ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

SAMUEL I. GOLDBERG
IZU VAISMAN

On compact locally conformal Kaehler manifolds with non-negative sectional curvature

Annales de la faculté des sciences de Toulouse 5e série, tome 2, nº 2 (1980), p. 117-123
<http://www.numdam.org/item?id=AFST_1980_5_2_2_117_0>

Annales Faculté des Sciences Toulouse
Vol II, 1980, P. 117 à 123

ON COMPACT LOCALLY CONFORMAL KAEHLER MANIFOLDS WITH NON-NEGATIVE SECTIONAL CURVATURE

Samuel I. Goldberg (1)(*) **and Izu Vaisman** (2)

(1) University of Illinois, Urbana, Illinois - Etats Unis.
(2) University of Haifa, Haifa - Israël.

Résumé : Une variété compacte, localement, mais pas globalement, kählérienne M, à forme de Lee ω jamais nulle et à courbure de Ricci positivement semi-définie et s'annulant dans la direction de B = $\#\,\omega$ seulement, a la forme de Lee parallèle. Si, de plus, M est régulière et a la courbure non-négative et la courbure sectionnelle sur $\omega = 0$ positive, alors elle a les nombres de Betti $b_1(M) = 1, b_2(M) = 0$. Si, encore de plus, M est quasi-Einstein et si elle a l'espace des feuilles M/B simplement connexe, M est une variété de Hopf.

Summary : A compact locally, but not globally, conformal Kaehler manifold M with nowhere vanishing Lee form ω, and with a positive semi-definite Ricci tensor which vanishes in the direction of B = $\#\,\omega$ only, has a parallel Lee form. If, moreover M is regular, and has non-negative curvature and positive sectional curvatures on $\omega = 0$, it has the Betti numbers $b_1(M) = 1, b_2(M) = 0$. And, if we also add the hypotheses that M is quasi-Einstein and has a simply connected leaf space M/B, then M is a Hopf manifold.

1 - INTRODUCTION

A locally conformal Kaehler manifold, henceforth called an l. c. K. manifold, is an Hermitian manifold (M, J, g) of complex dimension $m \geqslant 2$, where $J^2 = -I$ and $g(JX,JY) = g(X,Y)$, for which an open covering $\{U_\alpha\}$ exists, and for each α a differentiable function $\sigma_\alpha : U_\alpha \to \mathbb{R}$,

(*) This paper was written while the author was a Lady Davis Visiting Professor at the Technion-Israel Institute of Technology, Haifa, during the Spring of 1979.

such that $\tilde{g} = e^{-\sigma_\alpha}(g \mid U_\alpha)$ is a Kaehler metric on U_α, called a *locally conformal Kaehler metric.* It is then easy to see [7] that $\omega \mid U_\alpha = d\,\sigma_\alpha$ defines a global closed 1-form, and that M has the characteristic property $d\,\Omega = \omega \wedge \Omega$ where $\Omega(X,Y) = g(X,JY)$ is the fundamental form of M. If we can take $U_\alpha = M$, the manifold is *globally conformal Kaehler* (g. c. K.). The form ω is called the *Lee form* of M. It is exact if and only if M is g. c. K. An l. c. K. manifold is said to be *regular* if the trajectories of the *Lee vector field* B, defined by $\omega = g(B,\cdot\,)$, provide a regular foliation B on M, *with a Hausdorff quotient manifold* M/B.

Denoting by K(s) the sectional curvature of the 2-section s, we say that an l. c. K. manifold has *positive horizontal sectional curvature* if $K(s) > 0$ whenever s is orthogonal to B. An l. c. K. manifold is called *quasi-Einstein* if its Ricci curvature Q is given by $Q = ag + b\,\omega \otimes \omega$ for some functions a and b on M.

Pertinent examples are provided by the *Hopf manifolds* with the metric derived from the diffeomorphism with $S^{2m-1} \times S^1$ [7,8]. Let us recall that these are defined as quotient manifolds $H = (\mathbb{C}_m - \{0\}) / \Delta_k \approx S^{2m-1} \times S^1$, where Δ_k is the transformation group generated by $z^i \to kz^i$ (i=1,...,m), $|k| \neq 0,1$ and $(z^i) \in \mathbb{C}^m - \{0\}$. For $m > 1$, the second Betti number $b_2(H) = 0$, so H is not Kaehlerian. Its natural l. c. K. metric is [2,p.167;7,8]

$$g = \frac{1}{\lambda} \sum_{i=1}^{m} dz^i \otimes d\bar{z}^i, \quad \lambda = \sum_{i=1}^{m} |z^i|^2,$$

with the fundamental form

$$\Omega = -\frac{\sqrt{-1}}{\lambda} \sum_{i=1}^{m} dz^i \wedge d\bar{z}^i$$

satisfying $d\,\Omega = \omega \wedge \Omega$ for the Lee form

$$\omega = -\frac{1}{\lambda} \sum_{i=1}^{m} (\bar{z}^i\, dz^i + z^i\, d\bar{z}^i),$$

which is closed, not exact and it has no zeroes. Using results of [8], it follows that H is a compact l. c. K. manifold with non-negative curvature and horizontal positive sectional curvature, whose metric is quasi-Einstein and whose Ricci curvature vanishes in the direction of B only.

The purpose of this paper is to give the following differential geometric characterization of the Hopf manifolds.

THEOREM. *(i) Let* M *be a compact connected regular* l. c. K. *manifold which is not* g. c. K. *and has non-negative curvature and horizontal positive sectional curvature. If its Ricci curvature vanishes in the direction of* B *only, then* $b_1(M) = 1$ *and* $b_2(M) = 0$;

(ii) If, moreover, M *is quasi-Einstein and the leaf space* M $|B$ *is simply connected,* M *is a Hopf manifold.*

2 - THE LEE FORM

That Q vanishes in the direction of B only is required in order to ensure that the Lee form is parallel. Indeed, we have the following.

PROPOSITION 2.1. *Let* M *be a compact 1. c. K. manifold with Lee form* $\omega \neq 0$ *everywhere, and which is not* g. c. K. *Then, if its Ricci tensor is positive semi-definite, and vanishes in the direction of* B *only,* ω *is parallel.*

Proof. Let

$$\omega = df + h \tag{2.1}$$

be the Hodge decomposition of ω, where h is its harmonic part. Since Q is positive semi-definite, h is covariant constant (see [2,p.87]), and so from (2.1)

$$\nabla_j \omega_i = \nabla_j \nabla_i f, \tag{2.2}$$

where ∇_j denotes the covariant derivative operator with respect to the Levi Civita connection in the direction of the natural basis vector $\partial/\partial x^j$.

The integrability condition of (2.2) is

$$R^j{}_{ikh}(\omega_j - \nabla_j f) = 0$$

where the $R^j{}_{ikh}$ are the components of the curvature tensor with respect to the natural basis. Hence, by contraction $R^j_h(\omega_j - \nabla_j f) = 0$, that is $Q(B-F,X) = 0$ for every X, where $df = g(F, \cdot)$. In particular, $Q(B-F,B-F) = 0$, so since Q vanishes in the direction of B only, we must have

$$B - F = \lambda B. \tag{2.3}$$

Assume $\lambda \neq 1$. Then, (2.3) yields

$$\omega = \mu \; df \tag{2.4}$$

for some function μ. Since ω is closed, $d\mu \wedge df = 0$, and because ω does not vanish anywhere, $df \neq 0$ everywhere. We therefore have $d\mu = \rho\, df$. Again, since $df \neq 0$ everywhere, we see that μ depends only on f, so by (2.4), ω is exact. Since M is not g. c. K., this is impossible. Hence, $\lambda = 1$ and $f = $ const. Thus, from (2.1), ω is harmonic, and therefore covariantly constant.

Remark. The condition $\omega \neq 0$ at every point says that the Euler-Poincaré characteristic of M vanishes.

Proposition 2.1 has important consequences including the fact that $b_1(M)$ is odd [4]. Moreover, we can prove

PROPOSITION 2.2. *Let* M *be a compact connected non-Kaehler* l. c. K. *manifold with parallel Lee form. Then, (i) if* α *is a covariantly constant* p-*form on* M, $\alpha = k\omega$ (k = const.) *for* p = 1, *and* $\alpha = 0$ *for* $2 \leqslant p \leqslant 2m-2$, $\dim_{\mathbb{C}} M = m$; *(ii) If the Ricci curvature is positive semi-definite, then* $b_1(M) = 1$.

Proof. (ii) is a straightforward consequence of (i), and of the well-known fact (already used) that a harmonic 1-form on a compact Riemannian manifold with positive semi-definite Ricci curvature is covariant constant.

To get (i), we shall use a result of Kashivada and Sato [4] and adapt a computation of Blair and one the authors [1]. Namely, if we set $A = -JB$, we have, by the proof in [4] and in analogy with a known result for Sasakian manifolds [6], that $i(A)\alpha = 0$ for every harmonic p-form α with $p < m$. In particular, this is true for the covariantly constant forms α. ($i(X)$ denotes the interior product by X).

Furthermore, if we put $\theta = \omega \circ J$, the following formula is implicit in [4] and [8]

$$2\, \nabla_X A = \theta(X)B - \omega(X)A - |\omega|^2 JX. \tag{2.5}$$

Since for the α we consider $i(\nabla_X A)\alpha = \nabla_X(i(A)\alpha) = 0$, (2.5) yields

$$|\omega|^2 i(JX)\alpha = \theta(X) i(B)\alpha,$$

or equivalently

$$|\omega|^2 \alpha(X, Y_1, \dots, Y_{p-1}) = \omega(X)\alpha(B, Y_1, \dots, Y_{p-1}).$$

Here, $|\omega| = \text{const.} \neq 0$ since M is not Kaehlerian. For $p = 1$, this yields $\alpha = k\omega$ ($k = \text{const.}$). For $2 \leqslant p \leqslant m-1$, if we put $X = X' + \lambda B$ with $\omega(X') = 0$, we get $\alpha(X', Y_1, \ldots, Y_{p-1}) = 0$, and if $Y_1 = Y'_1 + \mu B$ with $\omega(Y'_1) = 0$, we obtain $\alpha(B, Y_1, \ldots, Y_{p-1}) = \alpha(B, Y'_1, Y_2, \ldots, Y_{p-1})$. Thus, $\alpha(X, Y_1, \ldots, Y_{p-1}) = \lambda\alpha(B, Y_1, \ldots, Y_{p-1}) = \lambda\alpha(B, Y'_1, \ldots, Y_{p-1}) = -\lambda\alpha(Y'_1, B, \ldots, Y_{p-1}) = 0$. Since $\nabla\alpha = 0$ implies $\nabla(*\alpha) = 0$, where $*$ is the Hodge star operator, $\alpha = 0$ for $m + 1 \leqslant p \leqslant 2m-2$ also. Finally, for $p = m$, we consider the form $\omega \wedge \alpha$ which is covariantly constant and of degree $m + 1$. Therefore it must be zero. It follows that $|\omega|^2 \alpha = \omega \wedge i(B)\alpha = 0$ since $i(B)\alpha$ is covariantly constant and of degree $m - 1$.

This completes the proof of Proposition 2.2.

3 - PROOF OF THEOREM

We now prove the theorem formulated in Section 1. We begin by noting that, under the hypotheses, and in view of Proposition 2.1, the Lee form is parallel. The geometric structure of such manifolds has been studied by one of the authors in [8] , where the following basic facts were obtained. First, the equation $\omega = 0$ defines a codimension one foliation of M, whose leaves L are totally geodesic submanifolds and inherit from M a generalized Sasakian structure (see below). Second, because of the regularity of the foliation B (see Section 1), this gives rise to an S^1-principal fibration $p : M \to S$, $S = M/B$, which is flat (with connection ω) and is such that $p \mid L : L \to S$ is a covering map for any leaf L defined above. This leads to a generalized Sasakian structure on S. By a *generalized Sasakian structure* we understand here a normal contact metric structure [6] whose contact form η and fundamental form Φ are related by $d\eta = |\omega|\Phi$, where $|\omega| = \text{const.}$ This is a Sasakian structure if $|\omega| = 1$.

Now, since L is orthogonal to B and it is totally geodesic, it has, in our case, positive sectional curvature. Thus, the same is true for S, and $b_1(S) = b_2(S) = 0$ follow by a known theorem [5,6] if $|\omega| = 1$. Moreover, in the same case, S is a sphere if it is a simply connected Einstein manifold [5,6] . The same results are true if $|\omega| \neq 1$ since this case can be reduced by a homothetic change to the previous one, while preserving the hypotheses.

Furthermore, since the fibration p is flat, it has vanishing Euler class, and hence a vanishing Gysin map. It then follows easily form the Gysin sequence that

$$b_i(M) = b_i(S) + b_{i-1}(S), \; i \geqslant 1, \tag{3.1}$$

from which (i) follows.

To prove (ii), it is sufficient to note that since L is totally geodesic and M is quasi-Einstein, S is Einstein. Therefore, S is a sphere and M can be identified with the product of this sphere by S^1 (see also [8]).

In connection with the last part of the Theorem, we also prove

PROPOSITION 3.1. *A compact connected non-Kaehler* l. c. K. *manifold with parallel Lee form is quasi-Einstein if and only if, the Ricci curvature* $\tilde{Q}$ *of the locally conformal Kaehler metrics* $\tilde{g}$ *vanish. In this case, the Ricci tensor* Q *is given by*

$$(3.2) \qquad Q(X,Y) = \frac{m-1}{2} [\, |\omega|^2 g(X,Y) - \omega(X)\,\omega(Y)],$$

and it is positive semi-definite.

Proof. Since g and $\tilde{g}$ are conformally related and $\nabla\omega = 0$, a well-known formula [2, p.115] gives

$$(3.3) \qquad Q(X,Y) = \tilde{Q}(X,Y) + \frac{m-1}{2} [\, |\omega|^2 g(X,Y) - \omega(X)\,\omega(Y)].$$

This shows that the stated condition is sufficient. Conversely, if $Q = ag + b\,\omega \otimes \omega$, (3.3) gives

$$\tilde{Q}(X,Y) = (a - \frac{m-1}{2} |\omega|^2) g(X,Y) + (b + \frac{m-1}{2})\, \omega(X)\,\omega(Y).$$

Since $\tilde{g}$ is a Kaehler metric, $\tilde{\psi}(X,Y) = \tilde{Q}(X,JY)$ is skew-symmetric, and this implies $b = -(m-1)/2$. Consequently, $\tilde{\psi}(X,Y) = \upsilon\,\Omega(X,Y)$. Since $\tilde{\psi}$ is closed, $d\upsilon \wedge \Omega + \upsilon\, d\Omega = 0$. But $d\Omega = \omega \wedge \Omega$ and Ω is non-degenerate, so

$$(3.4) \qquad d\upsilon + \upsilon\,\omega = 0.$$

If $\upsilon \neq 0$ everywhere, ω would be exact and M would therefore be g. c. K. This is impossible since by [4], $b_1(M)$ is odd. On the other hand, if υ vanishes at a point x_0, it is identically zero. Indeed, $\omega = d\tau/\tau$ about x_0, so from (3.4), $\upsilon = c/\tau$ for some constant c. Since $\upsilon(x_0) = 0$, c is zero. Thus, $\upsilon(x) = 0$ for all $x \in M$ as one sees by propagating the local result along a chain of consecutive intersecting neighborhoods joining x_0 to x. But, $\upsilon = 0$ means $a = (m-1)\,|\omega|^2/2$. The Ricci tensor is therefore given by (3.2).

REFERENCES

[1] D.E. BLAIR and S.I. GOLDBERG. *«Topology of almost contact manifolds»*. J. Differential Geometry, 1(1967), 347-354.

[2] S.I. GOLDBERG. *«Curvature and Homology»*. Academic Press, Inc., New York & London, 1962.

[3] T. KASHIWADA. *«Some properties of locally conformal Kähler manifolds»*. Hokkaido, Math. J.8(1979), 191-198.

[4] T. KASHIWADA and S. SATO. *«On harmonic forms in compact locally conformal Kähler manifolds with the parallel Lee form»*. Preprint.

[5] E.M. MOSKAL. *«Contact manifolds of positive curvature»*. Thesis, Univ. of Illinois, 1966.

[6] S. SASAKI. *«Almost contact manifolds»*. Part III, Lecture Notes, Math. Inst., Tôhoku Univ., 1968.

[7] I. VAISMAN. *«On locally conformal almost Kähler manifolds»*. Israel J. Math., 24 (1976), 338-351.

[8] I. VAISMAN. *«Locally conformal Kähler manifolds with parallel Lee form»*. Rend. di Mat. Roma, 12(1979), 263-284.

(Manuscrit reçu le 13 janvier 1980)