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A Cauchy problem for $u_t - \Delta u = u^p$ with $0 < p < 1$. Asymptotic behaviour of solutions.

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RÉSUMÉ. — Nous prouvons l'existence, l'unicité et la régularité de solutions globales pour le problème de Cauchy

$$\begin{aligned} &u_t - \Delta u = u^p && \text{dans } (0,\infty)\times \mathbf{R}^n \\ &u(0,x) = u_0(x) \geq 0 && \text{dans } \mathbf{R}^n \\ &u \geq 0 && \text{dans } (0,\infty)\times \mathbf{R}^n \end{aligned}$$

avec donnée initiale non identiquement nulle dans un ensemble assez large de fonctions. On démontre que ces solutions sont toutes bornées inférieurement par $((1-p)t)^{1/(1-p)}$. On prouve finalement l'existence de solutions auto-similaires décrivant le comportement asymptotique des solutions lorsque t va à l'infini.

ABSTRACT. — We prove existence, uniqueness and regularity of global solutions for the Cauchy problem

$$\begin{aligned} &u_t - \Delta u = u^p && \text{in } (0,\infty)\times \mathbf{R}^n \\ &u(0,x) = u_o(x) \geq 0 && \text{in } \mathbf{R}^n \\ &u \geq 0 && \text{in } (0,\infty)\times \mathbf{R}^n \end{aligned}$$

for a large class of non identically null initial data. Solutions are shown to be uniformly above $\big((1-p)t\big)^{1/(1-p)}$. Finally the existence of self-similar solutions (not necessarely radial) describing the asymptotic behaviour of solutions as t goes to infinity is shown.

§ 1. Introduction and Main Results

Consider the following Cauchy problem :

$$\begin{aligned} &u_t - \Delta u = |u|^{p-1}u && \text{in } (0,\infty)\times \mathbf{R}^n \\ &u(x,0) = u_o(x) && \text{in } \mathbf{R}^n \\ &u \geq 0 && \text{in } \mathbf{R}^n \end{aligned} \qquad (P)$$

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where $0 < p < 1$.

When looking for existence (local and global), uniqueness and regularity results, several remarks are in order. The first one is about uniqueness. If the initial data u_o is identicaly zero, there is a continuum of nonnegative space independent solutions, namely the null solution and the family of solutions.

$$u_\tau(t,x) = \left((1-p)(t-\tau)_+\right)^{1/(1-p)}, \tau \geq 0 \tag{0.1}$$

The second one is about regularity. Let k be the integer part of $1/(1-p)$ if it is not an integer, and $[1/(1-p)] - 1$ if it is. Then u_τ is of class C^k in $[0,\infty)$, but is not of classe C^l in $(\tau - \epsilon, \infty)$ for any $l > k$ and $\epsilon > 0$. That is, even with a C^∞ initial value ($u_o \equiv 0$), solutions are only C^k.

This lack of uniqueness and regularity is due to the fact that the function u^p is not Lipschitz in any interval of the form $[0,\epsilon)$ for $0 < p < 1$ and $\epsilon > 0$. However, when the initial value is not identically zero, we are able to prove uniqueness and regularity of solutions of (P). The difficulties coming from the nonlinear term not being Lipschitz are overcome by a uniform lower estimate on the growth of the solutions given by the following.

LEMMA .— *Denote by $S(t)$ the semigroup of the heat equation on $\mathbf{R}^n$. Let u be a nonnegative function on $(O,T) \times \mathbf{R}^n (T > 0$ fixed) and u_o a nonnegative function on $\mathbf{R}^n$ not identically null such that*

$$u(t,x) \geq S(t)u_o + \int_o^t S(t-s)u^p(s,.)ds$$

makes sense and holds on $\mathbf{R}^n$. Then

$$u(t,x) > ((1-p)t)^{1/(1-p)} \quad \textit{for } 0 < t < T \textit{ and } x \in \mathbf{R}^n \tag{0.2}$$

Inequality (0.2) seems to mean that diffusion in the sublinear case is unifom all the way up to infinity, which is not the case for the linear and superlinear case. On the other hand, it is clear from (0.2) that even for an initial data with compact support there is no solution u of (P) such that $u(t,.)$ belongs to $L^q(\mathbf{R}^n)$ for $t > 0$ and $1 \leq q < \infty$. However if the initial value u_o is such that ρu_o belongs to $L^\infty(\mathbf{R}^n)$ where ρ is a weight function of the form :

$$\rho(x) = (1+|x|)^{-\alpha} \quad 0 \leq \alpha$$

or

$$\rho(x) = exp(-\alpha|x|^\beta) \quad 0 \leq \alpha, 0 \leq \beta < 2$$

it is easy to see that for any $t > 0$, $\rho(x)S(t)u_o \in L^\infty$. Then defining

$$E_\rho(\mathbf{R}^n) = E_\rho = \{u : \mathbf{R}^n \to \mathbf{R} \mid \rho u \in L^\infty(\mathbf{R}^n)\}$$

problem (P) can be solved for $u_o \in E_\rho$, and the solution is such that $u(t,.) \in E_\rho$ for all $t > 0$.

Let us recall now some known results about problem (P) when $p > 1$. If $1 < p < 1 + 2/n$, any solution of (P) blows up in a finite time ([5]) and the same holds for $p = 1 + 2/n$ ([1], [8], [10], [12]). When $p > 1 + 2/n$, positive global solutions of (P) exist ([3], [7], [11]). For $0 < p < 1$ we prove in sections 1 and 2 the following.

THEOREM .— *For any nonnegative $u_o \in E_\rho$ there is a mild global solution u of (P) such that*

i) $u \in C((0,\infty) \times \mathbf{R}^n) \cap L^\infty_{loc}[(0,\infty); E_\rho]$

ii) For all $t > 0, 1 \le i \le n$, $\partial u/\partial x_i \in C(\mathbf{R}^n)$ *and* $\nabla_x u \in L^\infty_{loc}[(\epsilon,\infty); E_\rho] \quad \forall \epsilon > 0$

iii) $\lim_{t\to 0} u(t,x) = u_o(x)$ *a.e.* $x \in \mathbf{R}^n$

Moreover if u_o is not identically zero, this solution is unique and is in $C^\infty[(0,\infty) \times \mathbf{R}^n]$.

By mild solution we understand that u verifies the integral equation

$$u(t) = S(t)u_o + \int_o^t S(t-s)u^p(s)ds.$$

In section 3 we study the asymptotic behaviour of solutions of (P) as t goes to infinity. Consider the corresponding problem with the nonlinear term having the so called "good sign" :

$$\begin{aligned} &u_t - \Delta u = -|u|^{p-1}u && \text{in } (0,\infty) \times \mathbf{R}^n \\ &u(x,0) = u_o(x) && \text{in } \mathbf{R}^n \\ &u \ge 0 && \text{in } (0,\infty) \times \mathbf{R}^n \end{aligned} \qquad (P')$$

If $1 < p < 1 + 2/n$ and $u_o \in L^2(\mathbf{R}^n)$ the asymptotic behaviour of the solution is given by the so called self-similar solutions, i.e. solutions invariant under the transformations :

$$u_\lambda(t,x) = \lambda^{-2/(1-p)}u(\lambda^2 t, \lambda x) \ \forall \lambda > 0.$$

These solutions are of the form :

$$u(t,x) = t^{-1/(p-1)} f(x/\sqrt{t}) \tag{0.3}$$

where f satisfies :

$$\begin{aligned} &- \Delta f - \frac{1}{2} x.\nabla f = -|f|^{p-1} f - \frac{1}{1-p} f \\ &f > 0 \end{aligned} \tag{0.4}$$

More precisely, if the initial value u_o of problem (P') is radial and a is defined by :

$$a = \lim_{|x|\to\infty} |x|^{2/(p-1)} u_o(x)$$

where $a = +\infty$ is allowed, then the solution u of (P') satisfies :

$$\lim_{t\to\infty} t^{1/(p-1)} \| u(t,.) - t^{-1/(p-1)} f_a(./\sqrt{t}) \|_\infty = 0$$

where f_a is the unique radial nonnegative solution of (0.4) satisfying :

$$\lim_{|x|\to\infty} |x|^{2/(p-1)} f(x) = a$$

(see [4], [6]).

If $0 \le p < 1$ and $u_o \in L^q(\mathbf{R}^n)$, $1 < q < \infty$, then there is a $T > 0$ such that $u \equiv 0$ on $(T, \infty) \times \mathbf{R}^n$. If $p = 1$ then

$$\lim_{t\to\infty} e^t t^{n/2} u(t,x) = (4\pi)^{-n/2} \int u_o(y) dy$$

uniformly on compact subsets of $\mathbf{R}^n$ ([6]). In our case where the nonlinear term has the "bag sign", when $1+2/n < p < (n+2)/(n-2)$ and u_o satisfies :

$$\int (|u_o(x)|^2 + |\nabla u_o(x)|^2) e^{|x|^2/4} dx < \infty$$

it has been shown ([9]) that there is a sequence (t_n) going to infinity and a solution f of

$$\begin{aligned} &- \Delta f - \frac{1}{2} x.\nabla f = +|f|^{p-1} f - \frac{1}{1-p} f \\ &f > 0 \end{aligned}$$

such that

$$\lim_{t\to\infty} \|t_n^{1/(p-1}u(t_n,.) - f(x/\sqrt{t_n})\|_\infty = 0$$

i.e., $u(t_n, x)$ is close to a self-similar solution $t_n^{-1/(p-1)}f(x/\sqrt{t_n})$.

In the sublinear case $0 < p < 1$, the results of sections 1 and 2 show the existence of a family of self-similar solutions

$$W(t,x) = t^{1/(1-p)}f(x/\sqrt{t})$$

belonging to $L^\infty_{\mathrm{loc}}[(0,\infty); E_\rho]$ with $\rho(x) = (1 + |x|)^{-2/(1-p)}$ and with initial value $W(0, x)$ any homogeneous function of degree $2/(1-p)$ lying in E_ρ. Inequality (0.2) and (0.3) yield in that case

$$f(x) > (1-p)^{1/(1-p)} \qquad \forall x \in \mathbf{R}^n$$

if $W(0, x)$ is not identically null. If $W(0, x) \equiv 0$, then

$$f(x) = (1-p)^{1/(1-p)} \qquad \forall x \in \mathbf{R}^n$$

Asuming $W(0, x)$ to be continuous it is proved that :

$$\lim_{r\to\infty} \|f(r\sigma)r^{-2/(1-p)} - W(0,\sigma)\|_{L^\infty(S^{n-1})} = 0$$

where for $x \in \mathbf{R}^n$ we have written $x = r\sigma$ and $\sigma = x/|x| \in S^{n-1}$.

Finally, the asymptotic behaviour of the solutions of (P) for a certain class of initial values is given in the following.

THEOREM .— *Let $u_o \in E_\rho$ be such that for a given $\varphi \in C(S^{n-1})$*

$$\lim_{r\to\infty} \|u_o(r\sigma)r^{-2/(1-p)} - \varphi(\sigma)\|_{L^\infty(S^{n-1})} = 0$$

and let u be the solution of (P) with initial data u_o. If W_φ is the self-similar solution of (P) such that :

$$W_\varphi(0,x) = |x|^{2/(1-p)}\varphi(x/|x|)$$

then for any compact $K \subset \mathbf{R}^n$ we have :

$$\lim_{t\to\infty} t^{-1/(1-p)}\|u(t,.) - W_\varphi(t,.)\|_{L^\infty(K^t)} = 0$$

where $K^t = \{(t,x) | x/\sqrt{t} \in K\}$.

Notations.— All integrals will be with respect to Lebesgue measure, and all functions wil be asumed to be measurable. When no explicit domain of integration is indicated, it is to be understood that it is all of $\mathbf{R}^n$. $S(t)$ will denote the semigroup of the heat equation on $\mathbf{R}^n$, that is, if u is a function on $\mathbf{R}^n$, such that for all $\epsilon > 0$ there is a $C_\epsilon > 0$ for which :

$$|u(x)| \leq C_\epsilon e^{-\epsilon|x|^2}$$

then for $t > 0$ and $x \in \mathbf{R}^n$

$$S(t)u(x) = (4\pi t)^{-n/2} \int e^{-|x-y|^2/4t} u(y)dy.$$

We denote by $\rho(x)$ a fixed weight function belonging to one of the families

$$\rho_\alpha(x) = (1 + |x|)^{-\alpha} \quad 0 \leq \alpha$$

or

$$\rho_{\alpha,\beta}(x) = \exp(-\alpha|x|^\beta) \quad 0 \leq \alpha, 0 \leq \beta < 2$$

For this ρ we define $E_\rho = \{u : \mathbf{R}^n \to \mathbf{R} \mid \rho u \in L^\infty(\mathbf{R}^n)\}$. Endowed with the norm $\|u\|_\rho = \|\rho u\|_\infty$ it becomes a Banach space. It is easily seen that

$$\rho^{-1}(x+y) \leq \rho^{-1}(x)\rho^{-1}(y) \; \forall x, y \in \mathbf{R}^n$$

(where $\rho^{-1} = 1/\rho$) so that

$$\begin{aligned} S(t)\rho^{-1}(x) &= (4\pi t)^{-n/2} \int e^{-|x-y|^2/4t} \rho^{-1}(y)dy \\ &= \pi^{-n/2} \int e^{-|y|^2} \rho^{-1}(x + 2\sqrt{t}y)dy \end{aligned}$$

and therefore

$$S(t)\rho^{-1}(x) \leq \varphi(t)\rho^{-1}(x)$$

where

$$\varphi(t) = \pi^{-n/2} \int e^{-|y|^2} \rho^{-1}(2\sqrt{t}y)\,dy$$

is a positive continuous and increasing function on $[0, \infty)$ such that $\varphi(0) = 1$. If we define for $m \in \mathbf{N}$

$$\varphi_m(t) = \int e^{-|y|^2} |y| \rho^{-m}(2\sqrt{t}y)\,dy$$

then φ_m is an increasing and continuous function on $[0,\infty)$. Finally, p will be a fixed number, $0 < p < 1$, and we let $q = 1/(1-p)$.

§ 1. Existence

This section is devoted to the proof of the existence of nonnegative solutions $u(t,x)$ of the integral equation :

$$u(t,x) = S(t)u_o(x) + \int_o^t S(t-s)u^p(s,x)ds \tag{1.1}$$

for $u_o \geq 0$. Functions satisfying (1.1) are called mild solutions of the semilinear parabolic Cauchy problem :

$$\begin{aligned} &u_t - \Delta u = u^p && \text{in } (0,\infty)\times \mathbf{R}^n \\ &u(x,0) = u_o(x) \geq 0 && \text{in } \mathbf{R}^n \end{aligned} \tag{1.2}$$

It will be shown in section 2 that such solutions are in fact classical under rather weak conditions on the initial value u_o.

LEMMA (1.3).— *Let g be a non decreasing Lipschitz function with $g(0) = 0$. For each $u_o \in E_\rho$ there is a unique mild solution u of :*

$$\begin{aligned} &u_t - \Delta u = g(u) && \text{in } (0,\infty)\times \mathbf{R}^n \\ &u(0,x) = u_o(x) && \text{in } \mathbf{R}^n \end{aligned} \tag{1.4}$$

such that $u \in L^\infty_{loc}[(0,\infty); E_\rho]$.

Moreover, if u and v are two mild solutions of (1.4) with initial values u_o and v_o respectively satisfying $u_o \geq v_o$, then :

$$u(t,x) \geq v(t,x) \quad \forall t > 0,\ \forall x \in \mathbf{R}^n.$$

In particular, if $u_o \geq 0$ then $u \geq 0$.

Proof.— Let u_o be any element of E_ρ and $T > 0$ fixed. For u in $L^\infty[(0,T); E_\rho]$ define :

$$Fu(t,x) = S(t)u_o + \int_o^t S(t-s)g(u(s))ds \ \forall t \in (0,T),\ \forall x \in \mathbf{R}^n$$

F maps $L^\infty[(0,T);E_\rho]$ into itself since for all $u \in L^\infty[(0,T);E_\rho]$ ·

$$\begin{aligned}|Fu(t,x)| &\le (4\pi t)^{-n/2}\int e^{-|x-y|^2/4t}|u_o(y)|dy\\&+\int_0^t(4\pi s)^{-n/2}\int e^{-|x-y|^2/4t}|g(u(t-s,y))|dyds\\&\le \|u_o\|_\rho S(t)\rho^{-1}(x)+k\sup_{0\le s\le T}\|u(s,.)\|_\rho\int_0^t S(t-s)\rho^{-1}(x)ds\\&\le \varphi(t)\|u_o\|_\rho\rho^{-1}(x)+kt\varphi(t)\sup_{0\le s\le T}\|u(s,.)\|_\rho\rho^{-1}(x)\end{aligned}$$

where k is the Lipschitz constant of g. It follows that :

$$\sup_{0\le t\le T}\|Fu(t,.)\|_\rho \le \varphi(T)\{\|u_o\|_\rho + kT\sup_{0\le t\le T}\|u(t,.)\|_\rho\} \tag{1.5}$$

and therefore Fu is defined almost everywhere and belongs to $L^\infty[(0,T);E_\rho]$.

We will prove next that for t small enough, F is a contraction. To see this let u and v be two elements of $L^\infty[(0,T);E_\rho]$. Then for all $t \in (0,T)$ and $x \in \mathbf{R}^n$:

$$\begin{aligned}|Fu(t,x)-Fv(t,x)| &\le k\int_0^t S(t-s)|(u-v)(s)|ds\\&\le k\sup_{0\le s\le T}\|(u-v)(s)\|_\rho\int_0^t S(t-s)\rho^{-1}(x)ds\\&\le kt\varphi(t)\sup_{0\le s\le t}\|(u-v)(s)\|_\rho\ \rho^{-1}(x)\end{aligned}$$

and then :

$$\sup_{0\le t\le T}\|Fu(t,.)-Fv(t,.)\|_\rho \le kT\varphi(T)\sup_{0\le t\le T}\|(u-v)(t,.)\|_\rho$$

If we chose T so that $kT\varphi(T) < 1$, F is a contraction, and by Banach's fixed point theorem there is a unique u belonging to $L^\infty[(0,T);E_\rho]$ such that

$$u(t) = S(t)u_o + \int_0^t S(t-s)g(u(s))ds.$$

Next we prove that u is defined for all $t > 0$. In fact, the interval of existence of the solution depends on g and ρ but not on u_o, so that the solution can be continued as long as it does not blow up. But this can not happen since, by (1.6), (1.5) and Gronwall's lemma :

$$\|u(t,.)\|_\rho \leq \varphi(T)\|u_o\|_\rho e^{k\varphi(T)t} \quad \forall t \in (0,T)$$

whenever u is defined on $(0,T)$. This also proves that u belongs to $L^\infty_{\text{loc}}[(0,\infty);E_\rho]$. Now suppose that u and v belong to $L^\infty_{\text{loc}}[(0,\infty);E_\rho]$ and are such that :

$$u(t) = S(t)u_o + \int_0^t S(t-s)g(u(s))ds$$

$$v(t) = S(t)v_o + \int_0^t S(t-s)g(v(s))ds$$

with

$$u_o \in E_\rho,\ v_o \in E_\rho,\ u_o \geq v_o.$$

Then :

$$\begin{aligned}(v-u)(t) &= S(t)(v_o - u_o) + \int_0^t S(t-s)[g(v(s)) - g(u(s))]ds \\ &\leq \int_0^t S(t-s)[g(v(s)) - g(u(s))]_+ ds \\ &\leq k\int_0^t S(t-s)[v(s) - u(s)]_+ ds\end{aligned}$$

since g is non decrasing and Lipschitz with constant k. It follows that :

$$(v-u)_+ \leq k\int_0^t S(t-s)[v(s) - u(s)]_+ ds$$

and for all $t > 0$

$$\|(v-u)_+(t)\|_\rho \leq k\int_0^t \varphi(t-s)\|(v-u)_+(s)\|_\rho ds \leq k\,\varphi(t)\int_0^t \|(v-u)_+(s)\|_\rho ds$$

and by Gronwall's lemma $\|(v-u)_+(t)\|_\rho$ is null for any t so that $u \geq v$.

Now, we solve problem (1.2) by using an approximation procedure and the existence and comparison result given in the previous lemma.

THEOREM (1.7).— *For every nonnegative function $u_o \in E_\rho$ there is at least a mild nonnegative solution u of (1.2) in the space $L^\infty_{loc}[(0,\infty);E_\rho]$.*

Proof.— Let (g_n) be a sequence of nondecreasing Lipschitz functions such that $g_n(0) = 0$ and $g_n(r) = r^p \ \forall r \geq 1/2n$, and consider the problems :

$$\begin{aligned} &u_t - \Delta u = g_n(u) && \text{in } (0,\infty) \times \mathbf{R}^n \\ &u(x,0) = u_o(x) + 1/n && \text{in } \mathbf{R}^n. \end{aligned} \tag{1.8$_n$}$$

By lemma (1.3) for each n there is a unique nonnegative mild solution $u_n \in L^\infty_{loc}[(0,\infty); E_\rho]$ satisfying

$$u_n(t) = S(t)\{u_o + 1/n\} + \int_0^t S(t-s) g_n(u_n(s))ds \tag{1.9}$$

Since $u_n \geq 0$, it follows that $u_n(t) \geq S(t)(u_o + 1/n) \geq 1/n$. Now if $n > m$ we have by construction that $g_n(u) = g_m(u)$ if $u \geq 1/2m$. Since $u_m \geq 1/m$, it follows that u_m and u_n are mild solutions of the same equation with initial data $u_o + 1/m$ and $u_o + 1/n$ respectively. By lemma (1.3), $u_m \geq u_n$ and for almost every $(t,x) \in (0,\infty) \times \mathbf{R}^n, (u_n(t,x))$ is a lower bounded decreasing sequence of real numbers. Thus we can define :

$$u(t,x) = \lim_{n\to\infty} u_n(t,x) \qquad \text{a.e. in } (0,\infty) \times \mathbf{R}^n$$

By (1.9) and the fact that $u_n \geq 1/n$:

$$u_n(t,x) = S(t)u_o + \frac{1}{n} + \int_0^t (4\pi s)^{n/2} \int e^{-|x-y|^2/4t} u_n^p(t-s,y)dyds \tag{1.10}$$

for almost every $(t,x) \in (0,\infty) \times \mathbf{R}^n$. As n goes to infinity, the left hand side converges to $u(x,t)$. As for the right hand side, we have that the integrand converges monotonically to $(4\pi s)^{n/2} e^{-|x-y|^2/4t} u^p(t-s,y)$. Since :

$$\begin{aligned} \int_0^t S(t-s)([u_1(s)]^p)ds &= \int_0^t S(t-s) g_1(u_1(s))ds \\ &\leq k \sup_{0 \leq s \leq T} \|u_1(s)\|_\rho \rho^{-1}(x) \int_0^t \varphi(t-s)ds < \infty \end{aligned}$$

it follows from the monotone convergence theorem that :

$$u(t) = S(t)u_o + \int_0^t S(t-s)u^p(s)ds$$

and since $0 \leq u \leq u_1$ we have that u belongs to $L^\infty_{\text{loc}}[(0,\infty), E_\rho]$.

We give now a partial regularity result and postpone until section 2 the proof that mild solutions of (1.4) are in fact classical solutions.

THEOREM (1.11).— *Let $u \in L^\infty_{loc}[(0,\infty), E_\rho]$ be a mild solution of (1.2). Then :*

i) $u \in C[(0,\infty) \times \mathbf{R}^n]$

ii) $\forall t > 0,\ 1 \le i \le n,\ \partial u/\partial x_i(t,.) \in C(\mathbf{R}^n)$

iii) $\forall \epsilon > 0\ \nabla_x u \in L^\infty_{loc}[(\epsilon,\infty); E_\rho]$

iv) $\lim_{t\to 0} u(t,x) = u_o(x)$ *for a.e.* $x \in \mathbf{R}^n$.

If $u_o \in C(\mathbf{R}^n)$ the convergence is uniform on compact subsets of $\mathbf{R}^n$.

Proof.— Let $u_1 = S(t)u_o$ and $u_2 = \int_0^t S(t-s)u^p(s)ds$. By the standard theory of the linear heat equation, $u_1 \in C^\infty[(0,\infty) \times \mathbf{R}^n]$ and can be differentiated under the integral sign. Then for $0 < \epsilon < t$ and $1 \le i \le n$:

$$\begin{aligned}|\partial u_1/\partial x_i(t,x)| &\le (4\pi t)^{-n/2} \int e^{-|x-y|^2/4t}(|x_i - y_i|/2t)u_o(y)dy \\ &\le C\|u_o\|_\rho t^{-1/2} \int e^{-|y|^2}|y|\rho^{-1}(x + 2\sqrt{t}y)dy \\ &\le C\rho^{-1}(x)\|u_o\|_\rho \epsilon^{-1/2} \int e^{-|y|^2}|y|\rho^{-1}(2\sqrt{t}y)dy\end{aligned}$$

proving iii) for u_1. It is also a classical result that as t goes to zero, $S(t)u_o(x)$ converges a.e. to u_o if $u_o \in L^\infty(\mathbf{R}^n)$, the convergence being uniform on compact sets when $u_o \in C(\mathbf{R}^n)$. For $u_o \in E_\rho$ we write $u_o = f + g$ where f is bounded with compact support and $g \in E_\rho$ vanishes on the support of f. Then a straighforward argument proves iv) for u_1.

Let's consider now u_2. We prove first that $u_2(t,x)$ converges to zero uniformly on compact subsets of $\mathbf{R}^n$ as $t \to 0$. In fact

$$\begin{aligned}u_2(t,x) &= \int_0^t (4\pi s)^{-n/2} \int e^{-|x-y|^2/4s} u^p(t-s,y)dyds \\ &\le \sup_{0\le s\le t} [\|u(s)\|_\rho]^p \int_0^t (4\pi s)^{-n/2} \int e^{-|x-y|^2/4s}\rho^{-p}(y)dyds \\ &\le \sup_{0\le s\le t} [\|u(s)\|_\rho]^p\ \pi^{-n/2} \int_0^t \int e^{-|y|^2}\rho^{-1}(x + 2\sqrt{t}y)dyds \\ &\le \rho^{-1}(x)t\varphi(t) \sup_{0\le s\le t} [\|u(s)\|_\rho]^p\end{aligned}$$

where we have used that $\rho^{-p} \leq \rho^{-1}$ since $\rho \leq 1$ and $0 < p < 1$.

We claim next that $\partial u_2/\partial x_i$ exits and can be obtained by differentiation under the integral sign. As before, $S(t-s)u^p(s)$ can be differentiated under the integral sign. Thus

$$|\partial/\partial x_i S(t-s)u^p(s)| \leq s^{-1/2} \sup_{0\leq s\leq t} [\|u(s)\|_\rho]^p \int |y| e^{-|y|^2} \rho^{-p}(x + 2\sqrt{s}y)dy$$
$$\leq s^{-1/2}\rho^{-1}(x)\varphi_1(s) \sup_{0\leq s\leq t} [\|u(s)\|_\rho]^p$$

Since this last expression is integrable on $[0,t]$ uniformly for x on compact sets, the claim follows. Finally we have :

$$|\partial u_2/\partial x_i| \leq \int_0^t |\partial/\partial x_i S(t-s)u^p(s)|ds$$
$$\leq \rho^{-1}(x) \sup_{0\leq s\leq t} [\|u(s)\|_\rho]^p \int_0^t s^{-1/2}\varphi_1(s)ds$$

proving iii) (in fact u_2 verifies iii) on $(0,\infty)$, not only on (ϵ,∞)).

Next we prove that $\partial u_2/\partial x_i$ is continuous on $\mathbf{R}^n$. In fact, for any $x, \overline{x} \in \mathbf{R}^n$:

$$|\partial u_2/\partial x_i(t,x) - \partial u_2/\partial x_i(t,\overline{x})| \leq \int_0^t (2s)^{-1}(4\pi s)^{-n/2}$$
$$\times \int |e^{-|x-y|^2/4s}(x_i - y_i) - e^{-|\overline{x}-y|^2/4s}(\overline{x}_i - y_i)|\ u^p(t-s,y)dyds.$$

As x tends to $\overline{x}$, the integrand in the right hand side converges to 0 almost everywhere. Arguing as before, it is easely seen that for x and $\overline{x}$ in a compact subset of $\mathbf{R}^n$ it is uniformly bounded by an integrable function. The dominated convergence theorem proves then that the left hand side converges to zero.

To see i) we will prove something more, namely, that as a function on $(0,\infty)$ with values in $C(\mathbf{R}^n)$, u_2 is continuous, that is, for any $\overline{t} > 0$ and any compact K of $\mathbf{R}^n$:

$$\lim_{t\to\overline{t}} \|u_2(t,.) - u_2(\overline{t},.)\|_{L^\infty(K)} = 0 \tag{1.12}$$

To see this let $x \in \mathbf{R}^n$. Then :

$$
\begin{aligned}
&|u_2(t,x) - u_2(\bar t,x)| \le \\
&\le \int_0^{\bar t} (4\pi(t-s))^{-n/2} \int |e^{-|x-y|^2/4(t-s)} - e^{-|x-y|^2/4(\bar t - s)}| u^p(s,y)\,dyds \\
&\quad + \int_{\bar t}^{t} (4\pi(t-s))^{-n/2} \int e^{-|x-y|^2/4(t-s)} u^p(s,y)\,dyds \\
&\quad + \int_0^{\bar t} |(4\pi(\bar t - s))^{-n/2} - (4\pi(t-s))^{-n/2}| \\
&\qquad\qquad \times \int e^{-|x-y|^2/4(t-s)} u^p(s,y)\,dyds \\
&\le C\rho^{-1}(x) \sup_{0\le s\le t} [\|u(s)\|_\rho]^p \\
&\qquad\qquad \times \iint_0^{\bar t} |e^{-|y|^2} - e^{-(t-s)|y|^2/(\bar t - s)}| \rho^{-1}(2\sqrt{(t-s)}y)\,dyds \\
&\quad + \rho^{-1}(x) \sup_{0\le s\le t} [\|u(s)\|_\rho]^p |t - \bar t| \int e^{-|y|^2} \rho^{-1}(2\sqrt{t}y)\,dy \\
&\quad + \rho^{-1}(x) \sup_{0\le s\le t} [\|u(s)\|_\rho]^p \int_0^{\bar t} |1 - ((\bar t - s)/(t-s))^{n/2}| \\
&\qquad\qquad \times \int e^{-|y|^2} \rho^{-1}(2\sqrt{t}y)\,dyds
\end{aligned}
$$

and once again by the dominated convergence theorem we have (1.12) for any compact subset of $\mathbf{R}^n$.

Remark (1.13).— Using the same method it is easy to see that if ∇u_o belongs to E_ρ then $\nabla_x u$ belongs to $L^\infty_{loc}[(0,\infty); E_\rho]$.

Remark (1.14).— All the results in this section remain true if we consider a more general class of weight functions ρ satisfying :

i) $1 \ge \rho(x) \ge C\exp(-A|x|^2)$ $\forall x \in \mathbf{R}^n$ and some positive constants A and C.

ii) $S(t)\rho^{-1}$ is defined on $\mathbf{R}^n$ for any $t > 0$ and $S(t)\rho^{-1}(x) \le \varphi(t)\rho^{-1}(x)$ for some locally bounded φ.

iii) the function

$$\varphi_1(t) = \int e^{-|y|^2} |y| \rho^{-1}(2\sqrt{t}y)\,dy$$

is locally bounded.

§ 2. Uniqueness and regularity

Since u^p is not a Lipschitz function for $0 < p < 1$, a uniqueness result for (1.2) is not obvious. In fact, if $u_o \equiv 0$, there is a continuum of space independent solutions, namely the trivial solution $u \equiv 0$ and the family

$$u_\tau(t) = ((1-p)(t-\tau)_+)^q \quad \tau \geq 0 \tag{2.1}$$

where, here and in all the following, $q = 1/(1-p)$.

If $u_o \geq c$ for some constant $c > 0$, the same is true for the solution u, which then satisfies :

$$\begin{aligned} &u_t - \Delta u = g(u) && \text{in } (0,\infty) \times \mathbf{R}^n \\ &u(0) = u_o && \text{in } \mathbf{R}^n \end{aligned}$$

for some C^∞ function g, uniqueness being guaranteed in that case. The case when $u_o \geq 0$ is not bounded away from zero, for instance if u_o is of compact support, can be treated by means of a uniform lower estimate for the growth of the solution (lemma (2.2)). This inequality and its corollaries will then be used to prove the uniqueness and regularity results of this section and the asymptotic behaviour of solutions of (1.2) given in the next one.

LEMMA (2.2).— *Let u_o be a non identically null, nonnegative function on $\mathbf{R}^n$, $T > 0$, and u a nonnegative function on $(0,T) \times \mathbf{R}^n$ such that, for any t in $(0,T)$ and any x in $\mathbf{R}^n$:*

$$u(t,x) \geq S(t)u_0 + \int_0^t S(t-s)u^p(s)ds. \tag{2.3}$$

Then :

$$u(t,x) > ((1-p)t)^q \qquad \forall t \in (0,T),\ \forall x \in \mathbf{R}^n. \tag{2.4}$$

Proof.— We will consider first the case when, for any $x \in \mathbf{R}^n$ and for some positive constants c and a, $u_o(x)$ is greater of equal than $ce^{-a|x|^2}$. Use will be made of the identity :

$$S(t)(e^{a|\cdot|^2})(x) = (1+4at)^{-n/2}\exp\{-a|x|^2/(1+4at)\}$$

Since $u \geq 0$, (2.3) implies :

$$u(t,x) \geq S(t)u_o(x) \geq S(t)(ce^{-a|.|^2})(x) = c(1+4at)^{-n+/2}\exp\{-a|x|^2/(1+4at)\}.$$

Substituting this for u in (2.3) and using that $u_o \geq 0$ we get :

$$\begin{aligned} u &\geq c^p \int_0^t (1+4as)^{-np/2} S(t-s)\{-ap|.|^2/(1+4as)\}(x)ds \\ &= c^p \int_0^t (1+4as)^{n(1-p)/2}(1+4as+4ap(t-s))^{-n/2} \\ &\qquad \exp\left\{-\frac{ap|x|^2}{1+4as+4ap(t-s)}\right\} ds \\ &\geq c^p\ t(1+4at)^{-n/2}\exp(-ap|x|^2/(1+4apt)) \end{aligned}$$

where we have used that $1+4at \leq 1+4as+4ap(t-s) \leq 1+4at$ for $0 \leq s \leq t$ and $0 < p < 1$. Substituting again this estimate in (2.3) we obtain :

$$\begin{aligned} u &\geq c^{p^2} \int_0^t s^p(1+4as)^{-np/2} S(t-s)\{-ap^2|.|^2/(1+4apt)\}(x)ds \\ &= c^{p^2} \int_0^t s^p[(1+4aps)/(1+4as)^p]^{-n/2}(1+4aps+4ap^2(t-s))^{-n/2} \\ &\qquad \exp\left\{-\frac{ap^2|x|^2}{1+4aps+4ap^2(t-s)}\right\} ds \\ &\geq c^{p^2}(1+p)^{-1}t^{1+p}(1+4apt)^{-n/2}\exp\{-ap^2|x|^2/(1+4ap^2t)\} \end{aligned}$$

since as before $1+4ap^2t \leq 1+4aps+4ap^2(t-s) \leq 1+4apt$ and $(1+4aps)/(1+4as)^p \geq 1$ for $0 \leq s \leq t$ and $0 < p < 1$.

Iterating this procedure, an easy induction shows that for $k \in \mathbf{N}$:

$$u \geq c^{p^{k+1}} c(k) t^{1+p+\ldots+p^k}(1+4ap^kt)^{-n/2}\exp(-ap^k|x|^2/(1+4ap^kt)) \quad (2.5)$$

where

$$c(k) = (1+p+\ldots+p^k)^{-1} \ . \ (1+p+\ldots+p^{k-1})^{-p}\ldots(1+p)^{-p^{k-1}}$$

Taking logarithms we obtain :

$$\begin{aligned} \log c(k) &= -\sum_{j=1}^{k} p^{k-j}\log\Big(\sum_{h=0}^{j} p^j\Big) \\ &\geq \log(1-p)\sum_{j=1}^{k} p^{k-j} \geq q\log(1-p) \end{aligned}$$

and thus

$$c(k) \geq (1-p)^q.$$

Using this estimate in (2.5) and letting k go to infinity we obtain the desired result. For the general case, take $0 < t_o < T$ and define for $0 < t < T - t_o$ $v(t) = u(t_o + t)$. By the semigroup property of S, and since $u \geq 0$, it is easely seen that :

$$v(t) \geq S(t)v_o + \int_0^t S(t-s)v^p(s)d$$

where :

$$v_o(s) = S(t_o)u_o \geq ce^{-a|x|^2} \quad \forall x \in \mathbf{R}^n$$

for certain positive constants c and a (depending on u_o and t_o). It follows from the previous part that if $0 \leq t + t_o \leq T$ then

$$u(t+t_o, x) \geq ((1-p)t)^q \quad \forall x \in \mathbf{R}^n$$

Then given any $t > 0$ we have for all $\epsilon > 0$ small enough and all $x \in \mathbf{R}^n$:

$$u(t,x) = u(\epsilon + (t-\epsilon), x) \geq [(1-p)(t-\epsilon)]^q$$

and thus :

$$u(t,x) \geq ((1-p)t)^q \quad \forall t \geq 0,\ \forall x \in \mathbf{R}^n.$$

Strict inequality for $t > 0$ follows now from (2.3) and the fact that if $u_0 \not\equiv 0$ then $S(t)u_o(x) > 0$ for $t > 0$ and $x \in \mathbf{R}^n$.

COROLLARY (2.6).— *All nontrivial nonnegative mild solutions of (1.2) with $u_o \equiv 0$ in $L^\infty_{loc}[(0,\infty); E_\rho]$ are given by the family (2.1).*

Proof.— Let $w \in L^\infty_{\text{loc}}[(0,\infty); E_\rho]$ be a mild solution of (1.4) with $u_o \equiv 0$. Then w is continuous on $(0,\infty) \times \mathbf{R}^n$ and :

$$w(t) = \int_0^t S(t-s)w^p(ds) \quad w \geq 0.$$

Then

$$\|w(t)\|_\infty \leq \int_0^t [\|w(s)\|_\infty]^p ds$$

and therefore :

$$\|w(t)\|_\infty \leq ((1-p)t)^q.$$

Suppose that w is not identically null. Then there is a $t > 0$ and x in $\mathbf{R}^n$ such that $w(t,x) > 0$. Define $\tau = \inf\{t > 0 \mid w(t,x) > 0 \text{ for some } x \in \mathbf{R}^n\}$. By the diffusion property of the heat equation, $w(t,x) > 0$ for any x in $\mathbf{R}^n$ and any t greater than τ. Given any $\bar{t} > \tau$ let $\overline{w}(t,x) = w(t+\bar{t},x)$. We have :

$$\begin{aligned} &\overline{w}_t - \Delta\overline{w} = \overline{w}^p && \text{in } (0,\infty)\times\mathbf{R}^n \\ &\overline{w}(0) = w(\bar{t},x) && \text{in } \mathbf{R}^n. \end{aligned}$$

By lemma (2.2) we have for $t \geq 0$ and $x \in \mathbf{R}^n$

$$w(\bar{t}+t,x) \geq ((1-p)t)^q$$

and therefore, for all $t \geq 0$, $\bar{t} > \tau$ and $x \in \mathbf{R}^n$

$$w(t,x) \geq ((1-p)(t-\bar{t})_+)^q$$

which implies :

$$w(t,x) \geq ((1-p)(t-\tau)_+)^q \quad \forall t \geq 0,\ \forall x \in \mathbf{R}^n \tag{2.7}$$

Choose now $\underline{t} < \tau$ and define $\underline{w}(t,x) = w(t+\underline{t},x)$ for $(t,x) \in (0,\infty)\times\mathbf{R}^n$. $\underline{w}$ satisfies :

$$\begin{aligned} &\underline{w}_t - \Delta\underline{w} = \underline{w}^p && \text{in } (0,\infty)\times\mathbf{R}^n \\ &\underline{w}(0) = w(\underline{t},x) \equiv 0. \end{aligned}$$

and then

$$w(t+\underline{t},x) \leq ((1-p)t)^q \quad \forall t \geq 0,\ \forall x \in \mathbf{R}^n$$

so that

$$w(t,x) \leq ((1-p)(t-\underline{t})_+)^q \quad \forall t \geq 0,\ \forall x \in \mathbf{R}^n$$

Since this holds for all $\underline{t} < \tau$ one concludes that :

$$w(t,x) \leq ((1-p)(t-\tau)_+)^q \quad \forall t \geq 0,\ \forall x \in \mathbf{R}^n$$

Finally this and (2.7) give that $w \equiv u_\tau$.

Uniqueness of the solution of (1.4) when u_o is not identically null will be proved now as a corollory of the next theorem, which is a comparison result that will be used throughout the rest of the paper.

THEOREM (2.8).— *Let $u, v \in L^\infty_{loc}[(0,\infty);E_\rho]$ be nonnegative and such that for all $t > 0$*

$$u(t) \geq S(t)u_o + \int_0^t S(t-s)u^p(s)ds = U(t)$$

$$v(t) \leq S(t)v_o + \int_0^t S(t-s)v^p(s)ds = V(t)$$

where

$$u_o, v_o \in E_\rho, \quad u_o \geq v_o \geq 0, \quad u_o \not\equiv 0.$$

Then

$$u(t) \geq v(t) \qquad \forall t \geq 0.$$

Proof.— Let $g(t) = v(t) - u(t)$. We want to prove $g_+(t) \equiv 0$. We have

$$\begin{aligned} g(t) &\leq S(t)(v_o - u_o) + \int_0^t S(t-s)(v^p(s) - u^p(s))ds \\ &\leq \int_0^t S(t-s)(v^p(s) - u^p(s))_+ ds \\ &\leq \int_0^t S(t-s)[g_+(s)]^p ds \end{aligned}$$

and then

$$g_+(t) \leq \int_0^t S(t-s)[g_+(s)]^p ds \leq \rho^{-1}(x) \int_0^t [\|g_+(s)\|_\rho]^p \varphi(t-s)ds$$

from where

$$\|g_+(t)\|_\rho \leq \int_0^t [\|g_+(s)\|_\rho]^p \varphi(t-s)ds \leq \varphi(t) \int_0^t [\|g_+(s)\|_\rho]^p ds.$$

It follows that for $0 \leq t \leq T$

$$\|g_+(t)\|_\rho \leq [\varphi(T)(1-p)t]^q \tag{2.9}$$

On the other hand, by the mean value theorem

$$v^p(s,x) - u^p(s,x) = p(v(s,x) - u(s,x))\theta^{p-1} \tag{2.10}$$

for some θ between $v(s,x)$ and $u(s,x)$. If $v(s,x) < u(s,x)$ both sides of (2.10) are negative. If $v(s,x) \geq u(s,x)$ then $\theta \geq u(s,x) \geq ((1-p)s)^q$. It follows that

$$(v^p(s) - u^p(s))_+ \leq pqs^{-1}(v(s) - u(s))_+ \tag{2.11}$$

and

$$g_+(t) \leq pq \int_0^t s^{-1} S(t-s) g_+(s) ds \leq \rho^{-1}(x)\varphi(T) pq \int_0^t s^{-1} \|g_+(s)\|_\rho ds. \tag{2.12}$$

Observe that by (2.9) $s^{-1}\|g_+(s)\|_\rho$ is integrable on $(0,T)$. We finally obtain

$$\|g_+(t)\|_\rho \leq pq\varphi(T) \int_0^t s^{-1} \|g_+(s)\|_\rho ds \tag{2.13}$$

If we define for $0 \leq t \leq T$

$$f(t) = pq\varphi(T) \int_0^t s^{-1} \|g_+(s)\|_\rho ds$$

inequality (2.13) can be rewritten as

$$f'(t)/f(t) \leq pq\varphi(T) t^{-1} \qquad 0 \leq t \leq T$$

Given any $\epsilon > 0$ we have for $\epsilon \leq t \leq T$

$$f(t) \leq \epsilon^{-pq\varphi(T)} f(\epsilon) t^{pq\varphi(T)}. \tag{2.14}$$

But by the definition of f and by (2.9)

$$\begin{aligned} f(\epsilon) &= pq\varphi(T) \int_0^\epsilon s^{-1} \|g_+(s)\|_\rho ds \\ &\leq pq\varphi(T)^{1+q}(1-p)^q \int_0^\epsilon s^{-1+q} ds \\ &= p\varphi(T)^{1+q}(1-p)^q \epsilon^q. \end{aligned}$$

Carrying this into (2.14) we obtain

$$f(t) \leq C\epsilon^{q(1-p\varphi(T))} t^{pq\varphi(T)} \tag{2.15}$$

where C is a constant independent of ϵ and t. Since $\varphi(0) = 1$ and φ is continuous, we can choose $T > 0$ such that $1 - p\varphi(T) > 0$. It follows then from (2.15) that $f \equiv 0$ on $(0,T)$. Thus we have proved

$$v(t,x) \leq u(t,x) \qquad \forall (t,x) \in (0,T) \times \mathbf{R}^n. \tag{2.16}$$

It is clear from the proof that T depends only on p and ρ and not on u, v or the initial data u_o, v_o. Define $u'(t) = u(t + T/2)$ and $v'(t) = v(t + T/2)$. Then

$$u'(t) = u(t + T/2) \geq U(t + T/2) = S(t)U(T/2) + \int_0^t S(t-s)u'^p(s)ds$$

$$v'(t) = v(t + T/2) \leq V(t + T/2) = S(t)V(T/2) + \int_0^t S(t-s)v'^p(s)ds$$

and by (2.16) $V(T/2) \leq U(T/2)$. Therefore we can appy the same reasoning to get that (2.16) holds on $(0, 3T/2) \times \mathbf{R}^n$. Repeating the argument we finish the proof of the theorem.

Remark (2.17).— The conditions of theorem (2.8) are satisfied if for instance

$$u_t - \Delta u \geq u^p \quad u(0) = u_o$$

$$v_t - \Delta v \leq v^p \quad v(0) = v_o$$

$$u,\ v \geq 0 \quad ,\ u_o \geq v_o \geq 0 \quad ,\ u_o \not\equiv 0$$

We deduce now from theorem (2.8) three corollaries. The first one is about uniqueness of solutions of (1.2), the second about continuous dependence and the last one is an inequality that will be crucial in the investigation of the asymptotic behaviour of solutions of (1.2).

COROLLARY (2.18).— *For any nonnegative u_o in $E_\rho, u_o \not\equiv 0$, the solution of (1.2) whose existence is proved in theorem (1.7) is unique.*

Proof.— It is an inmediate consequence of the previous theorem.

COROLLARY (2.19).— *Let $E_{\rho,+} = \{u \in E_\rho \mid u \geq 0,\ u \not\equiv 0\}$. Then*

i) the maping taking $u_o \in E_{\rho,+}$ to the unique mild solution $u \in L^\infty_{loc}[(0,\infty); E_\rho]$ of (1.2) is concave, i.e., given $u_o, v_o \in E_{\rho,+}$ and $0 < \gamma < 1$, let u, v and w be the unique solutions of (1.2) with initial data u_o, v_o and $w_o = \gamma u_o + (1-\gamma)v_o$ respectively. Then

$$w(t,x) \geq \gamma u(t,x) + (1-\gamma)v(t,x)$$

ii) for each $(t,x) \in (0,\infty) \times \mathbf{R}^n$ the map taking $u_o \in E_{\rho,+}$ to $u(t,x)$ is continuous on $E_{\rho,+}$ with the topology inherited from E_ρ.

Proof.— Since $0 \leq p \leq 1$ we have $\gamma u^p + (1-\gamma)v^p \leq (\gamma u + (1-\gamma)v)^p$. Then

$$w(t) = S(t)w_o + \int_0^t S(t-s)w^p(s)ds$$

$$\begin{aligned}\gamma u(t) + (1-\gamma)v(t) &= S(t)w_o + \int_0^t S(t-s)(\gamma u^p(s) + (1-\gamma)v^p(s))ds \\ &\leq S(t)w_o + \int_0^t S(t-s)((\gamma u(s) + (1-\gamma)v(s))^p)ds\end{aligned}$$

and by remark (2.17) to theorem (2.8)

$$\gamma u(t) + (1-\gamma)v(t) \leq w(t)$$

proving i).

ii) is now a consequence of classical convex analysis (see [2]).

COROLLARY (2.20).— *Let u_o, v_o, $w_o \in E_{\rho,+}$ be such that $w_o \leq u_o + v_o$ and let u, v, w be the unique mild solutions of (1.2) with initial data u_o, v_o, w_o respectively. Then*

$$w(t) \leq u(t) + v(t) - ((1-p)t)^q \quad \forall t > 0 \tag{2.21}$$

Proof.— First of all we show that $w \leq u + v$. In fact we have

$$w(t) = S(t)w_o + \int_0^t S(t-s)w^p(s)ds$$

$$\begin{aligned}u(t) + v(t) &= S(t)(u_o + v_o) + \int_0^t S(t-s)(u^p(s) + v^p(s))ds \\ &\geq S(t)w_o + \int_0^t S(t-s)((u(s) + v(s))^p)ds\end{aligned}$$

and $w \leq u + v$ follows from theorem (2.8).

Next we prove (2.21) when $u_o \leq w_o, v_o \leq w_o$ and $u_o + v_o - w_o \not\equiv 0$. The proof is based on lemma (2.2) and the inequality

$$x^p + y^p - z^p \geq (x + y - z)^p$$

which holds when

$$0 \leq x \leq z \ , \ 0 \leq y \leq z \ , \ z \leq x + y$$

as can be easely seen by reducing it to the case $z = 1$. We have then

$$u + v - w = S(t)(u_o + v_o - w_o) + \int_0^t S(t-s)(u^p(s) + v^p(s) - w^p(s))ds$$
$$\geq S(t)(u_o + v_o - w_o) + \int_0^t S(t-s)((u(s) + v(s) - w(s))^p)ds$$

and (2.21) follows now from lemma (2.2).

Consider now the case $w_o = u_o + v_o$. Since $v_o \geq 0$ is not identically null, there is a set $A \subset \mathbf{R}^n$ of positive measure such that $w_o - u_o > 0$ on A. For $0 < \epsilon < 1$ let $u_{o,\epsilon} = u_o + \epsilon(w_o - u_o)\chi_A$ where χ_A is the characteristic function of A. Then $u_{o,\epsilon}$, v_o, w_o fall into the case treated above. Denoting by $u^\epsilon(t)$ the unique solution in $L^\infty_{\text{loc}}[(0,\infty); E_\rho]$ of (1.2) with initial data $u_{o,\epsilon}$ we have

$$w(t) \leq u^\epsilon(t) + v(t) - ((1-p)t)^q. \tag{2.22}$$

But since $u_o \in E_{\rho,+}$ and $u_{o,\epsilon}$ converges to u_o in the norm of E_ρ as ϵ goes to zero, it follows from ii) of corollary (2.19) that $u^\epsilon(t,x)$ converges to $u(t,x)$. Since (2.22) holds for all $\epsilon > 0$, passing to the limit as $\epsilon \to 0$ proves the result. The remaining cases follow now easely.

We finish this section by proving that solutions of (1.2) are in fact classical solutions.

THEOREM (2.23).— *Let $u \in L^\infty_{loc}[(0,\infty); E_\rho]$ be a nonnegative mild solution of (1.2) with $u_o \not\equiv 0$. Then $u \in C^\infty[(0,\infty) \times \mathbf{R}^n]$ and is a classical solution of (1.2).*

Proof.— We already know that for $t > 0$ and $1 \leq i \leq n$, $\partial u/\partial x_i(t,.)$ exists and is continuous on $\mathbf{R}^n$. We will show that the same is true for $\partial^2 u/\partial x_i \partial x_j$, $1 \leq i,j \leq n$. Let $\epsilon > 0$ be given and let $v(t) = u(t+\epsilon)$, $v_o = u(\epsilon)$. Then

$$v(t) = S(t)v_o + \int_0^t S(t-s)v^p(s)ds \qquad t > 0 \tag{2.24}$$

$$v(t) \geq ((1-p)t)^q \qquad t > 0 \tag{2.25}$$

Since $S(t)v_o \in C^\infty[(0,\infty) \times \mathbf{R}^n]$ we will just consider the second summand of the right hand side of (2.24), that will be denoted by $\overline{v}$. We know that $\partial \overline{v}/\partial x_i$ exists, is continuous on $\mathbf{R}^n$ and can be obtained differentiating under the integral sign. Thus

$$\partial \overline{v}/\partial x_i = p \int_0^t S(t-s) v^{p-1}(s) \partial v/\partial x_i(s) ds$$

and by (2.25)

$$|v^{p-1} \partial v/\partial x_i| \le p\epsilon^{-1} |\partial v/\partial x_i|$$

It follows from theorem (1.11) that $v^{p-1}(s)\partial v/\partial x_i \in L^\infty_{\text{loc}}[(0,\circ); E_\rho]$.

Arguing as in the mentioned theorem we get that $\overline{v}$ has continuous second derivatives with respect to the space variables and that they are also in $L^\infty_{\text{loc}}[(0,\infty); E_\rho]$. We prove next that $\overline{v}$ has continuous third derivatives with respect to the space variables. We start from

$$\partial^2 \overline{v}/\partial x_i \partial x_j = p \int_0^t S(t-s)((p-1)v^{p-2}(s)\partial v/\partial x_i \partial v/x_j + v^{p-1}(s)\partial^2 v/\partial x_i \partial x_j) ds$$

and observe that

$$v^{p-2}(s)\partial v/\partial x_i \partial v/x_j \in L^\infty_{\text{loc}}[(0,\infty); E_{\rho^2}]$$
$$v^{p-1}(s)\partial^2 v/\partial x_i \partial x_j \in L^\infty_{\text{loc}}[(0,\infty); E_\rho]$$

Arguing again as in theorem (1.11) and taking into account the fact that

$$\varphi_2(t) = \int e^{-|y|^2} |y| \rho^{-2}(\sqrt{t} y) dy$$

is locally bounded, we conclude that $\overline{v}$ has continuous third derivatives with respect to the x variables. Using the same method and the fact that for all positive integers m the function

$$\varphi_m(t) = \int e^{-|y|^2} |y| \rho^{-m}(\sqrt{t} y) dy$$

is locally bounded we obtain that for any positive integer $m, \overline{v}$ has derivatives of order m with respect to space variables and they are in $L^\infty_{\text{loc}}[(0,\infty); E_{\rho^{m-1}}]$.

Finally, a standard argument will show that v has derivatives of all orders with respect to t and that $v_t - \Delta v = v^p$.

§ 3. Asymptotic Behaviour

We determine in this section the asymptotic behaviour of solutions of (1.2) for a large class of initial conditions. As stated in the introduction, the asymptotic behaviour of global solutions of an evolution equation is given in many cases by the self- similar solutions of the equation. The equation we are studying is

$$u_t - \Delta u = u^p \tag{3.1}$$

It is invariant under the one parameter group of transformations

$$u_\lambda(t,x) = \lambda^{-2q} u(\lambda^2 t, \lambda x) \tag{3.2}$$

A solution of (3.1) is called self- similar if it is also invariant under (3.2), that is, if

$$W_\lambda(t,x) = \lambda^{-2q} W(\lambda^2 t, \lambda x) \tag{3.3}$$

or equivalently if

$$W(t,x) = t^q f(x/\sqrt{t})$$

where $f(x) = W(1,x)$ satisfies the elliptic equation

$$-\Delta f - \frac{1}{2} x \cdot \nabla f = |f|^{p-1} f - \frac{1}{1-p} f \tag{3.5}$$

The existence of nonnegative self-similar solutions of (3.1) is usually shown by proving that (3.5) admits nonnegative solutions. In our case it is an easy consequence of the results in previous sections.

From now on the weight function ρ will be taken to be $\rho(x) = (1+|x|)^{-2q}$. We define $\Sigma = \{\varphi \in L^\infty(S^{n-1}) \mid \varphi \geq 0 \text{ a.e.}, \varphi \not\equiv 0\}$. Given $\varphi \in \Sigma$ the function $|x|^{2q}\varphi(x/|x|)$ belongs to E_ρ and is homogeneous of degree $2q$. We denote by W_φ the unique solution in $L^\infty_{\text{loc}}[(0,\infty); E_\rho]$ of (1.2) with initial data $|x|^{2q}\varphi(x/|x|)$.

THEOREM (3.6).— *For all $\varphi \in \Sigma$, W_φ is a self similar solution of (3.1).*

Conversely, if $W \in L^\infty_{loc}[(0,\infty); E_\rho]$ is a self-similar solution of (3.1) such that $W(t,x)$ converges for all $x \in \mathbf{R}^n$ to a function $w_o \in E_\rho$ as t goes to 0, then w_o is homogeneous of degree $2q$.

Proof.— If W is the solution of (3.1) with initial data $w_o(x) = |x|^{2q}\varphi(x/|x|)$, then $W_\lambda(t,x) = \lambda^{-2q}W(\lambda^2 t, \lambda x)$ satisfies

$$(W_\lambda)_t - \Delta W_\lambda = (W_\lambda)^p$$
$$W_\lambda(0,x) = \lambda^{-2q}W(0,\lambda x) = w_o(x)$$

and by uniqueness $W_\lambda \equiv W$.

Conversely, if W is as stated in the theorem, then for all $\lambda > 0$

$$W(t,x) = W_\lambda(t,x) = \lambda^{2q}W(\lambda^2 t, \lambda x).$$

Letting $t \to 0$ we finally get $w_o(x) = \lambda^{-2q}w_o(\lambda x)$ as desired.

Given $\varphi \in \Sigma$ we let $f_\varphi(x) = W_\varphi(1,x)$, so that $\omega_\varphi(t,x) = t^q f_\varphi(x/\sqrt{t})$. By the results of the previous sections, f_φ is of class C^∞ on $\mathbf{R}^n$, verifies equation (3.5) and for any $x \in \mathbf{R}^n$ the map taking f to $f_\varphi(x)$ is continuous from Σ into $\mathbf{R}$. The behaviour at infinity of the functions f_φ is described in the next lemma.

LEMMA (3.7).— *Let $\varphi \in \Sigma \cap C(S^{n-1})$. Then*

$$\lim_{r\to\infty} \|r^{-2q}f_\varphi(r\sigma) - \varphi(\sigma)\|_{L^\infty(S^{n-1})} = 0 \tag{3.8}$$

Proof.— $f_\varphi(x) = t^{-q}W_\varphi(t, x/\sqrt{t})$ where W_φ is the solution of (1.2) with initial value $|x|^{2q}\varphi(x/|x|)$. Taking $t = |x|^{-2}$ we get

$$|x|^{-2q}f_\varphi(x) = W_\varphi(|x|^{-2}, x/|x|).$$

Therefore for all $\sigma \in S^{n-1}$ and for all $r > 0$

$$|r^{-2q}f_\varphi(r\sigma) - \varphi(\sigma)| = |W_\varphi(r^{-2},\sigma) - W_\varphi(0,\sigma)|$$

and (3.8) follows by iv) of theorem (1.11).

We can now describe the asymptotic behaviour of solutions of (1.2).

THEOREM (3.9).— *Let u_o be a nonnegative locally bounded function on $\mathbf{R}^n$ such that there exists $\varphi \in \Sigma \cap C(S^{n-1})$ for which*

$$\lim_{r\to\infty} \|r^{-2q}u_o(r\sigma| - \varphi(\sigma)\|_{L^\infty(S^{n-1})} = 0 \tag{3.10}$$

Then $u_o \in E_\rho$, and if u is the unique solution of (1.2) with u_o as initial data, for any compact $K \subset \mathbf{R}^n$ we have

$$\lim_{t\to\infty} t^{-q} \|u(t,\cdot) - W_\varphi(t,\cdot)\|_{L^\infty(K^t)} = 0$$

where $K^t = \{(t,x) \; |x/\sqrt{t} \in K\}$.

Proof.— By (3.10) and the local boundedness of u_o it is clear that $u_o \in E_\rho$. Moreover, $u_o \not\equiv 0$ since $\varphi \not\equiv 0$, so that u is well defined. Let $\delta > 0$ be given. By (3.10) there exists $R > 0$ such that if $|x| \geq R$

$$u_o(x) \leq |x|^{2q}(\varphi(x/|x|) + \delta)$$

and by lemma (3.7) there is $R' > 0$ such that if $|x| > R'$

$$|x|^{2q}(\varphi(x/|x|) + \delta) \leq f_{\varphi+2\delta}(x)$$

Thus if $|x| \geq \max(R, R')$ we have

$$u_o(x) \leq f_{\varphi+2\delta}(x)$$

and then there exists $C > 0$ (depending on δ) such that

$$u_o(x) \leq f_{\varphi+2\delta}(x) + C \qquad \forall x \in \mathbf{R}^n \tag{3.11}$$

It follows from corollary (2.20) that

$$u(t,x) \leq W_{\varphi+2\delta}(1+t,x) + ((1-p)t + C^{1-p})^q - ((1+p)t)^q \tag{3.12}$$

since clearly the unique solutions of (1.2) with initial values $f_{\varphi+2\delta}$ and C are $W_{\varphi+2\delta}(1+t,x)$ and $((1-p)t + C^{1-p})^q$ respectively.

Arguing in a similar way to obtain a lower estimate, we see that for all x large enough (3.10) implies

$$|x|^{2q}(\varphi(x/|x|) - \delta)_+) \leq u_o(x)$$

and by lemma (3.7)

$$f_{(\varphi-2\delta)} \leq |x|^{2q}(\varphi(x/|x|) - 2\delta)_+) + \delta) \leq |x|^{2q}(\varphi(x/|x|) - \delta)_+).$$

Thus, there exists $C' > 0$ (depending on δ) such that

$$f_{(\varphi-2\delta)_+} \leq u_o(x) + C' \qquad \forall x \in \mathbf{R}^n$$

and then by corollory (2.20)

$$W_{(\varphi-2\delta)}(1+t,x) \leq u(t,x) + ((1-p)t + C'^{1-p})^q - ((1-p)t)^q \tag{3.13}$$

Multiplying both sides of (3.12) by t^{-q} and substracting $f_\varphi(x/\sqrt{t})$ we obtain

$$t^{-q}u(t,x) - f_\varphi(x/\sqrt{t}) \leq (1+1/t)^q f_{\varphi+2\delta}(x/\sqrt{1+t}) - f_\varphi(x/\sqrt{t}) + g(t) \tag{3.14}$$

where $g(t) = t^{-q}\{((1-p)t + C^{1-p})^q - ((1-p)t)^q\} = O_\delta(1/t)$ as $t \to \infty$. Let $t = \lambda^2$ and $x/\sqrt{t} = z$. Then (3.14) can be rewritten as

$$u_\lambda(1,z) - f_\varphi(z) \leq f_{\varphi+2\delta}(\overline{z}) - f_\varphi(z) + 0_\delta(\lambda^{-2}) + 0(\lambda^{-2}) f_{\varphi+2\delta}(\overline{z}) \tag{3.15}$$

with $\overline{z} = z + 0(\lambda^{-2})z$. By the mean value theorem

$$f_{\varphi+2\delta}(\overline{z}) = f_{\varphi+2\delta}(z) + 0(\lambda^{-2}) \nabla f_{\varphi+2\delta}(\widetilde{z}) \cdot z$$

for some $\widetilde{z}$ between z and $\widetilde{z}$.

If $0 < \delta < 1$, the set $(f_{\varphi+2\delta})$ is bounded in Σ. It is then easy to see, using the estimates of theorem (1.11) that the family $(\nabla f_{\varphi+2\delta})$ is uniformly bounded on compact subsets of $\mathbf{R}^n$, so that $(f_{\varphi+2\delta})$ is equicontinuous on compact subsets of $\mathbf{R}^n$. Let $K \subset \mathbf{R}^n$ be compact. If $z \in K$ and $\lambda \geq 1$, then $\overline{z}$ and $\widetilde{z}$ remain in a fixed compact set. It follows that

$$f_{\varphi+2\delta}(\overline{z}) = f_{\varphi+2\delta}(z) + 0_{\delta,K}(\lambda^{-2})$$

and

$$u_\lambda(1,z) - f_\varphi(z) \leq f_{\varphi+2\delta}(z) - f_\varphi(z) + 0_{\delta,K}(\lambda^{-2}). \tag{3.16}$$

Given any $\epsilon > 0$, by the equicontinuity of $(f_{\varphi+2\delta})$ on compact subsets of $\mathbf{R}^n$ and the continuity of $f_{\varphi+2\delta}(z)$ as a function of δ, there is a $\delta > 0$ such that

$$f_{\varphi+2\delta}(z) - f_\varphi(z) \leq \epsilon/2.$$

For this δ, there is a $\Lambda \geq 1$ such that if $\lambda \geq \Lambda$ the second summand in the right hand side of (3.16) is smaller than $\epsilon/2$. So we have proved that for any $\epsilon > 0$ there is a $\Lambda \geq 1$ such that

$$u_\lambda(1,z) - f_\varphi(z) \leq \epsilon \qquad \forall \lambda \geq \Lambda,\ \forall z \in K. \tag{3.17}$$

Starting from (3.13) the same reasoning yields a $\Lambda' > 1$ such that

$$u_\lambda(1,z) - f_\varphi(z) \geq -\epsilon \qquad \forall \lambda \geq \Lambda',\ \forall z \in K. \tag{3.18}$$

Letting again $t = \lambda^2$ and $z = x/\sqrt{t}$, we get from (3.17) and (3.18) that for any $\epsilon > 0$ there is a $T > 0$ such that

$$t^{-q}\,|u(t,x) - W_\varphi(t,x)| \leq \epsilon \qquad \forall t \geq T,\ \forall x \in K^t \tag{3.19}$$

proving the theorem.

Note - After completion of the manuscript, the authors learned from J.L.VAZQUEZ he had obtained similar results for the initial- boundary problem on a bounded domain. (Work made in colaboration and as yet unpublished).

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Références

[1] ARONSON (D.G) and WEINBERGER (H.F).— Multidimensional nonlinear diffusion arising in population genetics, *Advances in Mathematics*, t. **30**, 1978, p. 33-76.

[2] EKELAND (I) and TEMANN (R.).— *Analyse Convexe et Problèmes Variationnels.* Dunod, Paris, 1974.

[3] ESCOBEDO (M) and KAVIAN (0).— *Variational problems related to self-similar solutions of the heat equation.* — to appear in J. of Nonlinear Analysis, Theory, Methods & Appl..

[4] ESCOBEDO (M) & KAVIAN (0) & MATANO (H).— in preparation.

[5] FUJITA (H).— On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. of Tokio, Sect. I*, t. **13**, 1966, p. 109-124.

[6] GMIRA (A) and VERON (L).— Large time bahaviour of the solutions of a semilinear parabolic equation in $\mathbf{R}^n$, *J. of Diff. Eq.*, t. **53**, 1984, p. 259-276.

[7] HARAUX (A) and WEISSLER (F.B.).— Non uniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.*, t. **31**, n°**2**, 1982, p. 167-189.

[8] HAYAKAWA (K).— On non-existence of global solutions of some semilinear parabolic equations, *Proc. Japan Acad.*, t. **49**, 1973, p. 503-505.

[9] KAVIAN (O).— *Remarks on the time behaviour of a non linear diffusion equation.* Prépublication du Laboratoire d'Analyse Numérique, Universit, P. et M. Curie (Paris VI), 1985.

[10] KOBAYASHI (K), SIRAO (T) and TANAKA (H).— On the growing up problem for semilinear heat equations, *J. Math. Soc. Japan*, t. **29**, 1977, p. 407-424.

[11] WEISSLER (F.B). — *Rapidly decaying solutions of an O.D.E. with applications to semilinear parabolic P.D.E.'s.* — to appear.

[12] WEISSLER (F.B). — Existence and non-existence of global solutions for a semilinear heat equation, *Israel J. of Math.*, t. **38**, **n°1-2**, 1981, p. 29-39.

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