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COMPARISON THEOREMS FOR A CLASS OF FIRST  
ORDER HAMILTON-JACOBI EQUATIONS<sup>(3)</sup>

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**Résumé :** Nous étudions une certaine classe d'équations de Hamilton-Jacobi du premier ordre. Tout d'abord, en utilisant des techniques de symétrisation, nous comparons une solution du problème considéré avec la solution à symétrie sphérique décroissante d'un problème symétrisé. Enfin nous démontrons un théorème d'existence des solutions de viscosité.

**Summary :** We study a certain class of first order Hamilton-Jacobi equations. First, by means of symmetrization technique, we compare a solution of the considered problem with the decreasing spherically symmetric solution of a symmetrized problem. Next we prove an existence theorem of viscosity solution.

## 1. - INTRODUCTION AND RESULTS

From the same point of view as in [9], making use of symmetrization techniques, we study the Dirichlet problem :

$$(1.1) \quad \begin{cases} |Du| - \lambda u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $x \equiv (x_1, \dots, x_n)$  is a point of  $\mathbb{R}^n$ ,  $Du = \text{grad } u$ , and  $f(x)$  is a measurable real-valued function defined in  $\Omega$ .

We assume that :

- a)  $\Omega$  is an open subset of  $\mathbb{R}^n$  with finite measure  $M$  ;
- b)  $f(x) \in L^p(\Omega)$ ,  $p \geq 1$  ;
- c)  $\lambda$  is a positive real number ;
- d) there exists a generalized solution  $u$  of (1.1), that is there exists a function  $u \in W_0^{1,p}(\Omega)$ ,  $p \geq 1$ , which satisfies the equation  $|Du| - \lambda u = f(x)$  a.e. in  $\Omega$ .

Existence theorems for (1.1) can be found in [11], so that the last assumption makes sense.

Herein our main goal is to compare a solution  $u(x)$  of (1.1) with the unique decreasing spherically symmetric solution of a problem :

- which is of the same type as (1.1) ;
- given in a ball  $\Omega^* \subset \mathbb{R}^n$  having measure  $M$  ;
- for which the right hand-side has the same distribution function as  $f(x)$ .

In order to state our results more precisely, let us recall that one denotes by

$$\mu(t) = \text{meas} \{ x \in \Omega : |u(x)| > t \}$$

the *distribution function* of a measurable real-valued function  $u$  defined in  $\Omega$ , that

$$u^*(s) = \inf \{ t \geq 0 : \mu(t) < s \}$$

is the *decreasing rearrangement* of  $u$ , and that

$$u^*(x) = u^*(C_n |x|^n),$$

where  $C_n$  is the measure of the  $n$ -dimensional unit-ball of  $\mathbb{R}^n$ , is the *spherically symmetric decreasing rearrangement* of  $u$ .

Also we consider the *increasing rearrangement* of  $u$  :

$$u_*(s) = u^*(\text{meas } \Omega - s)$$

and the *spherically symmetric increasing rearrangement* of  $u$  :

$$u_\star(x) = u_*(C_n |x|^n).$$

The function  $u$  and its rearrangements have the same distribution function and, as well known, the following inequality holds (see [10], [13]) :

$$(1.2) \quad \int_{\Omega} |u v| dx \leq \int_0^M u^*(s) v^*(s) ds = \int_{\Omega^\star} u^\star(x) v^\star(x) dx$$

Here and below  $\Omega^\star$  is the ball of  $\mathbb{R}^n$  centered at the origin with the same measure  $M$  as  $\Omega$ . Denoted by  $u(x)$  a generalized solution of (1.1), using auxiliary lemmas of section 2, in section 3 we prove the following results :

THEOREM 1.1. *If  $n > 1$ , we have :*

$$\|u\|_\infty \leq \|w\|_\infty$$

where, if  $M \leq C_n \left(\frac{n-1}{\lambda}\right)^n$ ,  $w(x)$  is the unique decreasing spherically symmetric solution of the problem :

$$(1.4) \quad \left\{ \begin{array}{ll} |Dw| - \lambda w = f^\star(x) & \text{in } \Omega^\star \\ w = 0 & \text{on } \partial\Omega^\star \end{array} \right.$$

while, if  $M > C_n \left(\frac{n-1}{\lambda}\right)^n$ ,  $w(x)$  is the unique decreasing spherically symmetric solution of the problem :

$$(1.5) \quad \left\{ \begin{array}{ll} |Dw| - \lambda w = \hat{f}(C_n |x|^n) & \text{in } \Omega^\star \\ w = 0 & \text{on } \partial\Omega^\star \end{array} \right.$$

$\hat{f}(s)$  being a function with the same distribution function as  $f(x)$ . (see REMARK 3.1 for an explicit definition of  $\hat{f}$ ).

THEOREM 1.2. If  $n > 1$ , we have :

$$(1.6) \quad \|u\|_1 \leq \|z\|_1$$

where  $z(x)$  is the unique decreasing spherically symmetric solution of the problem :

$$(1.7) \quad \begin{cases} |Dz| - \lambda z = \check{f}(C_n |x|^n) & \text{in } \Omega^\star \\ z = 0 & \text{on } \partial\Omega^\star \end{cases}$$

$\check{f}(s)$  being a fixed function having the same distribution function as  $f(x)$ .

THEOREM 1.3. If  $n \geq 1$  and  $\lambda > (n-1) [C_n / M]^{1/n}$ , we have :

$$(1.8) \quad u^\star(x) \leq q(x) \quad \text{in } \Omega^\star - \Omega_0^\star$$

where  $\Omega_0^\star$  is a ball of  $\mathbb{R}^n$  centered at the origin and with radius  $\frac{n-1}{\lambda}$ , and  $q(x)$  is the unique spherically symmetric decreasing solution of the problem :

$$(1.9) \quad \begin{cases} |Dq| - \lambda q = f_\star(x) & \text{in } \Omega^\star \\ q = 0 & \text{on } \partial\Omega^\star \end{cases}$$

In particular, from Theorem 1.3 we derive the following :

COROLLARY 1.1. If  $n = 1$ , then :

$$u^\star(x) \leq q(x) \quad \text{in } \left(-\frac{M}{2}, \frac{M}{2}\right)$$

$q(x)$  being the unique solution, depending only on  $|x|$ , of the problem :

$$\begin{cases} \left| \frac{dq}{ds} \right| - \lambda q = f_\star(x) & \text{in } \left(-\frac{M}{2}, \frac{M}{2}\right) \\ q\left(-\frac{M}{2}\right) = q\left(\frac{M}{2}\right) = 0 \end{cases}$$

Now, let us recall that, more generally, for a first order Hamilton-Jacobi equation :  $H(x, u(x), Du(x)) = 0$  ( $H$  being a continuous function in  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ ), besides the definition of generalized solutions, M.G. Crandall and P.L. Lions have introduced the notion of *viscosity solution*.

We refer to [5], [6], [11] for the exact definition and for the properties of viscosity solutions.

Only let us mention that a viscosity solution  $u$  of the equation  $H=0$  need to be continuous but not necessarily differentiable in anywhere ; however, if  $u$  is differentiable at some  $x_0$ , then  $H(x_0, u(x_0), Du(x_0)) = 0$ .

Furthermore some uniqueness and stability problems can be solved introducing this new notion of solution (see [5], [6], [11]).

In section 4, under more restrictive assumptions, we prove an existence theorem for viscosity solutions of (1.1) (see Th. 4.1 for the exact statement).

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## 2. - TWO LEMMAS

Henceforth let  $u(x)$  be a solution of (1.1) and  $\mu(t)$  its distribution function, for each  $s \in [0, \text{meas } \Omega]$  consider a measurable subset  $D(s)$  of  $\Omega$ , such that :

$$\begin{aligned} \text{meas } D(s) &= s ; \\ s_1 < s_2 &\Rightarrow D(s_1) \subset D(s_2) ; \\ D(s) &= \{ x \in \Omega : |u(x)| > t \} \quad \text{if } s = \mu(t). \end{aligned}$$

Then by b)  $\int_{D(s)} f(x) dx$  is an absolutely continuous function; and so there exists a function  $\tilde{f}(t)$  such that :

$$(2.1) \quad \int_0^s \tilde{f}(t) dt = \int_{D(s)} |f(x)| dx.$$

Furthermore the following lemma holds (see [1] for the proof) :

**LEMMA 2.1.** *There exists a sequence  $\{f_h(s)\}$  of functions which have the same distribution function as  $f(x)$  and such that, if  $p > 1$  :*

$$f_h(s) \rightarrow \tilde{f}(s) \quad \text{in } L^p([0, M]),$$

while if  $p = 1$  :

$$\lim_h \int_0^M f_h(s) g(s) ds = \int_0^M \tilde{f}(s) g(s) ds$$

for each function  $g(s)$  belonging to the space  $BV([0, M])$  of the functions of bounded variation.

Now we give a sketch of the proof of the well known :

LEMMA 2.2. Let  $\phi(s)$  and  $f(s)$  be two given measurable functions in  $[0, M]$ . Then there exists  $\hat{f}(s)$ , which has the same distribution function as  $f(s)$  and which depends on  $\phi(s)$  such that :

$$(2.2) \quad \int_0^M f(s)\phi(s) ds \leq \int_0^M \hat{f}(s)\phi(s) ds = \int_0^M f^*(s)\phi^*(s) ds$$

(compare for example with [4] or [12]).

- Denoted by  $\nu_\phi(t)$  the distribution function of  $\phi(s)$ , for each  $s \in [0, M]$  we can fix a measurable subset  $E(s) \subseteq [0, M]$ , such that :

$$\begin{aligned} \text{meas } E(s) &= s ; \\ s_1 < s_2 &\Rightarrow E(s_1) \subset E(s_2) ; \\ E(s) &= \{ \sigma : |\phi(\sigma)| > t \} \quad \text{if } s = \nu_\phi(t). \end{aligned}$$

Then let

$$(2.3) \quad s(\sigma) = \inf \{ \bar{s} \in [0, M] : \sigma \in E(\bar{s}) \}, \quad \sigma \in [0, M];$$

the required function is :

$$(2.4) \quad \hat{f}(\sigma) = f^*(s(\sigma)).$$

Moreover, denoting by  $\nu_f(t)$  the distribution function of  $f(s)$ , we have :

$$\begin{aligned} \int_0^M \hat{f}(\sigma)\phi(\sigma) d\sigma &= \int_0^\infty dt \int_{\hat{f} > t} \phi(\sigma) d\sigma = \int_0^\infty dt \int_{E(\nu_f(t))} \phi(\sigma) d\sigma = \\ &= \int_0^\infty dt \int_0^{\nu_f(t)} \phi^*(\sigma) d\sigma = \int_0^\infty dt \int_{f^* > t} \phi^*(\sigma) d\sigma = \int_0^M f^*(s)\phi^*(s) ds. \end{aligned}$$

3. - PROOF OF THEOREMS 1.1, 1.2, 1.3

For the sake of clearness, first of all we prove two lemmas.

LEMMA 3.1. *We have :*

$$(3.1) \quad u^*(x) \leq \frac{1}{nC_n^{1/n}} \int_{C_n |x|^n}^M (\lambda u^*(s) + \tilde{f}(s)) s^{1/n-1} ds$$

a.e. in  $\Omega^*$ .

*Proof.* By the isoperimetric inequality (see [7]) :

$$nC_n^{1/n} \mu(t)^{1-1/n} \leq P \{ x : |u(x)| > t \},$$

where P is the perimeter in the sense of De Giorgi, and by the Fleming-Rishel formula ([8]) :

$$\int_{|u|>t} |Du| dx = \int_0^\infty P \{ x : |u(x)| > \xi \} d\xi$$

we get :

$$(3.2) \quad nC_n^{1/n} \mu(t)^{1-1/n} \leq -\frac{d}{dt} \int_{|u|>t} |Du| dx.$$

On the other hand, since u is a solution of (1.1) :

$$\frac{1}{h} \int_{t < |u| \leq t+h} |Du| dx \leq \frac{1}{h} \int_{t < |u| \leq t+h} (\lambda |u| + |f|) dx, \quad h > 0,$$

hence for  $h \rightarrow 0$  :

$$(3.3) \quad -\frac{d}{dt} \int_{|u|>t} |Du| dx \leq -\frac{d}{dt} \int_{|u|>t} (\lambda |u| + |f|) dx.$$

Moreover, since

$$\int_{|u|>t} |u| dx = \int_0^{\mu(t)} u^*(s) ds,$$

(2.1), (3.2) and (3.3) give :

$$nC_n^{1/n} \mu(t)^{1-1/n} \leq [\lambda u^*(\mu(t)) + \tilde{f}(\mu(t))] (-\mu'(t))$$

that is :

$$1 \leq \frac{-\mu'(t)}{nC_n^{1/n} \mu(t)^{1-1/n}} [\lambda u^*(\mu(t)) + \tilde{f}(\mu(t))].$$

Now, integrating both sides of last inequality from 0 to t, we have :

$$\begin{aligned} t &\leq \frac{1}{nC_n^{1/n}} \int_0^t [\lambda u^*(\mu(\tau)) + \tilde{f}(\mu(\tau))] \frac{-\mu'(\tau)}{\mu(\tau)^{1-1/n}} d\tau \\ &\leq \frac{1}{nC_n^{1/n}} \int_{\mu(t)}^M [\lambda u^*(\tau) + \tilde{f}(\tau)] \tau^{1/n-1} d\tau, \end{aligned}$$

which implies, by the definition of decreasing rearrangement :

$$u^*(s) \leq \frac{1}{nC_n^{1/n}} \int_s^M [\lambda u^*(\tau) + \tilde{f}(\tau)] \tau^{1/n-1} d\tau,$$

which gives (3.1) replacing s by  $C_n |x|^n$ .

LEMMA 3.2. *We have a.e. :*

$$(3.4) \quad u^*(x) \leq v(x)$$

where  $v(x)$  is the unique decreasing spherically symmetric solution of the problem :

$$(3.5) \quad \left\{ \begin{array}{ll} |Dv| - \lambda v(x) = \tilde{f}(C_n |x|^n) & \text{in } \Omega^*, \\ v(x) = 0 & \text{on } \partial\Omega^* \end{array} \right.$$

*Proof.* Define a sequence  $\{v_k(x)\}$  of functions in  $\Omega$  in this way :

$$(3.6) \quad v_0(x) = u^*(x)$$

and

$$(3.7) \quad v_k(x) = \frac{1}{nC_n^{1/n}} \int_{C_n |x|^n}^M [\lambda v_{k-1}^*(s) + \tilde{f}(s)] s^{-1+1/n} ds.$$

Of course  $v_k(x) \leq v_{k+1}(x)$  by (3.1). Moreover, as we prove now,  $\{v_k(x)\}$  converges in  $L^p(\Omega^*)$ .

In fact first of all we derive from (3.7), changing the variable on the right-hand side, that :

$$(3.8) \quad v_k(x) = \int_{|x|}^{(M/C_n)^{1/n}} [\lambda v_{k-1}^*(C_n \tau^n) + f(C_n \tau^n)] d\tau.$$

Now for simplicity set  $\rho = |x|$  and

$$\omega_k(\rho) = \begin{cases} v_k(x) & \text{if } \rho = |x| \text{ and } \rho \leq (M/C_n)^{1/n} \\ 0 & \text{if } \rho > (M/C_n)^{1/n}. \end{cases}$$

By the following Hardy inequality ([10]) :

$$(3.9) \quad \int_0^\infty y^r \left( \int_y^\infty a(t) dt \right)^p dy \leq \left( \frac{p}{r+1} \right)^p \int_0^\infty y^r (ya(y))^p dy,$$

where  $p \geq 1$  and  $r > -1$ , we get from (3.8) :

$$\begin{aligned} \int_0^\infty \rho^r (\omega_{k+1}(\rho) - \omega_k(\rho))^p d\rho &= \int_0^\infty \rho^r \left( \int_\rho^\infty \lambda (\omega_k(\tau) - \omega_{k-1}(\tau)) d\tau \right)^p d\rho \\ &\leq \left( \frac{p}{r+1} \right)^p \int_0^\infty \rho^{r+p} \lambda^p (\omega_k - \omega_{k-1})^p d\rho \leq \left( \frac{\lambda p}{r+1} \left( \frac{M}{C_n} \right)^{1/n} \right)^p \int_0^\infty \rho^r (\omega_k - \omega_{k-1})^p d\rho. \end{aligned}$$

Then repeating the above  $k$  times, finally we have :

$$(3.10) \quad \int_0^\infty \rho^r (\omega_{k+1}(\rho) - \omega_k(\rho))^p d\rho \leq \left( \frac{\lambda p}{r+1} \left( \frac{M}{C_n} \right)^{1/n} \right)^{kp} \int_0^\infty \rho^r (\omega_1(\rho) - \omega_0(\rho))^p d\rho.$$

Hence, fixed  $\bar{r} > r = \lambda p \left( \frac{M}{C_n} \right)^{1/n} - 1$  and  $r \geq n-1$ , the sequence  $\{\omega_k(\rho)\}$  converges in the space of the functions  $\phi(\rho)$  such that :

$$\left( \int_0^\infty \rho^r |\phi(\rho)|^p d\rho \right)^{1/p} < +\infty.$$

Then if  $n-1 > \bar{r}$  we conclude that the sequence  $\{v_k(x)\}$  converges in  $L^p(\Omega^*)$  since :

$$nC_n \int_0^\infty \rho^{n-1} |\omega_k(\rho) - \omega_h(\rho)|^p d\rho = \int_{\Omega^*} |v_k(x) - v_h(x)|^p dx$$

If on the contrary  $n-1 \leq \bar{r}$ , we can fix a positive number  $r > \bar{r}$  and an integer  $m > 0$

such that  $r - mp = n - 1$  <sup>(1)</sup> and so, as above, by inequality (3.9) :

$$\int_0^\infty \rho^{r-p} |\omega_{k+1}(\rho) - \omega_{h+1}(\rho)|^p d\rho \leq \left( \frac{\lambda p}{r-p+1} \right)^p \int_0^\infty \rho^r |\omega_k(\rho) - \omega_h(\rho)|^p d\rho.$$

Then the sequence  $\{\omega_k(\rho)\}$  converges in the space of the functions  $\phi(\rho)$  such that :

$$\left( \int_0^\infty \rho^{r-p} |\phi(\rho)|^p d\rho \right)^{1/p} < +\infty.$$

In this way, after  $m$  steps, we obtain the convergence of  $\{\omega_k(\rho)\}$  in the space of the functions  $\phi(\rho)$  for which :

$$\left( \int_0^\infty \rho^{n-1} |\phi(\rho)|^p d\rho \right)^{1/p} < +\infty$$

that is the convergence of  $\{v_k(x)\}$  in  $L^p(\Omega^*)$ .

Say  $v(x) = \lim v_k(x)$  in  $L^p(\Omega^*)$ . Then of course also :

$$v(x) = \lim v_k(x) \quad \text{a.e. in } \Omega^*.$$

Hence we get from (3.6) :

$$(3.11) \quad u^*(x) \leq v(x).$$

On the other hand, interchanging integration and limit process on the right-hand side of (3.7), for  $v(x)$  we get :

$$(3.12) \quad v(x) = \frac{1}{nC_n^{1/n}} \int_{C_n |x|^n}^M [\lambda v^*(s) + \tilde{f}(s)] s^{1/n-1} ds.$$

Then  $v(x)$  solves (3.5) and so it is the unique solution of (3.5), which is spherically symmetric decreasing and by (3.11) we get our claim.

*Proof of Theorem 1.1.* The decreasing spherically symmetric solution of (3.5) is :

$$\begin{aligned} v(x) &= e^{-\lambda |x|} \int_{|x|}^{(M/C_n)^{1/n}} \tilde{f}(C_n t^n) e^{\lambda t} dt \\ &= \frac{e^{-\lambda |x|}}{nC_n^{1/n}} \int_{C_n |x|^n}^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt, \end{aligned}$$

---

(1) - In fact we can choose  $m > \frac{\bar{r} - (n-1)}{p}$  and  $r = (n-1) + mp$ .

then for  $s = C_n |x|^n$  :

$$(3.13) \quad v^*(s) = \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt.$$

$$\text{Set } \alpha = C_n \left( \frac{n-1}{\lambda} \right)^n.$$

The function  $\phi(t) = e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}$  decreases in  $(0, \alpha]$ , then, if  $M \leq \alpha$ , from Lemma 2.1 and inequality (1.2) we derive :

$$\begin{aligned} v^*(s) &\leq \lim_h \frac{1}{nC_n^{1/n}} \int_s^M f_h(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt \\ &\leq \frac{1}{nC_n^{1/n}} \int_0^M f^*(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt \\ &= \sup \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M f^*(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt \\ &= \sup w^*(s) \end{aligned}$$

and from here, since  $w(x) = w^*(C_n |x|^n)$ , by (3.4) it follows (1.3) in the case  $M \leq \alpha$ .

Now assume  $M > \alpha$ . As above :

$$v^*(s) \leq \lim_h \frac{1}{nC_n^{1/n}} \int_s^M f_h(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt.$$

Now, since the functions  $f_h(t)$  and  $f(x)$  are equidistributed, by (2.2), there exists a function  $\hat{f}(t)$  having the same distribution function as  $f(x)$  such that :

$$v^*(s) \leq \frac{1}{nC_n^{1/n}} \int_0^M \hat{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt.$$

From this inequality and from (3.4) we derive the result, since the function

$$w(x) = \frac{e^{-\lambda |x|}}{nC_n^{1/n}} \int_{C_n |x|^n}^M \hat{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt$$

is the unique decreasing spherically symmetric solution of the problem (1.5).

*Remark 3.1.* In this case we can define precisely the function  $\hat{f}$  given by (2.4). In fact, if  $\nu_\phi$  is the distribution function of

$$\phi(t) = e^{\lambda(t/C_n)^{1/n}} t^{1/n-1},$$

we have

$$\hat{f}(t) = f^*(v_\phi(\phi(t))) \quad , \quad t \in [0, M].$$

Such a function verifies the following properties :

- each level set of  $\hat{f}$  is a level set of  $\phi$  ;
- $\hat{f}$  is decreasing in  $[0, \alpha]$  and increasing in  $[\alpha, M]$  ;
- $\hat{f}$  is equimeasurable with  $f^*$ .

*Proof of Theorem 1.2.* By Lemma 2.1 and (3.13) we have :

$$\begin{aligned} \|u\|_1 &= \int_0^M u^*(s) ds \leq \int_0^M v^*(s) ds \\ &= \frac{1}{nC_n^{1/n}} \int_0^M e^{-\lambda(s/C_n)^{1/n}} ds \int_s^M \check{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt \\ &= \frac{1}{nC_n^{1/n}} \int_0^M \check{f}(s) e^{\lambda(s/C_n)^{1/n}} s^{1/n-1} ds \int_0^s e^{-\lambda(t/C_n)^{1/n}} dt. \end{aligned}$$

Then by Lemma 2.1 and Lemma 2.2, there exists a function  $\check{f}(t)$ , which has the same distribution function as  $f(x)$ , such that :

$$\begin{aligned} \|u\|_1 &\leq \frac{1}{nC_n^{1/n}} \int_0^M \check{f}(s) e^{\lambda(s/C_n)^{1/n}} s^{1/n-1} ds \int_0^s e^{-\lambda(t/C_n)^{1/n}} dt \\ &= \|z^*(s)\|_1, \end{aligned}$$

where

$$z^*(s) = \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M \check{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt$$

and from there we get the result, since  $z(x) = z^*(C_n |x|^n)$ .

*Remark 3.2.* We could exhibit the function  $\check{f}$  in the same way we did for  $\hat{f}$  in Remark 3.1.

*Proof of Theorem 1.3.* Set

$$\chi_s(t) = \begin{cases} 0, & 0 \leq t \leq s \\ 1, & s \leq t \leq M \end{cases}$$

Obviously :

$$\int_s^M \tilde{f}(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt = \int_0^M \tilde{f}(t) \chi_s(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt$$

On the other hand if  $s \geq C_n \left(\frac{n-1}{\lambda}\right)^n = \alpha$ , then the function  $\chi_s(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1}$

is increasing in  $[0, M]$  and hence, by Lemma 2.1 and (1.2), it follows that :

$$\int_0^M \tilde{f}(t) \chi_s(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt \leq \int_0^M f_*(t) \chi_s e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt.$$

Then by (3.4) and (3.13), for  $s \geq \alpha$  we have :

$$u^*(s) \leq \frac{e^{-\lambda(s/C_n)^{1/n}}}{nC_n^{1/n}} \int_s^M f_*(t) e^{\lambda(t/C_n)^{1/n}} t^{1/n-1} dt = q^*(s).$$

*Remark 3.3.* As in section 3 of [9] it is possible to extend the previous results to the more general problem :

$$\begin{cases} H(u, Du) = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $H(p, q)$  and  $f(x, p)$  are given real valued functions, satisfying the hypotheses :

$\bar{a}) = a)$  ;

$\bar{b}) \exists K : \mathbb{R} \rightarrow \mathbb{R}_+$  strictly increasing such that :

$$K(|q| - \lambda p) \leq H(p, q), \quad \forall p \in \mathbb{R} \text{ and } \forall q \in \mathbb{R}^n ;$$

$\bar{c}) K^{-1}(f(x, 0)) \in L^p, \quad p \geq 1 ;$

$\bar{d}) f(x, p) \leq f(x, 0), \quad \forall (x, p) \in \Omega \times \mathbb{R} ;$

$\bar{e})$  analogous to hypothesis d).

In fact we can compare the solution  $u(x)$  of such a problem with the unique spherically symmetric decreasing solution of a spherically symmetric problem in  $\Omega^*$  for which the right-hand side is a function depending only on  $|x|$ , equidistributed with  $f(x, 0)$ .

#### 4. - AN EXISTENCE THEOREM FOR VISCOSITY SOLUTIONS OF (1.1)

In all this paragraph we assume that :

- i)  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  ;
- ii)  $f(x) \in C(\bar{\Omega})$ ,  $0 \leq f(x) \leq L$  ;
- iii)  $\lambda > 0$ .

Then, denoting by  $B_R$  a ball of  $\mathbb{R}^n$ , centered at the origin, with radius  $R$  and containing  $\Omega$ , the function :

$$g(x) = -\frac{L}{\lambda} + \frac{L}{\lambda} e^{\lambda(R-|x|)}, \quad |x| \leq R,$$

is a viscosity (and generalized) solution of :

$$(4.1) \quad |Dg| - \lambda g = L \quad \text{in } B_R,$$

with boundary condition  $g = 0$  on  $\partial B_R$ .

Also  $g(x)$  is a viscosity (and generalized) supersolution of the equation :

$$(4.2) \quad |Du| - \lambda u = f(x) \quad \text{in } \Omega.$$

Of course the function  $u_0 \equiv 0$  is a viscosity (and generalized) subsolution of (4.2).

For (4.2) the following theorem holds :

**THEOREM 4.1.** *In the interval  $[0, g]$  there exists a minimal and a maximal viscosity solution of (4.2), which are in  $C(\bar{\Omega})$  and which are zero on  $\partial\Omega$ .*

*Proof.* Using an iterative process, by Prop. 7.3 and by Prop. 3.4 of [11], we can construct a sequence of functions belonging to  $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ , such that :

- $u_n$  satisfies the equation

$$(4.n) \quad |Du| + u = f(x) + (\lambda+1)u_{n-1} \quad \text{in } \Omega$$

almost everywhere, and  $u_n = 0$  on  $\partial\Omega$  ;

- $u_n$  is a viscosity solution of (4.n) ;
- $0 \leq u_0 \leq u_{n-1} \leq u_n$  .

Furthermore, applying Th. 1.11 of [11], we have also :

- $u_n \leq g(x)$ .

Then the sequence  $\{u_n\}$  is equibounded in  $W^{1,\infty}(\Omega)$  and so  $u_n \rightarrow \bar{u} \in C(\bar{\Omega})$  uniformly. By stability theorem 1.2 of [6]  $\bar{u}$  is a viscosity solution of the equation  $|Du| + u = (\lambda+1)\bar{u}$  in  $\Omega$ , that is  $\bar{u}$  is a viscosity solution of (4.2). Of course  $0 \leq \bar{u}(x) \leq g(x)$ , and  $\bar{u} = 0$  on  $\partial\Omega$ .

Now we show that  $\bar{u}$  is minimal in the interval  $[0, g(x)]$  in the sense that, if  $u$  is a viscosity solution of (4.2) such that :

- $\underline{u}(x) \in C(\bar{\Omega})$ ,  $\underline{u} = 0$  on  $\partial\Omega$ ,
- $0 \leq \underline{u}(x) \leq g(x)$ ,

then :

$$(4.3) \quad \bar{u}(x) \leq \underline{u}(x).$$

In fact, by Th. 1.11 of [11], we get :

$$u_n(x) \leq \underline{u}(x),$$

which implies (4.3) immediately.

Now set  $u'_0(x) = g(x)$ . As above, by Prop. 7.3 and Prop. 3.4 of [11], by an iterative process we can construct a sequence  $\{u'_n\}$  of functions belonging to  $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ , such that :

- $u'_n$  satisfies the equation :

$$(4.n)' \quad |Du| + u = f(x) + (\lambda+1)u'_{n-1} \quad \text{in } \Omega$$

a.e. and  $u'_n = 0$  on  $\partial\Omega$ .

- $u'_n$  is a viscosity solution of (4.n)';
- $0 \leq u'_n(x)$ .

Furthermore, by Th. 1.11 and by Prop. 3.4 of [10], we have :

- $u'_n \leq u'_{n-1} \leq u'_0 = g(x)$ .

Then, analogously as before, we can conclude that  $u'_n$  converges uniformly to a function  $\bar{u}'$ , which is a viscosity solution of (4.2), verifying the boundary condition  $\bar{u}' = 0$  on  $\partial\Omega$  and for which

$$\underline{u}'(x) \leq \bar{u}'(x),$$

for each viscosity solution  $\underline{u}'(x)$  of (4.2) such that :

- $\underline{u}'(x) \in C(\bar{\Omega})$ ,  $\underline{u}'(x) = 0$  on  $\partial\Omega$ ,
- $0 \leq \underline{u}'(x) \leq g(x)$ .

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