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# FULLY NONLINEAR INEQUALITIES AND CERTAIN QUESTIONS ABOUT THEIR FREE BOUNDARY

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**Résumé** : Dans cet article on résout un problème d'obstacle pour un opérateur de second ordre avec une nonlinéarité générale. La démonstration de l'existence et l'unicité de la solution utilise des techniques d'estimation a priori ainsi que propriétés d'accretivité sur les problèmes approchés. De plus, on obtient certaines propriétés sur la frontière libre.

**Summary** : In this paper we study an inequality with an obstacle, governed by a fully nonlinear second order elliptic operator. By this, we mean second order operators involving a general nonlinearity. By means of a priori estimates techniques and accretive operator methods on approximated problems we obtain existence and uniqueness for the inequality. We, also, obtain some properties about the free boundary involved to obstacle.

## 1. - INTRODUCTION

A wide field of «phenomena» in physics and control theory can be formulated in the class of second order partial differential inequalities. Many of them have been studied for large time. This is the case of those governed by linear or quasilinear operators (see for exemple Duvaut-Lions [7], Bensoussan-Lions [1], Kinderlehrer-Stampacchia [12]).

In this paper we study a class of inequalities governed by *fully nonlinear* - also called non quasilinear - second order elliptic operator. By this, we mean second order operators involving a general nonlinearity.

More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded open with smooth boundary $\Gamma$, and F be a function

$$F : \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$$

satisfying :

(1.1) F is continuous differentiable, with bounded gradient.

(1.2) for each $(p,q,r) \in \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R}$, the matrix

$$\left(\frac{\partial F}{\partial p_{ij}}(p,q,r)\right)_{ij}$$

is nonnegative definite.

(1.3) $$\lim_{|p|\to\infty} |F(p,q,r)|/_{|p|} = 0 \text{ , for each } (q,r) \in \mathbb{R}^N \times \mathbb{R}.$$

We are concerned with to study of the equation

$$(P) \quad \begin{cases} \max\left\{\lambda u(x) - \Delta u(x) - F(D^2u(x),Du(x),u(x)) - f(x),\ u(x) - \psi(x)\right\} = 0 & x \in \Omega \\ u(x) = 0, & x \in \Gamma . \end{cases}$$

Here $\psi$ and f are given function, and $\lambda$ is a given real number such that $\lambda > M = \sup |DF(p,q,r)|$.

We note that a large class of Partial Differential Inequalities such as, for example, certain Hamilton-Jacobi-Bellman inequalities

$$(\hat{P}) \quad \begin{cases} \sup_{v \in V} \left\{A(x,v)u(x) - f(x,v)\right\} \leqslant 0, & x \in \Omega \\ u(x) \leqslant \psi(x) , & x \in \Omega \\ (u(x) - \psi(x)) . \sup_{v \in V} \left\{A(x,v)u(x) - f(x,v)\right\} = 0 , & x \in \Omega \\ u(x) = 0, & x \in \Gamma \end{cases}$$

where A(x,v) are second order elliptic operators (see P.L. Lions [15] , G. Diaz [5] ) belongs to general fully nonlinear inequality kind.

The main results in this paper is

THEOREM 1.1. *Assume (1.1), (1.2), (1.3). Let* $f \in C(\overline{\Omega})$ *and* $\psi \in W^{2,\infty}(\Omega)$ *such that* $\psi|_\Gamma \geqslant 0$, *then there exists a unique solution* $u \in W^{2,p}(\Omega)$ (*for all* $1 \leqslant p < \infty$) *of* (P).

It is possible to obtain $W^{2,\infty}$ solutions for some nonlinear inequalities (we show $W^{2,\infty}$ estimates for a quasilinear inequality in Section 2), but for general fully nonlinear difficult occur due to nonquasilinear term. However, the fully nonlinear inequality $(\hat{P})$ admits $W^{2,\infty}$ solution, because it can be approximated by semilinear system. We can use such an approximation in order to obtain $W^{2,\infty}$ solution for some fully nonlinear inequality (see Section 2).

As is well known the solution of (P) determines two sets $\Omega_1 = \{x \in \bar{\Omega} : u(x) = \psi(x)\}$ called the *coincidence set,* and $\Omega_2 = \{x \in \bar{\Omega} : u(x) < \psi(x)\}$. Thus we have a *free boundary.* When we interpret certain Hamilton-Jacobi-Bellman inequalities as the dynamic programming equation in a stochastic control theory situation, the above sets play an important role ($\Omega_1$, is there called the *stopping set*) see N.V. Krylov [13], G. Diaz [5]. Our plan is as follows. In § 2 we study some problem approximating (P) and show a uniform estimates on its solution. In § 3 we solve problem (P) and derive certain properties. Finally, in § 4 we obtain some properties of the free boundary which show how under very simple assumption $\Omega_1$ includes the «heart» of $\Omega$, $\Omega_2$ is around the boundary $\Gamma$ and $\partial\Omega_1$ is a null measure set.

We use through the summation convention.

## 2. - PROBLEMS APPROXIMATING (P)

For each $0 < \epsilon < 1$, choose $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ given by $\beta_\epsilon(t) = \frac{1}{\epsilon}\hat{\beta}(t)$ being $\hat{\beta}$ a convex and smooth function verifying

$$\hat{\beta}(t) = 0 \quad \text{if} \quad t < 0 \;, \quad 0 < \hat{\beta}'(t) \leqslant 1 \quad \text{if} \quad t > 0. \tag{2.1}$$

In order to simplify the notation we denote

$$\mathcal{F}(p,q,r) = p_{ii} + F(p,q,r) \;, \quad \text{for } (p,q,r). \tag{2.2}$$

Consider now the approximate fully nonlinear problem

$$(P_\epsilon) \quad \begin{cases} \lambda u - \mathcal{F}(D^2u,Du,u) + \beta_\epsilon(u-\psi) = f \;, & x \in \Omega \\ u(x) = 0 \;, & x \in \Gamma. \end{cases}$$

$(P_\epsilon)$ may be solved by using results of Evans-Lions [10] for

$$\mathcal{F}_0(p,q,r,x) = \mathcal{F}(p,q,r) - \beta_\epsilon(r - \psi(x)) \tag{2.3}$$

but its gradient is bounded by a $\frac{c}{\epsilon}$-term and therefore difficulties occur when we want to pass to limit because in Evans-Lions [10] $\lambda$ depends increasly on that bound.

We are going to use a fixed point argument on a result of Evans [9] in order to obtain adequate results of existence.

LEMMA 2.1. *If* F *verifies (1.1) then*

$$(2.4)\quad F(p,q,r) = \max_{\bar{y}\in X}\ \min_{\bar{z}\in X} \left\{ \left[\int_0^1 \frac{\partial F}{\partial p_{ij}}((1-\lambda)y^1 + \lambda z^1, (1-\lambda)y^2 + \lambda z^2, (1-\lambda)y^3 + \lambda z^3)d\lambda\right] \right.$$

$$\left. (p_{ij} - y^1_{ij}) + \left[\int_0^1 \frac{\partial F}{\partial q_i}(-)d\lambda\right](q_i - y_i^2) + \left[\int_0^1 \frac{\partial F}{\partial r}(-)d\lambda\right](r - y^3) + F(y^1,y^2,y^3), \right.$$

where $\bar{y}, \bar{z} \in X = \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R}$. #

*Remark* 2.1. The proof of the above Lemma is a simple extension of the proof of Lemma 2.2 of Evans [9]. #

THEOREM 2.2. *Assume (1.1), (1.2), (1.3). Let* f, $\psi \in C(\bar{\Omega})$, *then for each* $0 < \epsilon < 1$ *there exists a unique solution* $u_\epsilon \in W^{2,p}(\Omega)$ *(for all* $1 \leqslant p < \infty$*) of* $(P_\epsilon)$.

*Proof.* Let us consider the function

$$(2.5)\qquad h(r) = -r + \hat{\beta}(r).$$

Then, for each $0 < \epsilon < 1$ and $u \in C(\bar{\Omega})$ there exists, by Evans [9], a unique solution $Tu \in W^{2,p}(\Omega)$ (for all $1 \leqslant p < \infty$) of

$$(2.6)\quad \begin{cases} \dfrac{1}{\epsilon} Tu + \lambda\, Tu - F(D^2Tu, DTu, Tu) = f + \dfrac{1}{\epsilon}\psi - \dfrac{1}{\epsilon}h(u-\psi) & x \in \Omega \\ Tu(x) = 0\ , & x \in \Gamma \end{cases}$$

(In the obtainement of Tu the «quasilinearization» representation of Lemma 2.1 and (1.3) plays an important role).

Reasoning in a classical sense, for $u, v \in C(\bar{\Omega})$ one has by (2.5)

$$(2.7)\qquad \| Tu - Tv \|_\infty \leqslant \frac{1}{(\lambda - M)\epsilon + 1} \| u - v \|_\infty$$

and the proof ends by Banach's Theorem. #

In order to obtain uniform estimates we are going to use for each v, $\tilde v \in W^{2,p}(\Omega)$ the first variation

$$F(D^2 v, Dv, v) - F(D^2\tilde v, D\tilde v, \tilde v) \equiv a_{ij}^{v,\tilde v} D_{ij}(v-\tilde v) + a_i^{v,\tilde v} D_i(v-\tilde v) + a_o^{v,\tilde v}(v-\tilde v) \equiv L^{v,\tilde v}\, v-\tilde v \tag{2.8}$$

where

$$\left\{\begin{aligned} a_{ij}^{v,\tilde v} &\equiv \int_0^1 \frac{\partial F}{\partial p_{ij}}\,((1-t)D^2\tilde v + tD^2 v, (1-t)D\tilde v + tDv, (1-t)\tilde v + tv)dt \\ a_i^{v,\tilde v} &\equiv \int_0^1 \frac{\partial F}{\partial q_i}\,(\text{——})\,dt \\ a_o^{v,\tilde v} &\equiv \int_0^1 \frac{\partial F}{\partial r}\,(\text{——})\,dt \end{aligned}\right. \tag{2.9}$$

LEMMA 2.3. *Let* $u_\epsilon = u_\epsilon(f,\psi)$ *and* $\hat u_\epsilon = u_\epsilon(\hat f,\hat\psi)$ *the solution of* $(P_\epsilon)$ *corresponding to* $(f,\psi)$ *and* $(\hat f,\hat\psi)$, *then*

$$\| u_\epsilon - \hat u_\epsilon \|_\infty \leqslant \max \left\{ \frac{\| f-\hat f \|_\infty}{\lambda - M},\quad \| \psi-\hat\psi \|_\infty \right\} \tag{2.10}$$

$$\| (u_\epsilon - \hat u_\epsilon)^+ \|_\infty \leqslant \max \left\{ \frac{\| (f-\hat f)^+ \|_\infty}{\lambda - M},\ \| (\psi-\hat\psi)^+ \|_\infty \right\} \tag{2.11}$$

$$f \leqslant \hat f \quad \textit{and} \quad \psi \leqslant \hat\psi \quad \textit{implies} \quad u_\epsilon \leqslant \hat u_\epsilon . \tag{2.12}$$

*Proof.* Let us consider the operator $L^{u_\epsilon,\hat u_\epsilon}$ involved to the first variation of $F(D^2u_\epsilon, Du_\epsilon, u_\epsilon) - F(D^2\hat u_\epsilon, D\hat u_\epsilon, \hat u_\epsilon)$. (We note that the coefficients of $L^{u_\epsilon,\hat u_\epsilon}$ are bounded and measurable).

Then, the function

$$w = u_\epsilon - \hat u_\epsilon - \max \left\{ \frac{\| f-\hat f \|_\infty}{\lambda - M},\ \| \psi-\hat\psi \|_\infty \right\} \equiv u_\epsilon - \hat u_\epsilon - K, \tag{2.13}$$

verifies

$$\lambda w - L^{u_\epsilon,\hat u_\epsilon} w = f - \hat f - (\beta_\epsilon(u_\epsilon - \psi) - \beta_\epsilon(\hat u_\epsilon - \hat\psi)) + (-\lambda + a_o^{u_\epsilon,\hat u_\epsilon})K \tag{2.14}$$

Since in $[w > 0]$ one has

$$u_\epsilon - \hat u_\epsilon > \max \left\{ \frac{\| f-\hat f \|_\infty}{\lambda - M},\ \| \psi-\hat\psi \|_\infty \right\} \tag{2.15}$$

we obtain

$$\lambda w - L^{u_\epsilon,\hat{u}_\epsilon} w \leqslant 0 \quad \text{in} \quad [w > 0] \text{ and } w \leqslant 0 \text{ in } \Gamma. \tag{2.16}$$

Then the maximum principle in Sobolev spaces (see Bony [2] ) implies (2.13).

By chossing $w = u_\epsilon - \hat{u}_\epsilon - \max \left\{ \frac{\| (f-\hat{f})^+ \|_\infty}{\lambda - M}, \| (\psi-\hat{\psi})^+ \|_\infty \right\}$ (for $r^+ = \max(r,0)$) a similar argument concludes (2.11) and (2.12).

LEMMA 2.4. *Assume* $\psi \in W^{2,\infty}(\Omega)$ *then*

$$\| (u_\epsilon - \psi)^+ \|_\infty \leqslant \hat{\beta}^{-1} (\epsilon \, \| (f - \lambda\psi + F(D^2\psi, D\psi, \psi))^+ \|_\infty). \tag{2.17}$$

*Proof.* We write $(P_\epsilon)$ in the way

$$\lambda(u_\epsilon - \psi) - (F(D^2u_\epsilon, Du_\epsilon, u_\epsilon) - F(D^2\psi, D\psi, \psi)) \tag{2.18}$$

$$= F(D^2\psi, D\psi, \psi) - \beta_\epsilon(u_\epsilon - \psi) + f - \lambda\psi$$

Let $x_0 \in \overline{\Omega}$ such that $(u_\epsilon - \psi)^+ (x_0) = \| (u_\epsilon - \psi)^+ \|_\infty$. With no loss of generality we may assume $x_0 \in \Omega$ and $(u_\epsilon - \psi)^+ (x_0) > 0$, because otherwise (2.17) is obvious.

By using once more the maximum principle, in the first variation of the operator in the left hand side of (2.18) we obtain

$$0 \leqslant \lim_{x \to x_0} \operatorname{ess\,sup}(-\lambda\psi + F(D^2\psi, D\psi, \psi) - \beta_\epsilon(u_\epsilon - \psi) + f). \tag{2.19}$$

and then (2.17) one derives from the definition of $\beta_\epsilon$. #

COROLLARY 2.5. *Assume* $\psi \in W^{2,\infty}(\Omega)$, *then for each* $1 \leqslant p < \infty$ *there exist a constant* C, *independent on* $\epsilon$ *such that*

$$\| u_\epsilon \|_{W^{2,p}(\Omega)} \leqslant C. \tag{2.20}$$

*Proof.* By using classical results one has

$$\| u_\epsilon \|_{W^{2,p}(\Omega)} \leq C \, \| \Delta u_\epsilon \|_p \leq C \, \| F(D^2u_\epsilon, Du_\epsilon, u_\epsilon) - \beta_\epsilon(u_\epsilon - \psi) + f \|_p ,$$

the monotonicity of $\beta_\epsilon$ and (2.19) imply for any $\eta > 0$

$$\| u_\epsilon \|_{W^{2,p}(\Omega)} \leq C\eta \, \| u_\epsilon \|_{W^{2,p}(\Omega)} + C(\eta) + C \tag{2.21}$$

finally for $\eta$ small enough we obtain (2.20). #

Next, we end this Section mentionating some bounds for general fully nonlinear operators.

THEOREM 2.6. (G. Diaz [6] ). *Let* $u_\epsilon$ *and* $\psi$ *smooth function verifying*

$$(\hat{P}_\epsilon) \qquad \begin{cases} \lambda u_\epsilon - F(D^2u_\epsilon, Du_\epsilon, u_\epsilon, x) + \beta_\epsilon(u_\epsilon - \psi) = 0, & x \in \Omega \\ u_\epsilon(x) = 0, & x \in \Gamma \end{cases}$$

*where* $F : \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ *is a smooth function such that :*

(2.22) $\dfrac{\partial F}{\partial p_{ij}}(p,q,r,x)\xi_i\xi_j \geqslant \alpha \, |\xi|^2$, *for some positive constant* $\alpha$ *and all* $\xi \in \mathbb{R}^N$, $(p,q,r,x) \in \mathbb{R}^{N^2} \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$,

$|F(0,0,0,x)|, \ |DF(p,q,r,x)|, \ |D^2F(p,q,r,x)| \leqslant M$

*for some constant* $M < \lambda$, *and all* $(p,q,r,x)$.

*Then :*

$$(2.23) \quad \| u_\epsilon \|_{W^{1,\infty}(\Omega)} \leqslant \max\left\{ C(M + \| (-\lambda\psi + F(D^2\psi, D\psi, \psi, x))^+ \|_\infty, \frac{M}{\lambda - M}, \ \| \psi \|_{W^{1,\infty}} \right\}$$

$$(2.24) \qquad |D^2u_\epsilon(x)| \leqslant C + C \, \| D^2u_\epsilon \|_\infty^{1/2}, \quad \forall x \in \Gamma$$

*where* C *are positive constant independent on* $\epsilon$. (*In the proof of (2.24) is required* $\psi_{|\Gamma} \geqslant 0$). #

However, when we difference twice $(P_\epsilon)$ occur certain products of third and second order derivatives of $u_\epsilon$ which stop up the obtainement of $W^{2,\infty}$ estimates of $u_\epsilon$ independent on $\epsilon$.

We, only, know

THEOREM 2.7. (G. Diaz [6] ). *Under assumption of Theorem 2.6 as well as*

$$(2.25) \qquad F(p,q,r,x) = a_{ij}(r,x)p_{ij} + b_i(r,x)q_i + c(r,x) \quad \textit{for all} \quad (p,q,r,x)$$

*there exists a constant* C, *independent on* $\epsilon$, *such that*

$$(2.26) \qquad \| u_\epsilon \|_{W^{2,\infty}(\Omega)} \leqslant C. \quad \#$$

*Remark* 2.2. The Hamilton-Jacobi-Bellman inequality $(\hat{P})$

$$(2.27)\qquad \begin{cases} \max\left\{ \max\limits_{1\leqslant k\leqslant m} (L^k u(x) - f^k(x)),\ u(x)-\psi(x)\right\} = 0, & x\in\Omega \\ u(x) = 0, & x\in\Gamma \end{cases}$$

can be approximated by the semilinear system

$$(2.28)\qquad \begin{cases} L^k u_\epsilon^k + \beta_\epsilon(u_\epsilon^k - u_\epsilon^{k+1}) + \beta_\epsilon(u^k - \psi) = f^k, & \text{in } \Omega,\ 1\leqslant k\leqslant m \\ u_\epsilon^k = 0 & \text{on } \Gamma \end{cases}$$

(we assume $m+1 \equiv 1$), where $L^k$ are second order uniformly elliptic differential operators which coefficients are functions on the variable x. Then $W^{2,\infty}_{loc}$ estimates, independent on $\epsilon$, are, availably, obtained by using method of [11] (see S. Lenhart [14]). #

*Remark* 2.3. If $-F : IR^{N^2} \times IR^N \times IR \times \bar{\Omega} \to IR$ is a *convex* function at the variable $p \in IR^{N^2}$, one has

$$(2.29)\qquad -\frac{\partial F}{\partial p_{ij}}(\tilde{p},q,r,x)\,(p_{ij}-\tilde{p}_{ij}) - F(p,q,r,x) \leqslant -F(\tilde{p},q,r,x)$$

for all $p, \tilde{p} \in IR^{N^2}$, $q\in IR^N$, $r \in IR$, $x\in\bar{\Omega}$.

Then, a general fully nonlinear inequality can be written as

$$(2.30)\qquad \begin{cases} \sup\limits_{p\in IR^{N^2}} \lambda u(x) - \dfrac{\partial F}{\partial p_{ij}}(p,Du(x),u(x),x)(D_{ij}u(x)-p_{ij}) - F(p,Du(x),u(x),x) \leqslant 0 & x\in\Omega \\ u(x) \leqslant \psi(x) & x\in\Omega \\ (u(x)-\psi(x)) \,.\, \sup\limits_{p\in IR^{N^2}} \Big\{\lambda u(x) - \dfrac{\partial F}{\partial p_{ij}}(p,Du(x),u(x),x)(D_{ij}\,u(x)-p_{ij}) & \\ \qquad - F(p,Du(x),u(x),x)\Big\} = 0, & x\in\Omega. \end{cases}$$

We, now, may approximate (2.30) by a system as (2.28). #

In particular, for $-F : IR^{N^2} \times IR \times \bar{\Omega} \to IR$, convex at the variable $p\in IR^{N^2}$ it is possible to obtain $W^{2,\infty}$ solutions by using Theorem 2.7. #

## 3. - THE EXISTENCE AND UNIQUENESS

In order to obtain the existence we are going to use an accretive operator method due to Evans [9].

By using the representation in $L^\infty(\Omega)$

$$[f,g]_+ \equiv \lim_{\delta \downarrow 0} \operatorname{ess\,sup}_{\Omega(f,\delta)} g(x) \operatorname{sign} f(x) \quad (f \not\equiv 0) \tag{3.1}$$

where $\Omega(f,\delta)$ is defined (up to a set of measure zero) by

$$\Omega(f,\delta) \equiv \left\{ x \in \Omega : |f(x)| > \|f\|_\infty - \delta \right\} \tag{3.2}$$

one has

LEMMA 3.1. *The operators* $\lambda\phi - F(D^2\phi, D\phi, \phi)$ *and* $\lambda\phi - F(D^2\phi, D\phi, \phi) + \beta_\epsilon(\phi - \psi)$ *defined in* $W^{2,p}(\Omega) \cap W^{1,p}(\Omega)$, *for some* $p > N$, *are accretive in* $L^\infty(\Omega)$.

*Proof.* Indeed, let $\phi$, $\hat{\phi} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for some $p > N$ then there exists $x_0 \in \bar{\Omega}$ such that (with no loss generality)

$$(\phi - \hat{\phi})(x_0) = \|\phi - \hat{\phi}\|_\infty$$

If $x_0 \in \Omega$, then the maximum principle in Sobolev spaces implies

$$0 \leqslant \lim_{x \to x_0} \operatorname{ess\,sup} \left\{ \lambda(\phi - \hat{\phi})(x) - (F(D^2\phi, D\phi, \phi) - F(D^2\hat{\phi}, D\hat{\phi}, \hat{\phi})) \right.$$

(Recall the representation (2.9)).

Thus

$$0 \leqslant [\phi - \hat{\phi}, \lambda(\phi - \hat{\phi}) - (FD^2\phi, D\phi, \phi) - F(D^2\hat{\phi}, D\hat{\phi}, \hat{\phi}))]_+ . \quad \# \tag{3.3}$$

*Proof of the existence in Theorem 1.1.* By Lemma 2.4 and Corollary 2.5 there exists a sequence $\{\epsilon_k\}_k \downarrow 0$ and a function $u \in W^{2,p}(\Omega)$ (for all $1 \leqslant p < \infty$) such that

$$\begin{aligned} &u_{\epsilon_k} \to u && \text{uniformly on } \bar{\Omega} \\ &Du_{\epsilon k} \to Du && \text{uniformly on } \bar{\Omega} \\ &u_{\epsilon_k} \to u && \text{weakly in } W^{2,p}(\Omega) \text{ (for all } 1 \leqslant p < \infty) \end{aligned} \tag{3.4}$$

and

(3.5) $$u(x) \leqslant \psi(x)\ , \text{ for all } x \in \overline{\Omega}$$

Then, by $(P_\epsilon)$ and the definition of $\beta_\epsilon$ one has

$$\lambda u_\epsilon - F(D^2 u_\epsilon, Du_\epsilon, u_\epsilon) \leqslant f \qquad \text{a.e.} \quad x \in \Omega$$

and the weak convergence in (3.4) implies

(3.6) $$\lambda u - F(D^2 u, Du, u) \leqslant f, \quad \text{a.e.} \quad x \in \Omega$$

The proof ends if

(3.7) $$\lambda u - F(D^2 u, Du, u) \geqslant f \quad \text{a.e.} \quad x \in \Omega_2 = \{x : u < \psi\}.$$

To see this, for a.e. $x_o \in \Omega_2$ one has

(3.8) $$u(x_o) \leqslant u_{\epsilon_k}(x_o) < \psi(x_o)$$

for k large enough. Then, there exists $n_o$ such that

(3.9) $$u_{\epsilon_k}(x_o) < \phi_n(x_o) < \psi(x_o)\ , \text{ for } n \geqslant n_o$$

being $\{\phi_n\}_n$ a sequence of smooth function verifying

i) $\phi_n(x_o) \to u_{\epsilon_k}(x_o)\ ,\ D\phi_n(x_o) \to Du_{\epsilon_k}(x_o)\ ,\ D^2\phi_n(x_o) \to D^2 u_{\epsilon_k}(x_o)$

ii) $-(u_{\epsilon_k} - \phi_n)(x_o) = \| u_{\epsilon_k} - \phi_n \|_\infty$

iii) $0 > (u_{\epsilon_k} - \phi_n)(x) > (u_{\epsilon_k} - \phi_n)(x_o) \quad \text{for} \quad x \in \Omega\ ,\ x \neq x_o$

(such a sequence exists by Lemma 2.2 of [9] ).

By Lemma 3.1, for $\phi \in C^2(\Omega) \cap C_o(\overline{\Omega})$ we have

(3.10) $$0 \leqslant [\phi - u_\epsilon, \lambda\phi - F(D^2\phi, D\phi, \phi) + \beta_\epsilon(\phi - \psi) - \lambda u_\epsilon + F(D^2 u_\epsilon, Du_\epsilon, x) - \beta_\epsilon(u_\epsilon - \psi)]_+$$

$$= [\phi - u_\epsilon, \lambda\phi - F(D^2\phi, D\phi, \phi) + \beta_\epsilon(\phi - \psi) - f]_+ \,.$$

Now, from (3.10) and (3.1) we have

$$\lambda\phi_n(x_o) - F(D^2\phi_n(x_o)\,,\ D\phi_n(x_o)\,, \phi_n(x_o)) \geqslant f(x_o) \tag{3.11}$$

for $n \geqslant n_o$, and we conclude (3.7) after a passage to limit. #

*Proof of the uniqueness in Theorem 1.1.*

LEMMA 3.2. *Given* $\phi$, $\hat\phi \in W^{2,p}(\Omega)$, *for some* $p > N$ *such that*

$$\begin{cases} \lambda\phi - F(D^2\phi,D\phi,\phi) + \gamma(\phi) \leqslant \lambda\hat\phi - F(D^2\hat\phi,D\hat\phi,\hat\phi) + \gamma(\hat\phi) \quad \text{a.e.} \quad x \in \Omega \\ \phi(x) \leqslant \hat\phi(x) \quad , \quad x \in \Gamma \end{cases} \tag{3.12}$$

*then,* $\phi(x) \leqslant \hat\phi(x)$ *for all* $x \in \overline{\Omega}$, *where* $\gamma$ *is a continuous non-decreasing function.*

*Proof.* If there exists $x_o \in \Omega$ such that

$$(\phi - \hat\phi)\ (x_o) = \|\, \phi - \hat\phi\, \|_\infty$$

Bony's maximum principle and (3.12) implies

$$0 < \lim_{x \to x_o} \operatorname{ess\,sup}\ (\lambda(\phi - \hat\phi)(x) - F(D^2\phi,D\phi,\phi) - F(D^2\hat\phi,D\hat\phi,\hat\phi))$$
$$\leqslant \gamma(\hat\phi(x_o)) - \gamma(\phi(x_o)) \leqslant 0. \quad \#$$

Now, it is easy to show

a) «The solution u of (P) obtained in (3.4) is maximal among the set of solution of (P)». Indeed, let $\hat u \in W^{2,p}$ be any solution of (P), then

$$\lambda\hat u - F(D^2\hat u,D\hat u,\hat u) + \beta_\epsilon(\hat u - \psi) \leqslant 0 = \lambda u_\epsilon - F(D^2 u_\epsilon, Du_\epsilon, u_\epsilon) + \beta_\epsilon(u_\epsilon - \psi).$$

Hence, Lemma 3.2 and the uniform convergence (3.4) give the maximality of u.

b) «$\lambda u - F(D^2u,\ Du,u) \leqslant \lambda\hat u - F(D^2\hat u,D\hat u,\hat u)$ for all $\hat u$ satisfying (P)». Indeed, in $\hat\Omega_2 = \{x : \hat u < \psi\}$ (3.7) implies b), and otherwise in $\hat\Omega_1 = \{x : \hat u = \psi\}$ since u is maximal one has $\psi \geqslant u \geqslant \hat u$ and then $\psi = u = \hat u$. If $\hat\Omega_1$ has a positive measure we conclude b).

Finally the maximality of u and b) yield the uniqueness. #

All reimander results hold for general fully nonlinear inequalities

$$\begin{cases} \max \left\{ \lambda u - F(D^2u,Du,u,x)\, ,\, u(x) - \psi(x) \right\} = 0 \;, \text{ a.e. } x \in \Omega \\ u(x) = 0 \;, \quad x \in \Omega \end{cases}$$

for $u \in W^{2,p}(\Omega)$, for some $p > N$. $\psi \in W^{2,\infty}(\Omega)$, $\psi_{\mid \Gamma} \geqslant 0$ and F satisfying (2.22) and (2.23).

THEOREM 3.1. *Let* $u = u(\psi)$ *and* $\hat{u} = \hat{u}(\phi)$ *be the corresponding solutions of* (P) *to those obstacles. Then*

*i)* $\| u - \hat{u} \|_{C(\bar{\Omega})} \leqslant \| \psi - \phi \|_{C(\bar{\Omega})}$

(3.12) *ii)* $\psi \leqslant \phi$ *implies* $u \leqslant \hat{u}$.

*iii)* $\| u - \psi \|_{C(\Omega)} \leqslant \max(\| \psi \|_{\infty} , \frac{1}{\lambda - M} \| (\lambda - M)\psi - F(D^2\psi,D\psi,\psi,x) \|_{\infty})$.

*Proof.* i) and ii) are easily obtained from Lemma 2.3. iii) consider

$$0 \geqslant \lambda u - F(D^2u,Du,u,x) = \lambda(u - \psi) - (F(D^2u,Du,u,x)$$

$$- F(D^2\psi,D\psi,\psi,x)) + \lambda\psi - F(D^2\psi,D\psi,\psi,x)$$

so, $\bar{u} = u - \psi \in W^{2,p}(\Omega)$ satisfies

(3.13) $$\max \left\{ \lambda u - L^{u,\psi} u + \lambda\psi - F(D^2\psi,D\psi,\psi,x) \,;\bar{u} \right\} = 0 \text{ a.e. } x \in \Omega$$

Since $\bar{u} \in C(\bar{\Omega})$, there exists $x_o \in \bar{\Omega}$ where $-\bar{u}$ attains positive maximum. If $x_o \in \Gamma$ iii) is obvious. Otherwise, let us consider Bony's maximum principle for $-\tilde{L}^{\psi,u}(-\bar{u}) \equiv M(-\bar{u}) - L^{\psi,u}(-\bar{u})$ (see (3.4) and (3.5)).

Then

$$\lim_{x \to x_o} \text{ess sup} \, (-\tilde{L}^{\psi,u}(-\bar{u})(x)) \geqslant 0,$$

thus, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\underset{B(x_o;\delta)}{\text{ess sup}} \, (-\tilde{L}^{\psi,u}(-\bar{u})(x)) \geqslant -\epsilon$$

Since

$$\tilde{L}^{u,\psi} \bar{u} = -\tilde{L}^{\psi,u}(-\bar{u})$$

one has

$$\operatorname*{ess\,inf}_{B(x_0;\delta)} \ (-\widetilde{L}^{u,\psi}\,\bar{u}(x)) \leqslant \epsilon$$

Without loss of generality we can assume $\delta$ so small that $\bar{u}(x) < 0$ in $B(x_0;\delta)$ and (3.13) then yields, for $\widetilde{\lambda} = \lambda - M$

$$\operatorname*{ess\,inf}_{B(x_0;\delta)} \ (\widetilde{\lambda}(-\bar{u})(x) - \widetilde{\lambda}\psi + F(D^2\psi,D\psi,\psi,x)) \leqslant \epsilon$$

or

$$-\,\|\widetilde{\lambda}\psi - F(D^2\psi,D\psi,\psi,x)\|_\infty + \inf_{B(x_0;\delta)} \ \widetilde{\lambda}(-\bar{u})(x) \leqslant \epsilon$$

finally the continuity of $\bar{u}$ ends the estimate. #

## 4. - ON THE FREE BOUNDARY

Next we shall determine some properties of the free boundary (already studied in [4] ). Throughout this Section $\widetilde{\lambda} = \lambda - M$.

THEOREM 4.1. *If there exists a constant* $\gamma > 0$ *such that*

$$F(D^2\psi,D\psi,\psi,x) - \lambda\psi \geqslant \gamma \ , \quad \text{a.e. in } \Omega \tag{4.1}$$

then $u(x^o) = \psi(x^o)$, *for* $x^o \in \Omega$ *such that* $d(x^o,\Gamma) \geqslant \left[\dfrac{6NM}{\gamma} \sup_{\Gamma} \psi\right]^{1/2}$.

*Proof.* Recalling (3.13) $\bar{u} = u - \psi$ satisfies

$$\max\left\{\lambda\bar{u} - L^{u,\psi}\,\bar{u} + \lambda\psi - F(D^2\psi,D\psi,\psi,x)\ ;\bar{u}\right\} = 0 \quad \text{a.e. in } \Omega$$

where the coefficients of $L^{u,\psi}$ are not «previously», known but are bounded by M. We consider the *barrier function* $v(x) = \dfrac{-\gamma}{6NM}\,|\,x - x^o|^2$, $x^o \in \Omega$. After some calculations and using (4.1) we obtain

$$\lambda v(x) - L^{u,\psi}\,v(x) + \lambda\psi - F(D^2\psi,D\psi,\psi,x) \leqslant -\,L^{u,\psi}\,v(x) - \gamma \leqslant 0 \quad \text{a.e.} \quad x \in \Omega \tag{4.2}$$

Furtheremore, if $d(x^o,\Gamma) = R$ then

$$v(x) \leqslant -\frac{\gamma}{6NM}\,R^2 \leqslant \inf(-\psi) \leqslant -\,\psi(x) \leqslant \bar{u}(x) \quad \text{for all} \quad x \in \Gamma. \tag{4.3}$$

Finally, from (4.2), (4.3) and considering Bony's maximum principle for $L^{u,\psi}$ as in Theorem 3.1 we obtain a comparison result and, hence,

$$0 = v(x^o) \leqslant \bar{u}(x^o) = u(x^o) - \psi(x^o) \leqslant 0. \quad \#$$

COROLLARY 4.1. *Under the assumption (4.1) over an open* $G \subset \Omega$, *for* $\widetilde{\lambda}$ *we have* $u(x^o) = \psi(x^o)$, *for* $x^o \in G$ *and such that*

$$d(x^o;\partial G) \leqslant \left[\frac{6NM}{\gamma} \cdot \max\left\{\|\psi\|_\infty \frac{1}{\widetilde{\lambda}} \|\widetilde{\lambda}\psi - F(D^2\psi, D\psi, \psi, x)\|_\infty\right\}\right]^{1/2}.$$

*Proof.* It is sufficient to work in an adequate ball $B(x^o;R)$ and to consider the estimate of Theorem 3.1. #

This method estimates some values of u

THEOREM 4.2. *Assume* $\inf_{\Omega} \psi > 0$, *and suppose that for each* $0 < \mu < \inf_{\Omega} \psi$ *there exists a constant* $\gamma$ *such that*

$$F(0,0,\mu,x) - \lambda\mu \geqslant \gamma \quad \textit{for all} \quad x \in \Omega, \tag{4.4}$$

*then* $u(x^o) \geqslant \mu$, *for* $x^o \in \Omega$ *and such that* $d(x^o,\Gamma) \leqslant \left[\frac{6NM}{\gamma}\mu\right]^{1/2}$. #

The proof consists in comparing u with a suitable solution of a new problem with obstacle $\mu$.

We also obtain estimates of $\Omega_1$ near of the boundary.

THEOREM 4.3. *Assume (4.1) and suppose that there exists* $x^o \in \Gamma$ *and* $r > \left[\frac{6NM}{\gamma} \sup_\Gamma \psi\right]^{1/2}$ *such that* $\psi(x) = 0$ *in* $\Gamma \cap B(x^o,r)$ *then* $u(x) = \psi(x)$ *in* $\Omega \cap B(x^o,s)$ *for* $s = r - \left[\frac{6NM}{\gamma} \sup_\Gamma \psi\right]^{1/2}$. #

The proof is analogous to the proof of the Theorem 4.1 for the new barrier function

$$V_s(x) = \begin{cases} -\dfrac{\gamma}{6NM}(|x-x^o|-s)^2, & \text{if } |x-x^o| > s \\ 0, & \text{if } |x-x^o| \leqslant s \end{cases}$$

*Remark* 4.1. The above calculations can be found in G. Diaz [5]. #

Assumption (4.1) gives, also, a very simple topological property of $\Omega_2 = \{x \in \bar{\Omega} : u(x) < \psi(x)\}$.

THEOREM 4.1. *Let us assume* $\Gamma$ *is connected and* $F(D^2\psi, D\psi, \psi, x) - \lambda\psi \geqslant 0$ a.e. *in* $\Omega$. *Suppose that* $\psi > 0$ *in* $\Gamma$, *then* $\Omega_2$ *is connected.*

*Proof.* Let U be the component of $\Omega_2$ where closure intersects $\Gamma$. Suppose U' to be another component of $\Omega_2$, then $\partial U' \subset \Omega_1 = \{x : u = \psi\}$, so

$$(4.5)\qquad \begin{cases} \lambda\psi - F(D^2\psi, D\psi, \psi, x) \leqslant 0 = \lambda u - F(D^2u, Du, u, x) & \text{a.e. in } U' \\ \psi(x) = u(x) \quad \text{in} \quad \partial U' \end{cases}$$

But, then (4.5) and Lemma 3.3 give the contradiction

$$\psi(x) \leqslant u(x) < \psi(x) \quad \text{for all} \quad x \in U'. \quad \#$$

*Remark 4.2.* In order to derive that $\partial\Omega_1$ is a null measure set we may adapt to (P) (under the hypotheses (4.1)) some density properties of $\Omega_2$, due to L.A. Caffarelli and N.M. Riviere (see [12,p. 179]).

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