $\delta > 0$) i.e.

$$z(x) = \int_\Omega G(x, y)h(y)dy.$$

As usual one can split $G(\cdot, \cdot)$ into a singular and regular part

$$G(x, y) = \frac{1}{4\pi|x - y|} + g(x, y)$$

where $g(x, y)$ is, for every $x \in \Omega$, the only solutions to

$$-\Delta g(x, \cdot) - \lambda g(x, \cdot) + \delta^2 Bg(x, \cdot) = \frac{1}{4\pi}(\lambda - \delta^2 B)\left(\frac{1}{|x - \cdot|}\right) \quad \text{in } \Omega$$
$$g(x, \cdot) = -\frac{1}{4\pi|x - \cdot|} \qquad\qquad\qquad\qquad\qquad \text{in } \partial\Omega$$

(5.4)

PROPOSITION 4.1. — *Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain such that $0 \in \Omega$.*

If
$$g(0,0) > 0$$

then
$$\widehat{S}_\lambda < S,$$

and consequently (4.1) is solvable.

Proof. — Since the proof is the same as in BREZIS [4] we only give the ideas involved.

Let $\epsilon > 0$ and consider

$$U_\epsilon(x) = \frac{\sqrt{\epsilon}}{(\epsilon^2 + |x|^2)^{1/2}} \quad x \in \mathbf{R}^3,$$

a direct calculation shows that

$$-\Delta U_1 = 3U_1^5 \qquad \text{in } \mathbf{R}^3.$$

We consider as "test" functions for estimating $\widehat{S}_\epsilon$ the solution to

$$
\begin{aligned}
-\Delta \varphi_\epsilon - \lambda \varphi_\epsilon + \delta^2 B(\varphi \epsilon) &= -\Delta U_\epsilon & \text{in } \Omega \\
\varphi_\epsilon &= 0 & \text{on } \partial\Omega
\end{aligned}
\tag{5.5}
$$

By introducing

$$h_\epsilon(x) = \frac{\varphi_\epsilon - U_\epsilon}{\sqrt{\epsilon}}$$

we see that h_ϵ satisfies

$$
\begin{aligned}
-\Delta h_\epsilon - \lambda h_\epsilon + \delta^2 B(h_\epsilon) &= \frac{\lambda}{(\epsilon^2 + |x|^2)^{1/2}} - \delta^2 B\left(\frac{1}{(\epsilon^2 + |x|^2)^{1/2}}\right) \text{ in } \Omega \\
h_\epsilon &= -\frac{1}{(\epsilon^2 + |x|^2)^{1/2}} \qquad \text{on } \partial\Omega
\end{aligned}
\tag{5.6}
$$

Since $\dfrac{1}{(\epsilon^2 + |x|^2)^{1/2}} \rightarrow \dfrac{1}{|x|}$ in L^2 as $\epsilon \to 0$, elliptic theory gives

$$h_\epsilon \to h_0 \text{ as } \epsilon \to 0$$

uniformly in $\overline{\Omega}$, where h_0 is the solution to

$$
\begin{aligned}
-\Delta h_0 - \lambda h_0 + \delta^2 B(h_0) &= \frac{\lambda}{|x|} - \delta^2 B\left(\frac{1}{|x|}\right) & \text{in } \Omega \\
h_0 &= -\frac{1}{|x|} & \text{on } \partial\Omega
\end{aligned}
\tag{5.7}
$$

that is $h_0(x) = 4\pi g(x,0)$. At this point the calculations are exactly the same as in BREZIS [4]. So we shall omit them.

COROLLARY 5.2. — *Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain such that $0 \in \Omega$.*

Assume that

$$(i) \qquad 2\delta < \lambda < \lambda_1 + \frac{\delta^2}{\lambda_1} \qquad \lambda_1 > \delta$$

and

$$(ii) \qquad \left(\lambda + \sqrt{\lambda^2 - 4\delta^2}\right)g_+(0,0) - \left(\lambda - \sqrt{\lambda^2 - 4\delta^2}\right)g_-(0,0) > 0$$

$\Big($ *here $g_\pm$ is the regular part of the Green function of the problem $-\Delta G - \left(\dfrac{\lambda \pm \sqrt{\lambda^2 - 4\delta^2}}{2}\right)G = \delta(\cdot - y)\Big)$ holds.*

Then (5.1) has a positive solution.

Proof. — The positivity of the solution is guaranted by (i). (see Prop. 1.1). For the existence part, it is sufficient to observe that if (i) holds then the solution of (5.7) is exactly

$$h_0(x) = \left(\frac{\lambda + \sqrt{\lambda^2 - 4\delta^2}}{2\sqrt{\lambda^2 - 4\delta^2}}\right) g_+(x,0) - \left(\frac{\lambda - \sqrt{\lambda^2 - 4\delta^2}}{2\sqrt{\lambda^2 - 4\delta^2}}\right) g_-(x,0). \quad (5.8)$$

Remark 5.3. — In the limiting case (from the point of view of the maximum principle) $\lambda = 2\delta$, for the existence of positive solution it is sufficient to assume that $g(0,0) > 0$ where in this case g is the regular part of G, defined by

$$\begin{aligned} -\Delta G - \frac{\lambda}{2}G &= \delta(\cdot - y) \qquad \text{in } \Omega \\ G &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned} \quad (5.9)$$

Remark 5.4. — In the special case $\Omega = B_1(0)$ and $\lambda > 2\delta$, $\lambda_1 > \delta$, we know by Prop. 1.3 that all positive solutions of (5.1) must be radial. In this case (ii) of Corollary 5.2 reads :

$$\left(\lambda + \sqrt{\lambda^2 - 4\delta^2}\right)^3 ctg \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4\delta^2}}{2}} <$$

$$< \left(\lambda - \sqrt{\lambda^2 - 4\delta^2}\right)^3 ctg\sqrt{\frac{\lambda - \sqrt{\lambda^2 - 4\delta^2}}{2}} \quad (5.10)$$

So we can see that in view of the nonexistence results given by Proposition 2.3, (5.10) is "almost sharp" in the sense that there is only a small interval of $\lambda's$ in which we do not know what happen. However, since by using the original technique of BREZIS-NIRENBERG [6] we can arrive at the same solvability condition (5.10), we can conjecture that (5.10) is sharp.

Remark 5.5. — If $\Omega = B_1(0)$ and $\lambda = 2\delta\ \lambda_1 > \delta$ (here $\lambda_1 = \pi^2$) then (5.1) has a positive radial solution if $\frac{\pi^2}{2} < \lambda < 2\pi^2$.

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