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THE SEGRE IMBEDDING AND ITS CONVERSE

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Résumé : En utilisant les coordonnées homogènes des espaces projectifs complexes, C. Segre a construit, en 1891, un plongement kaehlerien de $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ dans $\mathbb{C}P^{N(\alpha_1, \dots, \alpha_n)}$, où $N(\alpha_1, \dots, \alpha_n) = (1 + \alpha_1) \dots (1 + \alpha_n) - 1$. Dans cette Note, nous considérons le problème inverse, et nous obtenons le résultat suivant :

Si $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ est une sous-variété produit de $\mathbb{C}P^m$, et est le produit de n variétés kaehleriennes, alors $m \geq N(\alpha_1, \dots, \alpha_n)$. De plus, si $m = N(\alpha_1, \dots, \alpha_n)$, alors $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ est un ouvert de $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ et l'immersion considérée est le plongement de Segre.

Summary : Using homogeneous coordinates of complex projective spaces, C. Segre constructed in 1891 a Kaehler imbedding of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ in $\mathbb{C}P^{N(\alpha_1, \dots, \alpha_n)}$ where $N(\alpha_1, \dots, \alpha_n) = (1 + \alpha_1) \dots (1 + \alpha_n) - 1$. In this paper, we consider the converse problem to the Segre imbedding and obtain the following result : If $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ is a Kaehler submanifold of $\mathbb{C}P^m$ which is the product of n Kaehler manifolds, then $m \geq N(\alpha_1, \dots, \alpha_n)$. And if $m = N(\alpha_1, \dots, \alpha_n)$, then $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ is an open portion of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ and the immersion is obtained by the Segre imbedding.

0. - INTRODUCTION

Let $\mathbb{C}P^n$ be a (complex) n -dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. Using homogeneous coordinates, C. Segre [4] constructed in 1891 an imbedding from the product variety $\mathbb{C}P^\alpha \times \mathbb{C}P^\beta$ into $\mathbb{C}P^{N(\alpha,\beta)}$, $N(\alpha,\beta) = \alpha + \beta + \alpha\beta$, as follows

$$(0.1) \quad S_{\alpha,\beta} : \mathbb{C}P^\alpha \times \mathbb{C}P^\beta \rightarrow \mathbb{C}P^{N(\alpha,\beta)}$$

$$(x_i) \cdot (y_a) \mapsto (x_i y_a).$$

It is well-known that $S_{\alpha,\beta}$ is a Kaehler imbedding which is known as the Segre imbedding from $\mathbb{C}P^\alpha \times \mathbb{C}P^\beta$ into $\mathbb{C}P^{N(\alpha,\beta)}$.

In 1981, Chen [2] had considered the «converse» problem to the Segre imbedding and obtained the following.

THEOREM A. *If $\mathbb{C}P^m$ admits a Kaehler submanifold $M_1^\alpha \times M_2^\beta$ which is the product of two Kaehler manifolds of (complex) dimension α and β , respectively, then $m \geq N(\alpha,\beta)$. In particular, if $m = N(\alpha,\beta)$, then (a) $M_1^\alpha \times M_2^\beta$ is an open portion of $\mathbb{C}P^\alpha \times \mathbb{C}P^\beta$ and (b) the immersion is obtained by the Segre imbedding $S_{\alpha,\beta}$ up to holomorphic and isometric transformations of $\mathbb{C}P^m$.*

For the product variety $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$, using homogeneous coordinates, C. Segre defined the following imbedding

$$(0.2) \quad S_{\alpha_1 \dots \alpha_n} : \mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n} \rightarrow \mathbb{C}P^{N(\alpha_1, \dots, \alpha_n)}$$

$$(x_{i_1}) \dots (x_{i_n}) \mapsto (x_{i_1} \dots x_{i_n})$$

where

$$(0.3) \quad N(\alpha_1, \dots, \alpha_n) = s_1 + s_2 + \dots + s_n$$

where $s_1 = \sum_{i=1}^n \alpha_i$, $s_2 = \sum_{i < j} \alpha_i \alpha_j, \dots, s_n = \alpha_1 \dots \alpha_n$. It is clear that $S_{\alpha_1 \dots \alpha_n}$ is also a Kaehler imbedding. We call it the Segre imbedding from $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ into $\mathbb{C}P^{N(\alpha_1, \dots, \alpha_n)}$.

In view of Theorem A, it is natural and interesting to consider the following two problems :

Problem 1. Is $N(\alpha_1, \dots, \alpha_n)$ the smallest possible dimension of a complex projective space to admit a Kaehler submanifold $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ which is the product of n Kaehler manifolds ?

Problem 2. If $N(\alpha_1, \dots, \alpha_n)$ is the smallest possible dimension of a complex projective space to admit such a product submanifold, does this product submanifold have to be obtained from the Segre imbedding ?

In this paper we will solve these two problems completely. More precisely, we will obtain the following.

THEOREM 1. *If $\mathbb{C}P^m$ admits a Kaehler submanifold $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ which is the product of n Kaehler manifolds $M_1^{\alpha_1}, \dots, M_n^{\alpha_n}$ of complex dimensions $\alpha_1, \dots, \alpha_n$ respectively, then we have*

$$(1) \quad m \geq N(\alpha_1, \dots, \alpha_n),$$

$$(2) \quad \text{if } m = N(\alpha_1, \dots, \alpha_n), \text{ then}$$

$$(2.1) \quad M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n} \text{ is an open portion of } \mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}, \text{ and}$$

(2.2) *the immersion is given by the Segre imbedding $S_{\alpha_1 \dots \alpha_n}$ up to holomorphic and isometric transformations of $\mathbb{C}P^m$.*

Let h denote the second fundamental form of the immersion and $\bar{\nabla}^p h$ the p -th covariant derivative of h . Then we also have the following best possible inequalities.

THEOREM 2. *Let $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then we have*

$$(0.4) \quad \|\bar{\nabla}^{\ell-2} h\|^2 \geq \ell! 2^\ell \sum_{i_1 < \dots < i_\ell} \alpha_{i_1} \dots \alpha_{i_\ell}$$

for $\ell = 2, 3, \dots, n$. The equality of (0.4) holds for some ℓ if and only if $M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ is an open portion of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ and the immersion is given by the Segre imbedding $S_{\alpha_1 \dots \alpha_n}$ up to holomorphic and isometric transformations of $\mathbb{C}P^m$. Moreover, in this case, the equality of (0.4) holds for all ℓ , $\ell = 2, 3, \dots, n$.

It seems to be interesting to point out that $N(\alpha_1, \dots, \alpha_n) = (1 + \alpha_1) \dots (1 + \alpha_n) - 1$ is much bigger than the dimension of M in general. For example, Theorem 1 shows that if $\mathbb{C}P^m$ contains a Kaehler submanifold M which is the product of twenty 3-dimensional Kaehler manifolds, then M is only 60-dimensional, however, $\mathbb{C}P^m$ is at least 1,099,511,627,776-dimensional !!! Moreover, if m is 1,099,511,627,776, M has to be obtained by the Segre imbedding !!

1. - BASIC FORMULAS

Let M be a submanifold of a Riemannian manifold \tilde{M} with Riemannian metric \langle , \rangle and Riemannian connection ∇' . Denote by ∇ the induced connection on M . The second fundamental form h of the immersion is given by

$$(1.1) \quad h(X, Y) = \nabla'_X Y - \nabla_X Y$$

where X and Y are vector fields tangent to M . For a vector field ξ normal to M and X tangent to M , we put

$$(1.2) \quad \nabla'_X \xi = -A_\xi X + D_X \xi$$

where $-A_\xi X$ and $D_X \xi$ denote the tangential and normal components of $\nabla'_X \xi$, respectively. We have

$$(1.3) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

For the second fundamental form h , we define its first covariant derivative $\bar{\nabla}h$ to be a normal-bundle-valued tensor of type $(0,3)$ given by

$$(1.4) \quad (\bar{\nabla}h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Let R' , R and R^D denote the curvature tensors associated with ∇' , ∇ , and D , respectively. The equations of Gauss, Codazzi, and Ricci are then given respectively by

$$(1.5) \quad R'(X, Y; Z, W) = R(X, Y; Z, W) - \langle h(X, W), h(Y, Z) \rangle \\ + \langle h(X, Z), h(Y, W) \rangle,$$

$$(1.6) \quad R'(X, Y; Z, \xi) = \langle \bar{\nabla}h(X, Y, Z) - \bar{\nabla}h(Y, X, Z), \xi \rangle,$$

$$(1.7) \quad R'(X, Y; \xi, \eta) = R^D(X, Y; \xi, \eta) - \langle [A_\xi, A_\eta]X, Y \rangle$$

for vector fields X, Y, Z, W tangent to M and ξ, η normal to M .

If we define the p -th ($p \geq 1$) covariant derivative of h by

$$(1.8) \quad (\bar{\nabla}^p h)(X_1, X_2, \dots, X_{p+2}) = D_{X_1}((\bar{\nabla}^{p-1} h)(X_2, \dots, X_{p+2})) \\ - \sum_{i=2}^{p+2} (\bar{\nabla}^{p-1} h)(X_2, \dots, \nabla_{X_1} X_i, \dots, X_{p+2}),$$

then $\bar{\nabla}^p h$ is a normal-bundle-valued tensor of type $(0, p+2)$. Moreover, it can be proved that $\bar{\nabla}^p h$ satisfies

$$(1.9) \quad \begin{aligned} & (\bar{\nabla}^p h)(X_1, X_2, X_3, \dots, X_{p+2}) - (\bar{\nabla}^p h)(X_2, X_1, X_3, \dots, X_{p+2}) \\ &= R^D(X_1, X_2) ((\bar{\nabla}^{p-2} h)(X_3, \dots, X_{p+2})) \\ &+ \sum_{i=3}^{p+2} (\bar{\nabla}^{p-2} h)(X_3, \dots, R(X_1, X_2)X_i, \dots, X_{p+2}), \quad p \geq 2. \end{aligned}$$

We put $\bar{\nabla}^0 h = h$.

Let \tilde{M} be a Kaehler manifold with the complex structure J and M be a complex submanifold of \tilde{M} with the induced Kaehler metric. Then we also have the following

$$(1.10) \quad h(JX, Y) = h(X, JY) = Jh(X, Y),$$

$$(1.11) \quad A_{J\xi} = JA_\xi, \quad JA_\xi = -A_\xi J, \quad \text{and} \quad D_X J\xi = JD_X \xi.$$

Let \tilde{R} denote the curvature tensor of $\mathbb{C}P^m$. Then it is well-known that \tilde{R} takes the following form :

$$(1.12) \quad \begin{aligned} \tilde{R}(X, Y)Z &= \langle Y, Z \rangle X - \langle X, Z \rangle Y \\ &+ \langle JY, Z \rangle JX - \langle JX, Z \rangle JY + 2\langle X, JY \rangle JZ. \end{aligned}$$

In particular, if M is a complex submanifold of $\mathbb{C}P^m$, (1.6) and (1.12) imply

$$(1.13) \quad \begin{aligned} (\bar{\nabla} h)(X, Y, Z) &= (\bar{\nabla} h)(Y, X, Z) \\ &= (\bar{\nabla} h)(Z, X, Y). \end{aligned}$$

In section 2, we also denote

$$(\bar{\nabla} h)(X, Y, Z) \text{ by } (\bar{\nabla}_X h)(Y, Z).$$

2. - PRODUCT OF 3 KAEHLER MANIFOLDS

Throughout this section we shall assume that $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ is the Riemannian product of three Kaehler manifolds M_1^α, M_2^β and M_3^γ of (complex) dimensions α, β , and γ , respectively. Let $x : M \rightarrow \mathbb{C}P^m$ be a Kaehler immersion from M into the m -dimensional complex projective space $\mathbb{C}P^m$.

In the following, we assume that $\{X_1, \dots, X_\alpha, JX_1, \dots, JX_\alpha\}$ (respectively, $\{Y_1, \dots, Y_\beta, JY_1, \dots, JY_\beta\}$ and $\{Z_1, \dots, Z_\gamma, JZ_1, \dots, JZ_\gamma\}$) forms an orthonormal basis for M_1^α (respectively, for M_2^β and for M_3^γ). We regard these vector fields as vector fields in M in a natural way.

We need the following results for the proof of the Main Lemma.

LEMMA 1. Let $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then

$$(2.1) \quad \begin{aligned} &h(X_i, Y_a), Jh(X_i, Y_a), h(X_i, Z_r), Jh(X_i, Z_r), h(Y_a, Z_r), \\ &Jh(Y_a, Z_r), \quad i = 1, \dots, \alpha; a = 1, \dots, \beta; r = 1, \dots, \gamma; \end{aligned}$$

are orthonormal local vector fields in $T^\perp M$.

Proof. Let X and W be any unit vectors tangent to M_1^α and $M_2^\beta \times M_3^\gamma$, respectively. Then by (1.5) we have

$$(2.2) \quad \tilde{R}(X, W; W, X) = \langle h(X, W), h(X, W) \rangle - \langle h(X, X), h(W, W) \rangle,$$

$$(2.3) \quad \tilde{R}(X, JW; JW, X) = \langle h(X, JW), h(X, JW) \rangle - \langle h(X, X), h(JW, JW) \rangle.$$

Combining (1.10), (2.2), and (2.3) we find

$$\tilde{K}(X, W) + \tilde{K}(X, JW) = 2 \|h(X, W)\|^2,$$

where \tilde{K} denotes the sectional curvature of $\mathbb{C}P^m$. Since $X \wedge W$ is a totally real section, i.e., $\langle X, W \rangle = \langle X, JW \rangle = 0$, this implies that the length of $h(X, W)$ satisfies

$$(2.4) \quad \|h(X, W)\| = 1.$$

Therefore, by linearity, we obtain

$$(2.5) \quad \langle h(X_i, W), h(X_j, W) \rangle = 0, \quad i \neq j, \quad i, j = 1, \dots, 2\alpha,$$

where we put $X_{\alpha+k} = JX_k$, $k = 1, \dots, \alpha$. Let W_1, W_2 be any two of the orthonormal vectors $Y_1, \dots, Y_\beta, Z_1, \dots, Z_\gamma$. Then we find from (2.5) that

$$(2.6) \quad \langle h(X_i, W_1), h(X_j, W_2) \rangle + \langle h(X_i, W_2), h(X_j, W_1) \rangle = 0.$$

On the other hand, because $R(X_i, X_j; W_1, W_2) = 0$, (1.5) and (1.12) imply

$$(2.7) \quad \langle h(X_i, W_1), h(X_j, W_2) \rangle = \langle h(X_i, W_2), h(X_j, W_1) \rangle.$$

Combining (2.6) and (2.7) we get $\langle h(X_i, W_1), h(X_j, W_2) \rangle = 0$. From this, together with (2.4), we conclude that

$$\begin{aligned} & h(X_i, Y_a), Jh(X_i, Y_a), h(X_i, Z_r), Jh(X_i, Z_r), \\ & i = 1, \dots, \alpha; a = 1, \dots, \beta; r = 1, \dots, \gamma \end{aligned}$$

are orthonormal. Applying the same argument to $h(Y, W)$ for unit vectors Y, W tangent to M_2^β and $M_1^\alpha \times M_3^\gamma$, respectively, we obtain Lemma 1.

LEMMA 2. Let $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ be a Kaehler submanifold of $\mathbb{C}P^m$ and X, Y , and Z unit vector fields tangent to M_1^α , M_2^β , and M_3^γ , respectively. Then we have

$$(2.8) \quad \langle (\bar{\nabla}_X h)(Y, Z), h(Y, Z) \rangle = 0$$

(2.8) and

$$\langle (\bar{\nabla}_X h)(Y, Z), Jh(Y, Z) \rangle = 0.$$

Proof. The first equation of (2.8) follows from (1.4) and the identities $\nabla_X Y = \nabla_X Z = 0$ and $\|h(Y, Z)\| = 1$.

The second equation follows from the first equation and equation (1.14).

LEMMA 3. Under the hypothesis of Lemma 2, we have

$$(2.9) \quad A_{h(Y, Z)} X = 0.$$

Proof. Let U be any unit vector tangent to M , Lemma 1 implies

$$\langle A_{h(Y, Z)} X, U \rangle = \langle h(Y, Z), h(X, U) \rangle = 0.$$

This prove (2.9).

LEMMA 4. Let $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then we have

$$(2.10) \quad \|(\bar{\nabla}_X h)(Y, Z)\| = 1$$

for any unit vector fields X, Y and Z tangent to M_1^α, M_2^β and M_3^γ respectively.

Proof. From the hypothesis, we have $R(X, JX)Y = R(X, JX)Z = 0$. Thus (1.9), (1.11) and (1.12) imply

$$(2.11) \quad \begin{aligned} & \langle (\bar{\nabla}_{JX} \bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_X \bar{\nabla}_{JX} h)(Y, Z), Jh(Y, Z) \rangle \\ &= \langle R^D(JX, X)(h(Y, Z)), Jh(Y, Z) \rangle \\ &= \langle \tilde{R}(X, JX)Jh(Y, Z), h(Y, Z) \rangle - \langle [A_{h(Y, Z)}, A_{Jh(Y, Z)}]JX, X \rangle \\ &= 2 + 2 \|A_{h(Y, Z)}X\|^2 = 2 \end{aligned}$$

by virtue of Lemma 3. On the other hand, (1.8) and Lemma 2 give

$$(2.12) \quad \begin{aligned} & \langle (\bar{\nabla}_{JX} \bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_X \bar{\nabla}_{JX} h)(Y, Z), Jh(Y, Z) \rangle \\ &= \langle D_{JX}((\bar{\nabla}_X h)(Y, Z)) - D_X((\bar{\nabla}_{JX} h)(Y, Z)) - (\bar{\nabla}_{[JX, X]} h)(Y, Z), Jh(Y, Z) \rangle \end{aligned}$$

Thus, by (2.11), (2.12) and Lemma 2 we find

$$\begin{aligned} 2 &= \langle -(\bar{\nabla}_X h)(Y, Z) + (\bar{\nabla}_{JX} h)(Y, Z), D_{JX}(Jh(Y, Z)) \rangle \\ &= \langle -(\bar{\nabla}_X h)(Y, Z) + (\bar{\nabla}_{JX} h)(Y, Z), (\bar{\nabla}_{JX} h)(JY, Z) \rangle \\ &= 2 \|(\bar{\nabla}_X h)(Y, Z)\|^2 \end{aligned}$$

by virtue of (1.13). From this we obtain (2.10).

In the following, we put

$$(2.13) \quad V = \text{Span} \{h(X, Y), h(X, Z), h(Y, Z) \mid X \in TM_1^\alpha, Y \in TM_2^\beta, Z \in TM_3^\gamma\}.$$

Then V is a complex $(\alpha\beta + \beta\gamma + \alpha\gamma)$ -dimensional holomorphic subbundle of the normal bundle $T^\perp M$. Moreover, the vector fields given by (2.1) form an orthonormal local basis of V .

We need the following.

LEMMA 5. Under the hypothesis of Lemma 2, we have

$$(2.14) \quad (\bar{\nabla}_X h)(Y, Z) \text{ is perpendicular to } V.$$

Proof. Let Y and Y' (respectively, Z and Z'), be two unit vector fields tangent to M_2^β (respectively, M_3^γ). Then, for any unit vector field W tangent to M_1^α , Lemma 2 implies

$$(2.15) \quad \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z) \rangle + \langle (\bar{\nabla}_X h)(Y', Z), h(Y, Z) \rangle = 0.$$

On the other hand, from (1.4), (1.13), and Lemma 1, we get

$$(2.16) \quad \begin{aligned} \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z') \rangle &= \langle (\bar{\nabla}_Y h)(X, Z), h(Y', Z') \rangle \\ &= \langle D_Y h(X, Z), h(Y', Z') \rangle \\ &= -\langle h(X, Z), D_Y h(Y', Z') \rangle \\ &= -\langle h(X, Z), (\bar{\nabla}_Y h)(Y', Z') \rangle \\ &= -\langle h(X, Z), (\bar{\nabla}_Y, h)(Y, Z') \rangle \\ &= -\langle h(X, Z), D_Y h(Y, Z') \rangle \\ &= \langle (\bar{\nabla}_Y, h)(X, Z), h(Y, Z') \rangle. \end{aligned}$$

Consequently, we have

$$(2.17) \quad \begin{aligned} \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z') \rangle &= \langle (\bar{\nabla}_X h)(Y', Z), h(Y, Z') \rangle \\ &= \langle (\bar{\nabla}_X h)(Y', Z'), h(Y, Z) \rangle. \end{aligned}$$

Combining (2.15) and (2.17) we obtain

$$(2.18) \quad \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z) \rangle = 0.$$

By linearity, (2.18) implies

$$(2.19) \quad \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z') \rangle + \langle (\bar{\nabla}_X h)(Y, Z'), h(Y', Z) \rangle = 0.$$

Therefore, (2.17) and (2.19) give

$$(2.20) \quad \langle (\bar{\nabla}_X h)(Y, Z), h(Y', Z') \rangle = 0.$$

Since $(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_Z h)(X, Y)$, a similar argument yields $\langle (\bar{\nabla}_X h)(Y, Z), h(X', Y') \rangle = \langle (\bar{\nabla}_X h)(Y, Z), h(X', Z') \rangle = 0$ for any unit vectors X, X' tangent to M_1^α . These prove Lemma 5.

LEMMA 6. Let $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then

$$(2.21) \quad (\bar{\nabla}_{X_i} h)(Y_a, Z_r), \quad i = 1, \dots, 2\alpha, \quad a = 1, \dots, \beta; \quad r = 1, \dots, \gamma$$

are orthonormal local vector fields in $T^\perp M$.

Proof. From (1.9), (1.12), Lemmas 1 and 3 we have

$$(2.22) \quad \begin{aligned} & \langle (\bar{\nabla}_{X_i} \bar{\nabla}_{X_j} h)(Y_a, Z_r) - (\bar{\nabla}_{X_j} \bar{\nabla}_{X_i} h)(Y_a, Z_r), h(Y_b, Z_t) \rangle \\ &= \langle R^D(X_i, X_j)(h(Y_a, Z_r)), h(Y_b, Z_t) \rangle \\ &= \tilde{R}(X_i, X_j; h(Y_a, Z_r), h(Y_b, Z_t)) \\ & \quad + \langle [A_h(Y_a, Z_r), A_h(Y_b, Z_t)] X_i, X_j \rangle \\ &= 0 \end{aligned}$$

for $i, j = 1, \dots, 2\alpha; \quad a, b = 1, \dots, \beta; \quad r, t = 1, \dots, \gamma$.

On the other hand, (1.8) and Lemma 5 imply

$$(2.23) \quad \begin{aligned} & \langle (\bar{\nabla}_{X_i} \bar{\nabla}_{X_j} h)(Y_a, Z_r) - (\bar{\nabla}_{X_j} \bar{\nabla}_{X_i} h)(Y_a, Z_r), h(Y_b, Z_t) \rangle \\ &= \langle D_{X_i}((\bar{\nabla}_{X_j} h)(Y_a, Z_r)) - D_{X_j}((\bar{\nabla}_{X_i} h)(Y_a, Z_r)), h(Y_b, Z_t) \rangle \\ &= \langle (\bar{\nabla}_{X_i} h)(Y_a, Z_r), (\bar{\nabla}_{X_j} h)(Y_b, Z_t) \rangle \\ & \quad - \langle (\bar{\nabla}_{X_j} h)(Y_a, Z_r), (\bar{\nabla}_{X_i} h)(Y_b, Z_t) \rangle. \end{aligned}$$

Hence, (2.22) and (2.23) give

$$(2.24) \quad \begin{aligned} & \langle (\bar{\nabla}_{X_i} h)(Y_a, Z_r), (\bar{\nabla}_{X_j} h)(Y_b, Z_t) \rangle \\ & = \langle (\bar{\nabla}_{X_j} h)(Y_a, Z_r), (\bar{\nabla}_{X_i} h)(Y_b, Z_t) \rangle. \end{aligned}$$

From Lemma 4 and linearity we also have

$$(2.25) \quad \langle (\bar{\nabla}_{X_i} h)(Y, Z), (\bar{\nabla}_{X_j} h)(Y, Z) \rangle = 0, \quad i \neq j; \quad i, j = 1, \dots, 2\alpha.$$

Thus, by using linearity again, we find

$$(2.26) \quad \begin{aligned} & \langle (\bar{\nabla}_{X_i} h)(Y_a, Z), (\bar{\nabla}_{X_j} h)(Y_b, Z) \rangle \\ & + \langle (\bar{\nabla}_{X_i} h)(Y_b, Z), (\bar{\nabla}_{X_j} h)(Y_a, Z) \rangle = 0. \end{aligned}$$

Combining (2.24) and (2.26) we obtain

$$(2.27) \quad \begin{aligned} & \langle (\bar{\nabla}_{X_i} h)(Y_a, Z), (\bar{\nabla}_{X_j} h)(Y_b, Z) \rangle = 0, \\ & i \neq j; \quad i, j = 1, \dots, 2\alpha. \end{aligned}$$

Thus, by applying linearity, (2.24), (2.27) and Lemma 4, we obtain Lemma 6.

Combining Lemmas 1, 5 and 6, we obtain the following.

LEMMA 7. Let $M = M_1^\alpha \times M_2^\beta \times M_3^\gamma$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then

$$\begin{aligned} & h(X_i, Y_a), Jh(X_i, Y_a), h(X_i, Z_r), Jh(X_i, Z_r), \\ & h(Y_a, Z_r), Jh(Y_a, Z_r), (\bar{\nabla}h)(X_i, Y_a, Z_r), J(\bar{\nabla}h)(X_i, Y_a, Z_r); \\ & i = 1, \dots, \alpha; \quad a = 1, \dots, \beta; \quad r = 1, \dots, \gamma; \end{aligned}$$

are orthonormal local vector fields in $T^\perp M$.

3. - MAIN LEMMA

Let $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ be the Riemannian product of n Kaehler manifolds $M_1^{\alpha_1}, \dots, M_n^{\alpha_n}$ of complex dimensions $\alpha_1, \dots, \alpha_n$ respectively. Assume that M is a Kaehler submanifold of $\mathbb{C}P^m$.

In the following, we denote by X^i, Y^i, Z^i, \dots , etc. (with super-index i) vector fields tangent to $M_i^{\alpha_i}$. We shall also regard them as vector fields tangent to $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ in a natural way. Moreover, we assume that $X_1^i, \dots, X_{\alpha_i}^i, X_{\alpha_i+1}^i = JX_1^i, \dots, X_{2\alpha_i}^i = JX_{\alpha_i}^i$ form an orthonormal basis for $M_i^{\alpha_i}$.

We need the following Main Lemma.

LEMMA 8. Let $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ be a Kaehler submanifold of $\mathbb{C}P^m$. Then the following vectors

$$\begin{aligned}
 & h(X_{a_1}^{i_1}, X_{a_2}^{i_2}), Jh(X_{a_1}^{i_1}, X_{a_2}^{i_2}), (\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \\
 & J(\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \dots, (\bar{\nabla}^{n-2}h)(X_{a_1}^{i_1}, \dots, X_{a_n}^{i_n}), \\
 (*) \quad & J(\bar{\nabla}^{n-2}h)(X_{a_1}^{i_1}, \dots, X_{a_n}^{i_n}), \\
 & i_1 < i_2 < \dots < i_n; 1 \leq j, i_1, \dots, i_n \leq n; 1 \leq a_j \leq \alpha_{i_j};
 \end{aligned}$$

are $2(s_2 + s_3 + \dots + s_n)$ orthonormal vectors normal to M .

Proof. We will prove this lemma by induction.

If $n = 3$, this lemma is just Lemma 7. Now we assume that this lemma holds for $n \leq \ell - 1$, $\ell \geq 4$, we want to prove that it is also true for $n = \ell$.

Let $M = M_1^{\alpha_1} \times \dots \times M_\ell^{\alpha_\ell}$ be a Kaehler submanifold of $\mathbb{C}P^m$. We put

$$\bar{M}_{\ell-1}^{\alpha_{\ell-1}} = M_{\ell-1}^{\alpha_{\ell-1}} \times M_\ell^{\alpha_\ell} \quad \text{and} \quad \bar{M}_j^{\alpha_j} = M_j^{\alpha_j}, \quad j = 1, \dots, \ell - 2.$$

We consider $\bar{M}_1^{\alpha_1} \times \dots \times \bar{M}_{\ell-1}^{\alpha_{\ell-1}}$. Then $X_1^{\ell-1}, \dots, X_{\alpha_{\ell-1}}^{\ell-1}, X_1^\ell, \dots, X_{\alpha_\ell}^\ell, JX_1^{\ell-1}, \dots, JX_{\alpha_{\ell-1}}^{\ell-1}, JX_1^\ell, \dots, JX_{\alpha_\ell}^\ell$ form an orthonormal basis for $M_{\ell-1}$. Thus by induction, we know that

$$\begin{aligned}
 & h(X_{a_1}^{i_1}, X_{a_2}^{i_2}), Jh(X_{a_1}^{i_1}, X_{a_2}^{i_2}), h(X_{a_1}^{i_1}, X_{a_{\ell-1}}^{\ell-1}), Jh(X_{a_1}^{i_1}, X_{a_{\ell-1}}^{\ell-1}), \\
 & h(X_{a_1}^{i_1}, X_{a_\ell}^\ell), Jh(X_{a_1}^{i_1}, X_{a_\ell}^\ell), (\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \\
 & J(\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), (\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_{\ell-1}}^{\ell-1}), \\
 & J(\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), (\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_\ell}^\ell), \\
 & J(\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_\ell}^\ell), \dots, (\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-1}}^{\ell-1}), \\
 & J(\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-1}}^{\ell-1}), (\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-2}}^{\ell-2}, X_{a_{\ell-1}}^{\ell-1}) \\
 & J(\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-2}}^{\ell-2}, X_{a_{\ell-1}}^{\ell-1}), (\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-2}}^{\ell-2}, X_{a_\ell}^\ell) \\
 & J(\bar{\nabla}^{\ell-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{\ell-2}}^{\ell-2}, X_{a_\ell}^\ell); \\
 & i_1 < \dots < i_{\ell-2}; 1 \leq i_1, \dots, i_{\ell-2} \leq \ell-2; 1 \leq a_j \leq \alpha_{i_j},
 \end{aligned}$$

are orthonormal vectors normal to $M_1^{\alpha_1} \times \dots \times M_\ell^{\alpha_\ell}$. Applying the same argument to all other possible similar cases and by induction, we obtain the following.

Statement 1. The following normal vectors ;

$$\begin{aligned}
 & h(X_{a_1}^{i_1}, X_{a_2}^{i_2}), Jh(X_{a_1}^{i_1}, X_{a_2}^{i_2}), (\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \\
 & J(\bar{\nabla}h)(X_{a_1}^{i_1}, X_{a_2}^{i_2}, X_{a_3}^{i_3}), \dots, \\
 & (\bar{\nabla}^{n-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{n-1}}^{i_{n-1}}), J(\bar{\nabla}^{n-3}h)(X_{a_1}^{i_1}, \dots, X_{a_{n-1}}^{i_{n-1}}); \\
 & i_1 < i_2 < \dots < i_{n-1}, 1 \leq j, i_1, \dots, i_{n-1} \leq n, 1 \leq a_j \leq \alpha_{i_j};
 \end{aligned}$$

are orthonormal.

We need the following.

Statement 2. For $i_1 < \dots < i_{t+2}$, and $i \neq i_1, \dots, i_{t+2}$, and any permutation σ of (i_1, \dots, i_{t+2}) , we

have

$$(3.1) \quad A_{(\bar{\nabla}^t h)(X^{i_1}, \dots, X^{i_{t+2}})} X^i = 0$$

and

$$(3.2) \quad (\bar{\nabla}^t h)(X^{i_1}, \dots, X^{i_{t+2}}) = (\bar{\nabla}^t h)(X^{\sigma(i_1)}, \dots, X^{\sigma(i_{t+2})}).$$

Proof. First we mention that Lemma 3 implies the following

$$(3.3) \quad A_{h(X^{i_1}, X^{i_2})} X^i = 0, \quad i_1 \neq i_2.$$

If $k \neq i$, then Statement 1 yields

$$(3.4) \quad \begin{aligned} & \langle A_{(\bar{\nabla}^t h)(X^{i_1}, \dots, X^{i_{t+2}})} X^i, X^k \rangle \\ &= \langle h(X^i, X^k), (\bar{\nabla}^t h)(X^{i_1}, \dots, X^{i_{t+2}}) \rangle \\ &= 0. \end{aligned}$$

Hence, in order to prove (3.1), it suffices to prove

$$(3.5) \quad \langle A_{(\bar{\nabla}^t h)(X^{i_1}, \dots, X^{i_{t+2}})} X^i, Y^i \rangle = 0,$$

for any vector Y^i tangent to $M_i^{\alpha_i}$.

For $i \neq i_1, i_2, i_3$ and unit vector fields $X^{i_1}, X^{i_2}, X^{i_3}, X^i$, and Y^i , we have

$$\begin{aligned} & \langle A_{(\bar{\nabla} h)(X^{i_1}, X^{i_2}, X^{i_3})} X^i, Y^i \rangle \\ &= \langle (\bar{\nabla} h)(X^{i_1}, X^{i_2}, X^{i_3}), h(X^i, Y^i) \rangle \\ &= \langle D_{X^{i_1}} h(X^{i_2}, X^{i_3}), h(X^i, Y^i) \rangle \\ &= -\langle h(X^{i_2}, X^{i_3}), D_{X^{i_1}} h(X^i, Y^i) \rangle \end{aligned}$$

$$\begin{aligned}
 &= -\langle h(X^i_2, X^i_3), (\bar{\nabla}h)(X^i_1, X^i, Y^i) \rangle \\
 &= -\langle h(X^i_2, X^i_3), (\bar{\nabla}h)(X^i, X^i_1, Y^i) \rangle \\
 &= -\langle h(X^i_2, X^i_3), D_{X^i} h(X^i_1, Y^i) \rangle \\
 &= \langle D_{X^i} h(X^i_2, X^i_3), h(X^i_1, Y^i) \rangle \\
 &= \langle (\bar{\nabla}h)(X^i, X^i_2, X^i_3), h(X^i_1, Y^i) \rangle \\
 &= 0.
 \end{aligned}$$

This proves (3.1) for $t = 1$.

For (3.2), if $t = 1$, (3.2) follows from (1.13). Now, we assume that both (3.1) and (3.2) are true for $t \leq r-1$, $r \geq 2$, that is we have

$$(3.6) \quad A_{(\bar{\nabla}^t h)(X^i_1, \dots, X^i_{t+2})} X^i = 0$$

and

$$(3.7) \quad (\bar{\nabla}^t h)(X^i_1, \dots, X^i_{t+2}) = (\bar{\nabla}^t h)(X^{\sigma(i_1)}, \dots, X^{\sigma(i_{t+2})})$$

for $t \leq r-1$. Then we have from Statement 1 and (3.7) that

$$\begin{aligned}
 (3.8) \quad &\langle A_{(\bar{\nabla}^r h)(X^i_1, \dots, X^i_{r+2})} X^i, Y^i \rangle \\
 &= \langle (\bar{\nabla}^r h)(X^i_1, \dots, X^i_{r+2}), h(X^i, Y^i) \rangle \\
 &= \langle D_{X^i_1} ((\bar{\nabla}^{r-1} h)(X^i_2, \dots, X^i_{r+2})), h(X^i, Y^i) \rangle \\
 &= -\langle (\bar{\nabla}^{r-1} h)(X^i_2, \dots, X^i_{r+2}), D_{X^i_1} h(X^i, Y^i) \rangle \\
 &= -\langle (\bar{\nabla}^{r-1} h)(X^i_2, \dots, X^i_{r+2}), (\bar{\nabla}h)(X^i_1, X^i, Y^i) \rangle \\
 &= -\langle (\bar{\nabla}^{r-1} h)(X^i_2, \dots, X^i_{r+2}), (\bar{\nabla}h)(X^i, X^i_1, Y^i) \rangle \\
 &= \langle (\bar{\nabla}^r h)(X^i, X^i_2, \dots, X^i_{r+2}), h(X^i_1, Y^i) \rangle.
 \end{aligned}$$

If $i < i_2$, then from Statement 1, we obtain (3.5) and hence (3.6) for $t = r$.

If $i > i_2$, then (3.8) implies

$$\begin{aligned}
& \langle A_{(\bar{\nabla}^r h)(X^{i_1}, \dots, X^{i_{r+2}})} X^i, Y^i \rangle \\
&= \langle (\bar{\nabla}^r h)(X^{i_2}, X^i, X^{i_3}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i) \rangle \\
&\quad + \langle R^D(X^i, X^{i_2})(\bar{\nabla}^{r-2} h)(X^{i_3}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i) \rangle \\
&= \langle (\bar{\nabla}^r h)(X^{i_2}, X^i, X^{i_3}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i) \rangle \\
&\quad + \tilde{R}(X^i, X^{i_2}; (\bar{\nabla}^{r-2} h)(X^{i_3}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i)) \\
&\quad + \langle [A_{(\bar{\nabla}^{\ell-2} h)(X^{i_3}, \dots, X^{i_{r+2}})}]_{h(X^{i_1}, Y^i)}^A X^i, X^{i_2} \rangle.
\end{aligned}$$

Thus, by (1.8), (1.12), and (3.7) we find

$$\begin{aligned}
& \langle A_{(\bar{\nabla}^r h)(X^{i_1}, \dots, X^{i_{r+2}})} X^i, Y^i \rangle \\
&= \langle (\bar{\nabla}^r h)(X^{i_2}, X^i, X^{i_3}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i) \rangle \\
&= \langle D_{X^{i_2}}^{i_2} ((\bar{\nabla}^{r-1} h)(X^i, X^{i_3}, \dots, X^{i_{r+2}})), h(X^{i_1}, Y^i) \rangle \\
&= \langle D_{X^{i_2}}^{i_2} ((\bar{\nabla}^{r-1} h)(X^{i_3}, \dots, X^{i_s}, X^i, X^{i_{s+1}}, \dots, X^{i_{r+2}})), h(X^{i_1}, Y^i) \rangle \\
&= \langle (\bar{\nabla}^r h)(X^{i_2}, \dots, X^{i_s}, X^i, X^{i_{s+1}}, \dots, X^{i_{r+2}}), h(X^{i_1}, Y^i) \rangle,
\end{aligned}$$

where $i_s < i < i_{s+1}$. Thus by Statement 1, we obtain (3.5) and hence (3.6) for $t = r$.

Now, we shall prove (3.7) for $t = r$.

Let ξ be any normal vector field normal to M . Then by (1.9), (3.7) and induction we have

$$\begin{aligned}
 & \langle (\bar{\nabla}^r h)(X^{i_1}, \dots, X^{i_{r+2}}), \xi \rangle \\
 &= \langle (\bar{\nabla}^r h)(X^{i_2}, X^{i_1}, X^{i_3}, \dots, X^{i_{r+2}}), \xi \rangle \\
 &+ \langle R^D(X^{i_1}, X^{i_2})(\bar{\nabla}^{r-2} h)(X^{i_3}, \dots, X^{i_{r+2}}), \xi \rangle \\
 &= \langle (\bar{\nabla}^r h)(X^{i_2}, X^{i_1}, X^{i_3}, \dots, X^{i_{r+2}}), \xi \rangle \\
 &+ \tilde{R}(X^{i_1}, X^{i_2}; \bar{\nabla}^{r-2} h)(X^{i_3}, \dots, X^{i_{r+2}}), \xi \\
 &+ \langle [A_{\bar{\nabla}^{r-2} h}(X^{i_3}, \dots, X^{i_{r+2}}), A_\xi] X^{i_1}, X^{i_2} \rangle \\
 &= \langle (\bar{\nabla}^r h)(X^{i_2}, X^{i_1}, X^{i_3}, \dots, X^{i_{r+2}}), \xi \rangle.
 \end{aligned}$$

This shows that

$$(3.9) \quad (\bar{\nabla}^r h)(X^{i_1}, X^{i_2}, \dots, X^{i_{r+2}}) = (\bar{\nabla}^r h)(X^{i_2}, X^{i_1}, X^{i_3}, \dots, X^{i_{r+2}}).$$

If $1 < s$, then we also have from induction

$$\begin{aligned}
 (3.10) \quad & (\bar{\nabla}^r h)(X^{i_1}, \dots, X^{i_s}, X^{i_{s+1}}, \dots, X^{i_{r+2}}) \\
 &= D_{X^{i_1}}^{i_1} ((\bar{\nabla}^{r-1} h)(X^{i_2}, \dots, X^{i_s}, \dots, X^{i_{r+2}})) \\
 &= D_{X^{i_1}}^{i_1} ((\bar{\nabla}^{r-1} h)(X^{i_2}, \dots, X^{i_{s+1}}, X^{i_s}, \dots, X^{i_{r+2}})) \\
 &= (\bar{\nabla}^r h)(X^{i_1}, \dots, X^{i_{s-1}}, X^{i_{s+1}}, X^{i_s}, X^{i_{s+2}}, \dots, X^{i_{r+2}}).
 \end{aligned}$$

Consequently, (3.9) and (3.10) imply (3.7) for $t = r$. Thus, by induction, we obtain (3.6) and (3.7) for any t . This proves Statement 2.

Statement 3. For unit vectors $X^{i_1}, \dots, X^{i_{p+2}}$, $i_1 < \dots < i_{p+2}$, $1 \leq s \leq p+2$, we have

$$(3.11) \quad (\bar{\nabla}^p h)(X^{i_1}, \dots, J X^{i_s}, \dots, X^{i_{p+2}}) = J (\bar{\nabla}^p h)(X^{i_1}, \dots, X^{i_{p+2}}).$$

Proof. If $p = 0$, (3.7) follows from (1.10). If $p = 1$, we obtain from (1.4) and (1.13) that

$$\begin{aligned}
 & (\bar{\nabla} h)(J X^{i_1}, X^{i_2}, X^{i_3}) = (\bar{\nabla} h)(X^{i_2}, J X^{i_1}, X^{i_3}) \\
 &= D_{X^{i_2}}^{i_2} (h(J X^{i_1}, X^{i_3})) = D_{X^{i_2}}^{i_2} J (h(X^{i_1}, X^{i_3}))
 \end{aligned}$$

$$\begin{aligned}
&= JD_{X^i_2} h(X^i_1, X^i_3) = J(\bar{\nabla}h)(X^i_2, X^i_1, X^i_3) \\
&= J(\bar{\nabla}h)(X^i_1, X^i_2, X^i_3).
\end{aligned}$$

Similar argument also yields

$$(\bar{\nabla}h)(X^i_1, JX^i_2, X^i_3) = (\bar{\nabla}h)(X^i_1, X^i_2, JX^i_3) = J(\bar{\nabla}h)(X^i_1, X^i_2, X^i_3).$$

These proves (3.11) for $p = 1$.

Now, assume that (3.11) holds for $1 \leq p \leq t-1$. If $s > 1$, then, by (1.9) and induction, we have

$$\begin{aligned}
(\bar{\nabla}^t h)(X^i_1, \dots, JX^i_s, \dots, X^i_{t+2}) &= D_{X^i_1} (\bar{\nabla}^{t-1} h)(X^i_2, \dots, JX^i_s, \dots, X^i_{t+2}) \\
&= D_{X^i_1} J(\bar{\nabla}^{t-1} h)(X^i_2, \dots, X^i_s, \dots, X^i_{t+2}) \\
&= JD_{X^i_1} (\bar{\nabla}^{t-1} h)(X^i_2, \dots, X^i_{t+2}) \\
&= J(\bar{\nabla}^t h)(X^i_1, \dots, X^i_{t+2}).
\end{aligned}$$

If $s = 1$, then by (1.9), Statements 1 and 2, we find that for any unit normal vector field ξ , we have

$$\begin{aligned}
&\langle (\bar{\nabla}^t h)(JX^i_1, X^i_2, \dots, X^i_{t+2}), \xi \rangle \\
&= \langle (\bar{\nabla}^t h)(X^i_2, JX^i_1, \dots, X^i_{t+2}), \xi \rangle \\
&\quad + \langle R^D(JX^i_1, X^i_2)(\bar{\nabla}^{t-2} h)(X^i_3, \dots, X^i_{t+2}), \xi \rangle \\
&= \langle J(\bar{\nabla}^t h)(X^i_2, X^i_1, \dots, X^i_{t+2}), \xi \rangle \\
&\quad + \langle \tilde{R}(JX^i_1, X^i_2; \bar{\nabla}^{t-2} h)(X^i_3, \dots, X^i_{t+2}), \xi \rangle \\
&\quad + \langle [A_{(\bar{\nabla}^{t-2} h)(X^i_3, \dots, X^i_{t+2})}, A_\xi] JX^i_1, X^i_2 \rangle \\
&= \langle J(\bar{\nabla}^t h)(X^i_2, X^i_1, \dots, X^i_{t+2}), \xi \rangle \\
&= \langle J(\bar{\nabla}^t h)(X^i_1, X^i_2, \dots, X^i_{t+2}), \xi \rangle \\
&\quad - \langle R^D(X^i_2, X^i_1)(\bar{\nabla}^{t-2} h)(X^i_3, \dots, X^i_{t+2}), J\xi \rangle \\
&= \langle J(\bar{\nabla}^t h)(X^i_1, X^i_2, \dots, X^i_{t+2}), \xi \rangle.
\end{aligned}$$

Since this is true for any ξ , we obtain $(\bar{\nabla}^t_h)(JX^1, X^2, \dots, X^{t+2}) = J(\bar{\nabla}^t_h)(X^1, \dots, X^{t+2})$. Consequently, we obtain (3.11) for $p = t$. Thus by induction, we obtain Statement 3.

Statement 4. For unit vectors X^1, \dots, X^n , we have

$$\|(\bar{\nabla}^{n-2}_h)(X^1, \dots, X^n)\| = 1.$$

Proof. We have

$$\begin{aligned}
(3.12) \quad & \langle (\bar{\nabla}^{n-1}_h)(JX^1, X^1, X^2, \dots, X^n) - (\bar{\nabla}^{n-1}_h)(X^1, JX^1, X^2, \dots, X^n), \\
& J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n) \rangle \\
& = \langle R^D(JX^1, X^1)((\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)), J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n) \rangle \\
& = \langle \tilde{R}(JX^1, X^1; (\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)), J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n) \rangle \\
& \quad + \langle [{}^A_{(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)} {}^A_{J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)}] JX^1, X^1 \rangle \\
& = 2 + 2 \| {}^A_{(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)} X^1 \|^2 = 2.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(3.13) \quad & \langle (\bar{\nabla}^{n-1}_h)(JX^1, X^1, X^2, \dots, X^n) - (\bar{\nabla}^{n-1}_h)(X^1, JX^1, X^2, \dots, X^n), \\
& J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n) \rangle \\
& = \langle D_{JX^1}((\bar{\nabla}^{n-2}_h)(X^1, \dots, X^n)) - D_{X^1}((\bar{\nabla}^{n-2}_h)(JX^1, X^2, \dots, X^n)) \\
& \quad - (\bar{\nabla}^{n-2}_h)([JX^1, X^1], X^2, \dots, X^n), J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n) \rangle \\
& = \langle (\bar{\nabla}^{n-2}_h)(JX^1, X^2, \dots, X^n), D_{X^1}(J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)) \rangle \\
& \quad - \langle (\bar{\nabla}^{n-2}_h)(X^1, \dots, X^n), D_{JX^1}(J(\bar{\nabla}^{n-3}_h)(X^2, \dots, X^n)) \rangle \\
& = \|(\bar{\nabla}^{n-2}_h)(JX^1, X^2, \dots, X^n)\|^2 + \|(\bar{\nabla}^{n-2}_h)(X^1, \dots, X^n)\|^2 \\
& = 2 \|(\bar{\nabla}^{n-2}_h)(X^1, X^2, \dots, X^n)\|^2.
\end{aligned}$$

Combining (3.12) and (3.13), we obtain Statement 4.

Statement 5. The following vectors

$$(\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n), J(\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n);$$

$$a_i = 1, \dots, \alpha_i; \quad i = 1, \dots, n;$$

are orthonormal.

Proof. From (1.7), (1.9), (1.12), and Statements 1 and 2, we find

$$\begin{aligned} & \langle (\bar{\nabla}^{n-1}h)(X_{a_1}^1, X_{b_1}^1, X_{a_2}^2, \dots, X_{a_n}^n) - (\bar{\nabla}^{n-1}h)(X_{b_1}^1, X_{a_1}^1, X_{a_2}^2, \dots, X_{a_n}^n), \\ & \quad (\bar{\nabla}^{n-3}h)(X_{b_2}^2, \dots, X_{b_n}^n) \rangle \\ &= \langle R^D(X_{a_1}^1, X_{b_1}^1)((\bar{\nabla}^{n-3}h)(X_{a_2}^2, \dots, X_{a_n}^n)), (\bar{\nabla}^{n-3}h)(X_{b_2}^2, \dots, X_{b_n}^n) \rangle \\ &= \tilde{R}(X_{a_1}^1, X_{b_1}^1; (\bar{\nabla}^{n-3}h)(X_{a_2}^2, \dots, X_{a_n}^n), (\bar{\nabla}^{n-3}h)(X_{b_2}^2, \dots, X_{b_n}^n)) \\ & \quad + \langle [{}^A(\bar{\nabla}^{n-3}h)(X_{a_2}^2, \dots, X_{a_n}^n), {}^A(\bar{\nabla}^{n-3}h)(X_{b_2}^2, \dots, X_{b_n}^n)] X_{a_1}^1, X_{b_1}^1 \rangle \\ &= 0. \end{aligned}$$

Hence, by using (1.8) and Statement 1, we may obtain

$$(3.14) \quad \begin{aligned} & \langle (\bar{\nabla}^{n-2}h)(X_{b_1}^1, X_{a_2}^2, \dots, X_{a_n}^n), (\bar{\nabla}^{n-2}h)(X_{a_1}^1, X_{b_2}^2, \dots, X_{b_n}^n) \rangle \\ &= \langle (\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n), (\bar{\nabla}^{n-2}h)(X_{b_1}^1, \dots, X_{b_n}^n) \rangle. \end{aligned}$$

By continuing this process sufficient by many times, we will obtain

$$(3.15) \quad \begin{aligned} & \langle (\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n), (\bar{\nabla}^{n-2}h)(X_{b_1}^1, \dots, X_{b_n}^n) \rangle \\ &= \langle (\bar{\nabla}^{n-2}h)(X_{c_1}^1, \dots, X_{c_n}^n), (\bar{\nabla}^{n-2}h)(X_{e_1}^1, \dots, X_{e_n}^n) \rangle, \end{aligned}$$

where $\{c_i, e_i\} = \{a_i, b_i\}$, $i = 1, \dots, n$. Thus, by using linearity, (3.15), and Statement 4, we may conclude that

$$(\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n); \quad a_i = 1, \dots, \alpha_i; \quad i = 1, \dots, n;$$

are orthonormal. Therefore, it suffices to prove that

$$(3.16) \quad \langle (\bar{\nabla}^{n-2}h)(X_{a_1}^1, \dots, X_{a_n}^n), J(\bar{\nabla}^{n-2}h)(X_{b_1}^1, \dots, X_{b_n}^n) \rangle = 0.$$

If $a_i = b_i$ for all i , then (3.16) is trivial. Suppose that there is an i such that $a_i \neq b_i$, then we just replace $X_{b_i}^i$ by $JX_{b_i}^i$, and applying the previous case, we obtain (3.16). Consequently, we obtain Statement 5.

In the following, we put

$$(3.17) \quad N_0 = \text{Span} \{ h(X_{a_1}^{i_1}, X_{a_2}^{i_2}), Jh(X_{a_1}^{i_1}, X_{a_2}^{i_2}) \mid i_1 < i_2, \\ a_1 = 1, \dots, \alpha_{i_1}, \quad a_2 = 1, \dots, \alpha_{i_2} \}$$

and

$$(3.18) \quad N_r = \text{Span} \{ (\bar{\nabla}^r h)(X_{a_1}^{i_1}, \dots, X_{a_{r+2}}^{i_{r+2}}), J(\bar{\nabla}^r h)(X_{a_1}^{i_1}, \dots, X_{a_{r+2}}^{i_{r+2}}) \mid \\ i_1 < \dots < i_{r+2}; \quad a_t = 1, \dots, \alpha_{i_t} \}$$

for $r = 1, \dots, n-2$.

Statement 6. We have

$$N_{n-2} \perp N_t,$$

for $t = 0, 1, \dots, n-3$.

Proof. If $n = 3$, this statement is already proved in Lemma 5. Now, assume that $n \geq 4$.

If $t \leq n-4$ and $i_1 < \dots < i_t$, then we may find one j such that $j \neq i_1, \dots, i_t$. Using Statements 1 and 5, we have

$$\begin{aligned} & \langle (\bar{\nabla}^{n-2}h)(X^1, \dots, X^n), (\bar{\nabla}^t h)(Y^{i_1}, \dots, Y^{i_{t+2}}) \rangle \\ &= \langle (\bar{\nabla}^{n-2}h)(X^j, X^1, \dots, \hat{X}^j, \dots, X^n), (\bar{\nabla}^t h)(Y^{i_1}, \dots, Y^{i_{t+2}}) \rangle \\ &= \langle D_{X^j}((\bar{\nabla}^{n-2}h)(X^1, \dots, \hat{X}^j, \dots, X^n)), (\bar{\nabla}^t h)(Y^{i_1}, \dots, Y^{i_{t+2}}) \rangle \\ &= - \langle (\bar{\nabla}^{n-3}h)(X^1, \dots, \hat{X}^j, \dots, X^n), D_{X^j}((\bar{\nabla}^t h)(Y^{i_1}, \dots, Y^{i_{t+2}})) \rangle \\ &= - \langle (\bar{\nabla}^{n-3}h)(X^1, \dots, \hat{X}^j, \dots, X^n), (\bar{\nabla}^{t+1} h)(X^j, Y^{i_1}, \dots, Y^{i_{t+2}}) \rangle \\ &= 0. \end{aligned}$$

Consequently we have $N_{n-2} \perp N_t$ for $t = 0, 1, \dots, n-4$.

Now, we want to prove that $N_{n-2} \perp N_{n-3}$. Let X^1, \dots, X^n be unit vector fields. Then we obtain from Statement 1 that

$$\|(\bar{\nabla}^{n-3}h)(X^2, \dots, X^n)\| = 1.$$

Thus, by (1.8), we have

$$\langle (\bar{\nabla}^{n-2}h)(X^1, X^2, \dots, X^n), (\bar{\nabla}^{n-3}h)(X^2, \dots, X^n) \rangle = 0.$$

Hence, by linearity, we get

$$(3.19) \quad \begin{aligned} & \langle (\bar{\nabla}^{n-2}h)(X^1, X^2, X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(X^2, X^3, \dots, X^n) \rangle \\ & + \langle (\bar{\nabla}^{n-2}h)(X^1, Y^2, X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(X^2, X^3, \dots, X^n) \rangle = 0. \end{aligned}$$

On the other hand, using Statements 1 and 5, we also have

$$\begin{aligned} & \langle (\bar{\nabla}^{n-2}h)(X^1, \dots, X^n), (\bar{\nabla}^{n-3}h)(Y^2, X^3, \dots, X^n) \rangle \\ & = \langle (\bar{\nabla}^{n-2}h)(X^2, X^1, X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(Y^2, X^3, \dots, X^n) \rangle \\ & = \langle D_{X^2}((\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n)), (\bar{\nabla}^{n-3}h)(Y^2, X^3, \dots, X^n) \rangle \\ & = -\langle (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n), (\bar{\nabla}^{n-2}h)(X^2, Y^2, X^3, \dots, X^n) \rangle \\ & \quad - \langle (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(\nabla_{X^2}Y^2, X^3, \dots, X^n) \rangle \\ & = -\langle (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n), (\bar{\nabla}^{n-2}h)(X^2, Y^2, X^3, \dots, X^n) \rangle \\ & = -\langle (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n), (\bar{\nabla}^{n-2}h)(Y^2, X^2, X^3, \dots, X^n) \rangle \\ & \quad + R^D(X^2, Y^2)((\bar{\nabla}^{n-4}h)(X^3, \dots, X^n)) \rangle \\ & = -\langle (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n), D_{Y^2}(\bar{\nabla}^{n-3}h)(X^2, \dots, X^n) \rangle \\ & \quad - \tilde{R}(X^2, Y^2; (\bar{\nabla}^{n-4}h)(X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n)) \\ & = -\langle [{}^A(\bar{\nabla}^{n-4}h)(X^3, \dots, X^n), {}^A(\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n)]X^2, Y^2 \rangle \end{aligned}$$

$$\begin{aligned}
 &= \langle D_{Y^2}((\bar{\nabla}^{n-3}h)(X^1, X^3, \dots, X^n)), (\bar{\nabla}^{n-3}h)(X^2, \dots, X^n) \rangle \\
 &= \langle (\bar{\nabla}^{n-2}h)(X^1, Y^2, X^3, \dots, X^n), (\bar{\nabla}^{n-3}h)(X^2, \dots, X^n) \rangle.
 \end{aligned}$$

Combining this with (3.19), we obtain

$$(3.20) \quad \langle (\bar{\nabla}^{n-2}h)(X^1, X^2, \dots, X^n), (\bar{\nabla}^{n-3}h)(Y^2, X^3, \dots, X^n) \rangle = 0.$$

Continuing this process $n-1$ times, we will find

$$\langle (\bar{\nabla}^{n-2}h)(X^1, X^2, \dots, X^n), (\bar{\nabla}^{n-2}h)(Y^2, \dots, Y^n) \rangle = 0.$$

Therefore, by using Statement 6, we conclude that N_{n-2} is perpendicular to N_{n-3} . Thus Statement 6 is proved.

From Statements 1, 5, and 6, we obtain Lemma 8.

(Q.E.D.)

4. - PROOF OF THEOREM 1

Let $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ be a Kaehler submanifold of $\mathbb{C}P^m$ which is the product of n Kaehler manifolds. Then by the Main Lemma, we see that $m \geq N(\alpha_1, \dots, \alpha_n)$.

Now, we assume that m is the smallest possible dimension $N(\alpha_1, \dots, \alpha_n)$. We want to prove that each $M_i^{\alpha_i}$ is an open portion of a $\mathbb{C}P^{\alpha_i}$.

Let X^i, Y^i be any two unit vector fields tangent to $M_i^{\alpha_i}$. We need the following Lemmas.

LEMMA 9. For any X^i, Y^i tangent to $M_i^{\alpha_i}$, we have

$$(4.1) \quad h(X^i, Y^i) \perp N_o,$$

where N_o is defined by (3.17).

Proof. Let X^i, Y^i , and Z^i be tangent to $M_i^{\alpha_i}$ and W^j tangent to $M_j^{\alpha_j}$, $j \neq i$. Then from (1.5) of

Gauss, (1.10) and (1.12), we find

$$\begin{aligned} & \langle h(X^i, Y^i), h(JX^i, W^j) \rangle \\ &= \langle h(X^i, W^j), h(JX^i, Y^i) \rangle \\ &= -\langle h(JX^i, W^j), h(X^i, Y^i) \rangle \end{aligned}$$

Hence, we have $\langle h(X^i, Y^i), h(X^i, W^j) \rangle = 0$. By applying linearity and using the equation of Gauss again, we obtain

$$(4.2) \quad \langle h(X^i, Y^i), h(Z^j, W^j) \rangle = 0, \quad i \neq j.$$

If $j, k \neq i$, then Main Lemma and equation (1.5) of Gauss also yield

$$(4.3) \quad \langle h(X^i, Y^i), h(W^j, V^k) \rangle = 0.$$

Combining (4.2) and (4.3) we obtain (4.1). This proves Lemma 9.

LEMMA 10. For X^i and Y^i tangent to M_i^{α} , we have

$$(4.4) \quad h(X^i, Y^i) \perp N_1.$$

Proof. Let j_1, j_2, j_3 be distinct. Then i is distinct from at least two of j_1, j_2, j_3 , say $i \neq j_2, j_3$. Then from Lemmas 9 and 10, we have

$$\begin{aligned} (4.5) \quad & \langle h(X^i, Y^i), (\bar{\nabla}h)(X^{j_1}, X^{j_2}, X^{j_3}) \rangle \\ &= \langle h(X^i, Y^i), D_{X^{j_1}} h(X^{j_2}, X^{j_3}) \rangle \\ &= -\langle D_{X^{j_1}} h(X^i, Y^i), h(X^{j_2}, X^{j_3}) \rangle \\ &= -\langle (\bar{\nabla}h)(X^{j_1}, X^i, Y^i), h(X^{j_2}, X^{j_3}) \rangle \\ &= -\langle D_{X^i} h(X^{j_1}, Y^i), h(X^{j_2}, X^{j_3}) \rangle \\ &= \langle h(X^{j_1}, Y^i), (\bar{\nabla}h)(X^i, X^{j_2}, X^{j_3}) \rangle. \end{aligned}$$

If $i \neq j_1$, then (4.5) and Lemma 8 imply that

$$(4.6) \quad \langle h(X^i, Y^i), (\bar{\nabla}h)(X^{j_1}, X^{j_2}, X^{j_3}) \rangle = 0.$$

Assume that $i = j_1$. Then using the same method as the proof of (4.6), we may find

$$(4.7) \quad \begin{aligned} & \langle h(X^i, Y^i), (\bar{\nabla}h)(Z^i, X^{j_2}, X^{j_3}) \rangle \\ & = \langle (\bar{\nabla}h)(Z^i, X^i, Y^i), h(X^{j_2}, X^{j_3}) \rangle. \end{aligned}$$

On the other hand, Lemma 9 implies

$$(4.8) \quad \begin{aligned} & \langle (\bar{\nabla}h)(Z^i, X^i, Y^i), h(X^{j_2}, X^{j_3}) \rangle \\ & + \langle h(X^i, Y^i), (\bar{\nabla}h)(Z^i, X^{j_2}, X^{j_3}) \rangle = 0. \end{aligned}$$

Combining (4.7) and (4.8), we get

$$(4.9) \quad \langle h(X^i, Y^i), (\bar{\nabla}h)(Z^i, X^{j_2}, X^{j_3}) \rangle = 0.$$

Therefore, by (4.6) and (4.9) we obtain (4.4). This proves Lemma 10.

LEMMA 11. For X^i and Y^i tangent to $M_i^{\alpha_i}$, we have

$$(4.10) \quad h(X^i, Y^i) \perp N_t$$

for $t = 2, 3, \dots, n-2$.

Proof. We shall prove this lemma by induction. Assume that $h(X^i, Y^i) \perp N_t$ for $t \leq \ell-1$. We want to prove that $h(X^i, Y^i) \perp N_\ell$ for $\ell = 2, \dots, n-2$.

Let $j_1, \dots, j_{\ell+2}$ be distinct. Then we may assume that $i \neq j_2, \dots, j_{\ell+2}$. Then by Lemma 9 and induction we have

$$\begin{aligned} & \langle h(X^i, Y^i), (\bar{\nabla}^\ell h)(Z^{j_1}, \dots, Z^{j_{\ell+2}}) \rangle \\ & = \langle h(X^i, Y^i), D_{Z^{j_1}}((\bar{\nabla}^{\ell-1} h)(Z^{j_2}, \dots, Z^{j_{\ell+2}})) \rangle \\ & = -\langle (\bar{\nabla}h)(Z^{j_1}, X^i, Y^i), (\bar{\nabla}^{\ell-1} h)(Z^{j_2}, \dots, Z^{j_{\ell+2}}) \rangle \\ & = \langle h(Z^{j_1}, Y^i), (\bar{\nabla}^\ell h)(X^i, Z^{j_2}, \dots, Z^{j_{\ell+2}}) \rangle. \end{aligned}$$

If $j_1 \neq i$, this implies

$$\langle h(X^i, Y^i), (\bar{\nabla}^\ell h)(Z^{j_1}, \dots, Z^{j_{\ell+2}}) \rangle = 0.$$

If $j_1 = i$, then we have

$$\begin{aligned} & \langle h(X^i, Y^i), (\bar{\nabla}^\ell h)(Z^i, Z^{i_2}, \dots, Z^{i_{\ell+2}}) \rangle \\ &= \langle (\bar{\nabla} h)(Z^i, X^i, Y^i), (\bar{\nabla}^{\ell-1} h)(Z^{i_2}, \dots, Z^{i_{\ell+2}}) \rangle. \end{aligned}$$

On the other hand, because $h(X^i, Y^i) \perp N_{\ell-1}$, we also have

$$\begin{aligned} & \langle h(X^i, Y^i), (\bar{\nabla}^\ell h)(Z^i, Z^{i_2}, \dots, Z^{i_{\ell+2}}) \rangle \\ &+ \langle (\bar{\nabla} h)(Z^i, X^i, Y^i), (\bar{\nabla}^{\ell-1} h)(Z^{i_2}, \dots, Z^{i_{\ell+2}}) \rangle = 0. \end{aligned}$$

Thus, we find $h(X^i, Y^i) \perp N_\ell$. Consequently, by induction, we obtain Lemma 10.

From Lemmas 9, 10, and 11, we conclude that $h(X^i, Y^i) \equiv 0$. Since $M_i^{\alpha_i}$ is totally geodesic in $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$. We see that $M_i^{\alpha_i}$ is also totally geodesic in $\mathbb{C}P^m$, $m = N(\alpha_1, \dots, \alpha_n)$. Therefore, $M_i^{\alpha_i}$ is an open portion of a linear subspace $\mathbb{C}P^{\alpha_i}$. Consequently, $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ is an open portion of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$. This proves statement (2.1) of Theorem 1. Statement (2.2) then follows by local rigidity theorem of Kaehler submanifolds. This proves Theorem 1. (Q.E.D.)

5. - PROOF OF THEOREM 2

(0.4) follows immediately from Lemma 8. Moreover, it is clear that if we have

$$(5.1) \quad \|\bar{\nabla}^{\ell-2} h\|^2 = \ell! 2^\ell \sum_{i_1 < \dots < i_\ell} \alpha_{i_1} \dots \alpha_{i_\ell}$$

for some ℓ , $2 \leq \ell \leq n$, then

$$(5.2) \quad (\bar{\nabla}^{\ell-2} h)(X_{a_1}^{j_1}, \dots, X_{a_\ell}^{j_\ell}) = 0$$

whenever two or more of j_1, \dots, j_ℓ are equal.

If $\ell = 2$, (5.2) implies

$$(5.3) \quad h(X^i, Y^i) = 0$$

for any X^i, Y^i tangent to $M_i^{\alpha_i}$. Because $M_i^{\alpha_i}$ sits in $M = M_1^{\alpha_1} \times \dots \times M_n^{\alpha_n}$ as a totally geodesic submanifold, (5.3) shows that $M_i^{\alpha_i}$ is totally geodesic in $\mathbb{C}P^m$. Thus, each $M_i^{\alpha_i}$ is an open portion of $\mathbb{C}P^{\alpha_i}$. Therefore, M is an open portion of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$. By applying Calabi's rigidity theorem, we see that the immersion is obtained by the Segre imbedding $S_{\alpha_1 \dots \alpha_n}$.

Now, assume that (5.1) holds for some ℓ with $\ell = 3, 4, \dots$, or n . Then we have (5.2) whenever two or more of j_1, \dots, j_ℓ are equal.

If two or more of $k_1, \dots, k_{\ell-1}$ are equal, then we may choose one i with $i \neq k_1, \dots, k_{\ell-1}$. From (1.8) and (5.2) we find

$$\begin{aligned}
 (5.4) \quad & (\bar{\nabla}^{\ell-1} h)(JX^i, X^i, X_{a_1}^{k_1}, \dots, X_{a_{\ell-1}}^{k_{\ell-1}}) \\
 &= (\bar{\nabla}^{\ell-1} h)(X^i, JX^i, X_{a_1}^{k_1}, \dots, X_{a_{\ell-1}}^{k_{\ell-1}}) \\
 &= 0.
 \end{aligned}$$

Thus, by using (1.7), (1.9), (1.12) and (5.4), we get

$$\begin{aligned}
 0 &= \langle R^D(JX^i, X^i)\xi, J\xi \rangle \\
 &= \tilde{R}(JX^i, X^i; \xi, J\xi) + \langle [A_\xi, A_{J\xi}]JX^i, X^i \rangle \\
 &= 2 \|\xi\|^2 + 2 \|A_\xi X^i\|^2,
 \end{aligned}$$

where $\xi = (\bar{\nabla}^{\ell-3} h)(X_{a_1}^{k_1}, \dots, X_{a_{\ell-1}}^{k_{\ell-1}})$. Therefore, we find

$$(5.5) \quad (\bar{\nabla}^{\ell-3} h)(X_{a_1}^{k_1}, \dots, X_{a_{\ell-1}}^{k_{\ell-1}}) = 0$$

whenever two or more of $k_1, \dots, k_{\ell-1}$ are equal. Continuing this process $\ell-2$ times, we obtain (5.3) for X^i, Y^i tangent to $M_i^{\alpha_i}$. Applying the same argument as before, we conclude that M is an open portion of $\mathbb{C}P^{\alpha_1} \times \dots \times \mathbb{C}P^{\alpha_n}$ and the immersion is obtained by the Segre imbedding. Moreover, if this is the case, we can see that the equality of (0.4) holds for all ℓ , $\ell = 2, \dots, n$. (Q.E.D.)

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