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COHOMOLOGY OF CR-SUBMANIFOLDS

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Résumé : Nous introduisons canoniquement une classe de cohomologie de Rham pour une CR-sous-variété compacte d'une variété kaehlérienne. Cette classe de cohomologie est utilisée pour montrer que si un certain groupe de cohomologie de dimension paire d'une CR-sous-variété, N est trivial, alors, soit la distribution holomorphe de N n'est pas intégrable, soit la distribution totalement réelle de N n'est pas minimale.

Summary : We introduce a canonical de Rham cohomology class for a closed CR-submanifold in a Kaehler manifold. This cohomology class is used to prove that if some even-dimensional cohomology group of a CR-submanifold N is trivial, then either the holomorphic distribution of N is not integrable or the totally real distribution of N is not minimal.

1. - INTRODUCTION

Let \tilde{M} be a Kaehler manifold with complex structure J and N a Riemannian manifold isometrically immersed in \tilde{M} . Let \mathcal{D}_x be the maximal holomorphic subspace of the tangent space $T_x N$, i.e., $\mathcal{D}_x = T_x N \cap J(T_x N)$. If the dimension of \mathcal{D}_x is constant along N , then \mathcal{D}_x defines a differentiable distribution \mathcal{D} , called the *holomorphic distribution* of N . A submanifold N in \tilde{M} is called a *CR-submanifold* [1,2] if there exists on N a holomorphic distribution \mathcal{D} such that its orthogonal complement \mathcal{D}^\perp is a distribution satisfying $J\mathcal{D}_x^\perp \subset T_x^\perp N$, $x \in N$. \mathcal{D}^\perp is called the *totally real distribution* of N .

Let \mathcal{H} be a differentiable distribution on a Riemannian manifold N with Levi-Civita connection ∇ . We put

$$(1.1) \quad \circ_{\sigma}(X, Y) = (\nabla_X Y)^\perp$$

for any vector fields X, Y in \mathcal{H} , where $(\nabla_X Y)^\perp$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution \mathcal{H}^\perp in N . Let X_1, \dots, X_r be an orthonormal basis of \mathcal{H} , $r = \dim_{\mathbb{R}} \mathcal{H}$. If we put

$$(1.2) \quad \mathring{H} = \frac{1}{r} \sum_{i=1}^r \circ_{\sigma}(X_i, X_i).$$

Then \mathring{H} is a well-defined \mathcal{H}^\perp -valued vector field on N (up to sign), called the *mean-curvature vector* of \mathcal{H} . A distribution \mathcal{H} on N is called *minimal* if the mean-curvature vector \mathring{H} of \mathcal{H} vanishes identically.

The main purpose of this paper is to introduce a canonical cohomology class and use it to prove the following.

THEOREM 1. *Let N be a closed CR-submanifold of a Kaehler manifold \tilde{M} . If $H^{2k}(N; \mathbb{R}) = 0$ for some $k \leq \dim_{\mathbb{C}} \mathcal{D}$, then either \mathcal{D} is not integrable or \mathcal{D}^\perp is not minimal.*

2. - THE CANONICAL CLASS OF CR-SUBMANIFOLDS

Let M be a Kaehler manifold and N a CR-submanifold of \tilde{M} . We denote by $\langle \cdot, \cdot \rangle$ the metric tensor of \tilde{M} as well as that induced on N . Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on N and \tilde{M} , respectively. The Gauss and Weingarten formulas are given respectively by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for any vector fields X, Y tangent to N and any vector field ξ normal to N . The second fundamental form σ and the second fundamental tensor A_ξ satisfy $\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle$. We recall the following.

PROPOSITION 2 [2]. *The totally real distribution \mathcal{D}^\perp of any CR-submanifold in any Kaehler manifold is integrable.*

For a CR-submanifold N of a Kaehler manifold M , we choose an orthogonal local

frame $e_1, \dots, e_h, Je_1, \dots, Je_h$ of \mathcal{D} . Let $\omega^1, \dots, \omega^h, \omega^{h+1}, \dots, \omega^{2h}$ be the $2h-1$ -forms on N satisfying

$$(2.3) \quad \omega^j(Z) = 0, \omega^i(e_j) = \delta_{ij}, i, j = 1, \dots, 2h$$

for any $Z \in \mathcal{D}^\perp$ where $e_{h+j} = Je_j$. Then

$$(2.4) \quad \omega = \omega^1 \wedge \dots \wedge \omega^{2h}$$

defines a $2h$ -form on N . This form is a well-defined global $2h$ -form on N because \mathcal{D} is orientable. We give the following.

THEOREM 3. *For any closed CR-submanifold N of a Kaehler manifold \tilde{M} , the $2h$ -form ω is closed which defines a canonical deRham cohomology class given by*

$$(2.5) \quad c(N) = [\omega] \in H^{2h}(N; \mathbb{R}), h = \dim_{\mathbb{C}} \mathcal{D} .$$

Moreover, this cohomology class is nontrivial if \mathcal{D} is integrable and \mathcal{D}^\perp is minimal.

Proof. First we give the following.

LEMMA 4. *If N is a CR-submanifold of a Kaehler manifold \tilde{M} , then the holomorphic distribution \mathcal{D} is minimal.*

Let X and Z be vector fields in \mathcal{D} and \mathcal{D}^\perp , respectively. Then we have

$$(2.6) \quad \langle Z, \nabla_X X \rangle = \langle JZ, \tilde{\nabla}_X JX \rangle = -\langle \tilde{\nabla}_X JZ, JX \rangle = \langle A_{JZ} X, JX \rangle .$$

Thus we find

$$(2.7) \quad \langle Z, \nabla_{JX} JX \rangle = -\langle A_{JZ} JX, X \rangle = -\langle A_{JZ} X, JX \rangle .$$

Combining (2.6) and (2.7) we get $\langle \nabla_X X + \nabla_{JX} JX, Z \rangle = 0$ from which we conclude that the holomorphic distribution \mathcal{D} is minimal. This proves the lemma.

From (2.4) we have

$$(2.8) \quad d\omega = \sum_{i=1}^{2h} (-1)^i \omega^1 \wedge \dots \wedge d\omega^i \wedge \dots \wedge \omega^{2h} .$$

It is clear from (2.3) and (2.8) that $d\omega = 0$ if and only if

$$(2.9) \quad d\omega(Z_1, Z_2, X_1, \dots, X_{2h-1}) = 0$$

$$(2.10) \quad d\omega(Z_1, X_1, \dots, X_{2h}) = 0$$

for any vectors $Z_1, Z_2 \in \mathcal{D}^\perp$ and $X_1, \dots, X_{2h-1} \in \mathcal{D}$. However, it follows from straight-forward computation that (2.9) holds when and only when \mathcal{D}^\perp is integrable and (2.10) holds when and only when \mathcal{D} is minimal. But for a CR-submanifold in a Kaehler manifold these two conditions hold automatically (Proposition 2 and Lemma 4). Therefore, the $2h$ -form ω is closed. Consequently, ω defines a deRham cohomology class $c(N)$ given by (2.5).

Let $e_{2h+1}, \dots, e_{2h+p}$ be an orthonormal local frame of \mathcal{D}^\perp and let $\omega^{2h+1}, \dots, \omega^{2h+p}$ be the p 1-forms on N satisfying $\omega^\alpha(X) = 0$ and $\omega^\alpha(e_\beta) = 0$ for any X in \mathcal{D} , where $\alpha, \beta = 2h+1, \dots, 2h+p$. Then by a similar argument for ω , we may conclude that if \mathcal{D} is integrable and \mathcal{D}^\perp is minimal, then the p -form $\omega^\perp = \omega^{2h+1} \wedge \dots \wedge \omega^{2h+p}$ is closed. Thus, the $2h$ -form ω is coclosed, i.e., $\delta\omega = 0$. Since N is a closed submanifold, ω is harmonic. Because ω is nontrivial, the cohomology class $[\omega]$ represented by ω is nontrivial in $H^{2h}(N; \mathbb{R})$. This proves the Theorem.

2. - PROOF OF THEOREM 1

Let N be a closed CR-submanifold of a complex m -dimensional Kaehler manifold M . Let $h = \dim_{\mathbb{C}} \mathcal{D}$ and $p = \dim_{\mathbb{R}} \mathcal{D}^\perp$. We choose an orthonormal local frame

$$e_1, \dots, e_h, e_{h+1}, \dots, e_{h+p}, e_{h+p+1}, \dots, e_m, |e_1, \dots, |e_m$$

in \tilde{M} in such a way that, restricted to N , $e_1, \dots, e_h, |e_1, \dots, |e_h$ are in \mathcal{D} and e_{h+1}, \dots, e_{h+p} are in \mathcal{D}^\perp . We denote by $\omega^1, \dots, \omega^m, \omega^{1*}, \dots, \omega^{m*}$, the dual frame of $e_1, \dots, e_m, |e_1, \dots, |e_m$. We put

$$\theta^A = \omega^A + \sqrt{-1} \omega^{A*}, \bar{\theta}^A = \omega^A - \sqrt{-1} \omega^{A*}, A = 1, \dots, m.$$

Then, restrict $\theta^{A,s}$ and $\bar{\theta}^{A,s}$ to N , we have

$$(3.1) \quad \theta^\alpha = \bar{\theta}^\alpha = \omega^\alpha \quad \text{for } \alpha = h+1, \dots, h+p$$

$$\theta^r = \bar{\theta}^r = 0 \quad \text{for } r = h+p+1, \dots, m.$$

The Kaehler form $\tilde{\Omega}$ of \tilde{M} is a closed 2-form on \tilde{M} given by

$$(3.2) \quad \tilde{\Omega} = \frac{\sqrt{-1}}{2} \sum_A \theta^A \wedge \bar{\theta}^A.$$

Let $\Omega = i^*\tilde{\Omega}$ be the 2-form on N induced from $\tilde{\Omega}$ via the immersion $i : N \rightarrow \tilde{M}$. Then, (3.1) and (3.2) give

$$(3.3) \quad \Omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^h \theta^i \wedge \bar{\theta}^i.$$

It is clear that Ω is a closed 2-form on N and it defines a cohomology class $[\Omega]$ in $H^2(N ; \mathbb{R})$. (2.4) and (3.3) imply that the canonical class $c(N)$ and the class $[\Omega]$ satisfy

$$(3.4) \quad [\Omega]^h = (-1)^h (h!) c(N).$$

If \mathcal{D} is integrable and \mathcal{D}^\perp is minimal, then Theorem 3 and (3.4) imply that $H^{2k}(N ; \mathbb{R}) \neq 0$ for $k = 1, 2, \dots, h$. (Q.E.D).

Because every hypersurface of a Kaehler manifold is a CR-hypersurface, Theorem 1 implies the following.

COROLLARY 5. *Let N be a $(2m-1)$ -dimensional closed manifold with $H^{2k}(N ; \mathbb{R}) = 0$ for some $k < m$. Then any immersion from N into a (complex) m -dimensional Kaehler manifold \tilde{M} is a CR-hypersurface such that either its holomorphic distribution is not integrable or its totally real distribution is not minimal.*

Remark. CR-products of a Kaehler manifold are examples of CR-submanifold whose holomorphic distributions are integrable and whose totally real distributions are minimal. Therefore, the assumption on cohomology groups are necessary for Theorem 1.

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