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SINGULAR PERTURBATIONS FOR A CLASS OF QUASI-LINEAR HYPERBOLIC EQUATIONS

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Résumé : Nous étudions le comportement pour $\epsilon \to 0_+$ de la solution d'un problème aux limites relatif à $\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$ où L_j $(j = 1,2)$ est un opérateur linéaire hyperbolique d'ordre j et G une fonction lipschitzienne.

Dans le cas «temporel» nous obtenons la convergence de u_ϵ vers u et des dérivées de u_ϵ dans des espaces de Sobolev locaux où u est la solution d'un problème aux limites relatif à $L_1(u) + G(u) = f$.

Summary : We study the behavior for $\epsilon \to 0_+$ of the solution of a boundary value problem relative to $\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$ where L_j $(j = 1,2)$ is a linear hyperbolic operator of order j and G a lipschitzian function.

In the «time like» case, we obtain the convergence of u_ϵ to u and of the derivatives of u_ϵ in local Sobolev spaces where u is the solution of a boundary value problem relative to $L_1 u + G(u) = f$.

We study a problem of singular perturbations for a class of hyperbolic quasi-linear partial differential equations which are of the type :

$$\epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f$$

where $L_2 = \dfrac{\partial^2}{\partial t^2} - \Delta$, $L_1 = a\dfrac{\partial}{\partial t} + \sum_{k=1}^{n} b_k \dfrac{\partial}{\partial x_k}$ and $G : \mathbb{R} \to \mathbb{R}$ is a lipschitzian function.

In particular, this type of equation includes the Gordon's equation with damping. A similar non linear problem has been studied by R. Geel [3] with a function G(x,t,v) whose derivative, with respect to v, satisfies a Hölder condition with exponent $\alpha > 0$, the solutions being taken in the classical sense.

We consider the problem in the «time-like» case, that is : when operator L_1 divides operator L_2 in the sense of J. Leray [5], L. Garding [2]. The results of convergence are obtained in Sobolev spaces of local type and are analogous, with some supplementary results, to those established in the case when the non-linear term is $G(v) = |v|^{\rho} v$ [4]. Moreover the theory of non linear interpolation has the interest to give here a theorem of convergence with weakened assumptions.

The following is an outline of this work :

1. Notations hypotheses and two examples
2. Convergence of u_ϵ and $L_1 u_\epsilon$
3. Convergence of the derivatives of u_ϵ
4. Application of the non linear interpolation
5. Some remarks about correctors.

1. NOTATIONS HYPOTHESES AND TWO EXAMPLES

Ω is a bounded open set in $\mathbb{R}^n$ of class $\mathcal{N}^{(1),1}$ (J. Nečas [9]) with boundary $\Gamma = \partial\Omega$.
We set $Q = \Omega \times]0,T[$, T real > 0, $\Sigma = \Gamma \times [0,T]$ and for every $t \in [0,T]$, $Q_t = \Omega \times]0,t[$, $\Sigma_t = \Gamma \times [0,t]$.

We represent the norm of the usual Sobolev spaces, by :

$$\|\cdot\|_{L^p(\Omega)} = |\cdot|_p \qquad \|\cdot\|_{H^1(\Omega)} = \|\cdot\|_2$$

$$\|\cdot\|_{L^p(Q)} = |\cdot|_p \qquad \|\cdot\|_{L^2(0,T;H^1(\Omega))} = \|\cdot\|_2$$

and the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$. We keep the same notation $(\cdot,\cdot)$ for the duality beetween $L^p(\Omega)$, $L^{p'}(\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$ and $H^{-1}(\Omega)$, $H^1_0(\Omega)$.

We note u', u'',... the derivatives of u in the sense of vector-value distributions on]0,T[and $\alpha(u,v)$ the bilinear form $\int_\Omega \overrightarrow{\text{grad}}\, u . \overrightarrow{\text{grad}}\, v \, dx$.

We consider the following initial boundary value problem :

$$P_\epsilon \begin{cases} \epsilon L_2 u_\epsilon + L_1 u_\epsilon + G(u_\epsilon) = f & (1.1) \\ u_\epsilon(x,0) = u_o\,,\ u'_\epsilon(x,0) = u_1 & (1.2) \\ u_\epsilon|_\Sigma = 0 & (1.3) \end{cases}$$

(two examples are given p. 141 and p. 143).

and the corresponding variational problem :

$$\mathscr{P}_\epsilon \begin{cases} \epsilon\,(u''_\epsilon, v) + \epsilon\,\alpha(u_\epsilon, v) + (L_1 u_\epsilon, v) + (G(u_\epsilon), v) = (f, v) \qquad (1.4) \\ \qquad \forall v \in H^1_o(\Omega), \text{ a.e in } t \in\,]0,T[\\ u_\epsilon \in L^\infty(0,T\,; H^1_o(\Omega)),\ u'_\epsilon \in L^\infty(0,T\,; L^2(\Omega)), \\ u_\epsilon(x,0) = u_o\,,\ u'_\epsilon(x,0) = u_1 \\ u_o\,, u_1\,, f \text{ given such that :} \\ u_o \in H^1_o(\Omega)\,,\ u_1 \in L^2(\Omega)\,,\ f \in L^2(Q) \\ \text{The variable coefficients } a\,,\ b_k \text{ and the function } G \text{ satisfy the hypothesis} \\ H_1 \begin{cases} \text{(i)} & a\,,\ b_k \in W^{1,\infty}(Q) \cap C^o(\overline{Q}) \\ \text{(ii)} & \inf_{\overline{Q}} a(x,t) = \delta > 0 \\ \text{(iii)} & G : \mathbb{R} \to \mathbb{R} \text{ is a lipschitzian function i.e. :} \\ & \forall (\lambda,\mu) \in \mathbb{R}^2\,,\quad |G(\lambda) - G(\mu)| \leqslant \ell\, |\lambda - \mu|,\ \ \ell \text{ positive constant.} \end{cases} \end{cases}$$

The condition H_1 (iii) implies (see M. Marcus and V.J. Mizel [7]) the :

LEMME 1.1. $G' \in L^\infty(\mathbb{R})$ *and for every* $v \in H^1(Q)$, *we have* :

$$\frac{\partial}{\partial t} G(v) = G'(v)v' \in L^2(Q) \text{ and } \| G'(v)v' \|_2 \leqslant \ell \| v' \|_2$$

$$\frac{\partial}{\partial x_k} G(v) = G'(v)\frac{\partial v}{\partial x_k} \in L^2(Q) \text{ and } \left\| G'(v)\frac{\partial v}{\partial x_k} \right\|_2 \leqslant \ell \left\| \frac{\partial v}{\partial x_k} \right\|_2 \quad (k = 1,2,\ldots,n)$$

EXISTENCE AND REGULARITY OF THE SOLUTION U_ϵ OF $\mathscr{P}_\epsilon$:

Taking into account hypothesis about a, b_k and f, and lemma 1.1, one can show thanks to Galerkin's method (in the case a, $b_k = 0$ see J.C. Saut [10]), the

THEOREM 1.2. *The problem* $\mathscr{P}_\epsilon$ *has a unique solution, for each* $\epsilon > 0$.

(In fact there exists a solution as soon as G is a Hölder function with exponent α, $0 < \alpha \leqslant 1$).

THEOREM 1.3. *Under hypothesis*

$H_2 : H_1$ with $u_o \in H^1_o(\Omega) \cap H^2(\Omega)$, $u_1 \in H^1_o(\Omega)$, $f' \in L^2(Q)$

for each $\epsilon > 0$, *there exists a unique solution to the problem* $\mathcal{P}_\epsilon$ *such that* :

$$u_\epsilon \in L^\infty(0,T\,;H^1_o(\Omega)\cap H^2(\Omega))\,,\ u'_\epsilon \in L^\infty(0,T\,;H^1_o(\Omega))\,,\ u''_\epsilon \in L^\infty(0,T\,;L^2(\Omega)).$$

In order to study the convergence, we have to introduce :

(1) The fundamental hypothesis :

The results of convergence are obtained in the «time-like» case that is with the condition :

$$\text{(A)}\qquad \sum_{k=1}^{n} b_k^2(x,t) < a^2(x,t) \qquad \forall (x,t) \in \overline{Q}$$

One can deduce from (A) the two properties :

If $\quad \Phi(\xi_1,\xi_2,\dots,\xi_n,\xi_o) = \xi_o^2 + 2\sum_{k=1}^{n} a^{-1} b_k \xi_k\, \xi_o + \sum_{k=1}^{n} \xi_k^2$

then $\quad \Phi(\xi_1,\xi_2,\dots,\xi_n,\xi_o) \geqslant \dfrac{\omega}{2}\sum_{k=o}^{n} \xi_k^2$ where $\omega = \inf\limits_{\overline{Q}} (1 - \sum_{k=1}^{n} a^{-2} b_k^2)$ (1.5)

For every functions $v \in L^2(Q)$, $\theta \in C^o(\overline{Q})$, $\theta \geqslant 0$, such that $\theta\,|\overrightarrow{\text{grad}}\, v|$ and $\theta\, v' \in L^2(Q)$, we have :

$$\int_o^t (|\theta\,|\overrightarrow{\text{grad}}\, v\|_2^2 - |\theta\, v'|_2^2)\, ds \geqslant -\omega_1 \int_o^t |\theta L_1 v|_2^2\, ds + \frac{3\,\omega}{4}\int_o^t |\theta|\, \text{grad}\, v\|_2^2\, ds \quad (1.6)$$

where the positive constant ω_1 depends only on the coefficients.

(2) Weight functions :

Let $\nu = (\nu_1,\nu_2,\dots,\nu_n, 0)$ the unit normal outward vector to Σ when it exists.

We represent by Λ the null-subset of Σ where ν is not defined, and by Σ_-, Σ_+, Σ_o, the subsets of $\Sigma - \Lambda$ corresponding respectively to :

$$\sum_{k=1}^{n} b_k \nu_k < 0\,,\quad \sum_{k=1}^{n} b_k \nu_k > 0,\quad \sum_{k=1}^{n} b_k \nu_k = 0.$$

Under the hypothesis H_1 (ii) and the assumption :

H_1' : L_1 is a vector-field of class C^1 on an open set of $\mathbb{R}^{n+1}$ which contains $\overline{Q}$,we may use functions φ (F. Mignot and J.P. Puel [8]) satisfying the condition :

$$\mathcal{A}_1 \begin{cases} \text{(i)} & \varphi \in C^o(Q) \cap W^{1,\infty}(Q)\,,\ 0 \leqslant \varphi \leqslant 1 \\ & \varphi = 0 \text{ in a neighbourhood of } \Sigma_+ \text{ in } \Sigma \\ \text{(ii)} & L_1\, \varphi \leqslant 0 \text{ on } Q. \end{cases}$$

These functions are such that :

There is a null-set $Z \subset Q$ such that $\forall (x,t) \in (Q-Z) \cup \Sigma_-$, there existe a function φ satisfying $\mathcal{A}_1$ such that $\varphi(x,t) \neq 0$.

For each compact $K \subset \overline{Q}$ with $K \cap \mathcal{V}(\Sigma_+) = \phi$, where $\mathcal{V}(\Sigma_+)$ denotes a neigh bourhood of Σ_+ in Σ, there exists a function φ satisfying $\mathcal{A}_1$ such that

$$\varphi(x,t) \geqslant m > 0 \text{ on } K.$$

GREEN'S FORMULA FOR OPERATOR L_1 :

Under the hypothesis H_1' and the condition

(B) : $\partial \Sigma_-$ is a finite reunion of (n–1) dimensional C^1 submanifolds
$\forall w \in L^2(Q_t)$ such that $L_1 w \in L^2(Q_t)$, $w\big|_{(\Sigma_t)_-} = 0$, $w(x,0) = u_o$, we have :

$$\int_0^t (L_1 w,w)ds = \frac{1}{2}\int_\Omega aw^2\,dx - \frac{1}{2}\int_\Omega a(x,0)u_o^2\,dx + \frac{1}{2}\int_{(\Sigma_t)_+} (\sum_{k=1}^n b_k \nu_k)w^2\, d\Gamma\, ds$$
$$-\frac{1}{2}\int_{Q_t} (a' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k})\, w^2\, dx\, ds \qquad (1.7)$$

We will start the study of the general case investigated below with an illustration through two simple examples which are Klein-Gordon equation with G(u) = sin u.

EXEMPLE 1. We consider the problem $\mathcal{P}_\epsilon$ where Ω is the square]0,1[×]0,1[in $\mathbb{R}^2$ (fig. 1)

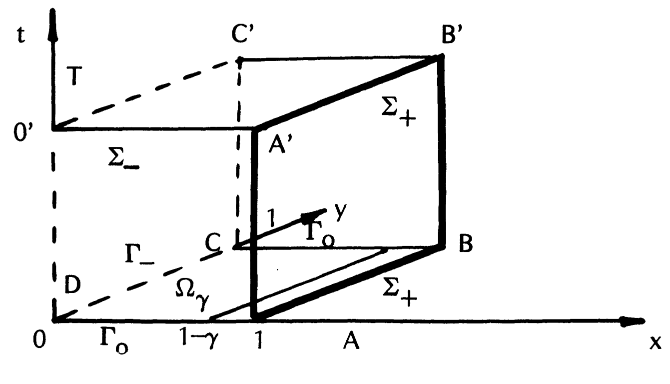

Fig. 1

$$\mathscr{P}_\epsilon \begin{cases} \epsilon\left(u''_\epsilon - \dfrac{\partial^2 u_\epsilon}{\partial x^2} - \dfrac{\partial^2 u_\epsilon}{\partial y^2}\right) + u'_\epsilon + b\,\dfrac{\partial u_\epsilon}{\partial x} + \sin u_\epsilon = f \\ u_\epsilon(x,y,0) = u_0\,,\ u'_\epsilon(x,y,0) = u_1 \\ u_\epsilon(x,y,t) = 0 \text{ on } \Sigma \end{cases}$$

b constant with $0 < b < 1$.

We note L_1 the operator of first order $u \mapsto L_1 u = u' + b\,\dfrac{\partial u}{\partial x}$.

We have seen in the general case that if $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(Q)$ the problem $\mathscr{P}_\epsilon$ has a unique solution u_ϵ, for all $\epsilon > 0$, such that $u_\epsilon \in L^\infty(0,T\,;H_0^1(\Omega))$, $u'_\epsilon \in L^\infty(0,T\,;L^2(\Omega))$.

For the weight fonctions and the limit problem the subsets of the boundary taken into account are :

$$\Gamma_- = \{(x,y),\ x=0,\ 0<y<1\},\ \Gamma_+ = \{(x,y)\,;\ x=1,\ 0<y<1\}$$

$$\Gamma_0 = \{(x,y)\,;0<x<1,\ y=0\} \cup \{(x,y)\,;0<x<1,\ y=1\}.$$

and $\Sigma_- = \Gamma_- \times [0,T]$, $\Sigma_+ = \Gamma_+ \times [0,T]$, $\Sigma_0 = \Gamma_0 \times [0,T]$.

(We remark that the subset Λ of Σ where the outward normal is not defined is composed of the four edges, 00', AA', BB' and CC').

The weight fonctions φ satisfy :

$$\mathscr{A}_1 \begin{cases} \varphi \in C^0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)\,,\ 0 \leqslant \varphi \leqslant 1 \text{ on } \Omega \\ \varphi(1,y) = 0\ ,\ 0 \leqslant y \leqslant 1 \\ \dfrac{\partial \varphi}{\partial x} \leqslant 0 \text{ on } \Omega \end{cases}$$

Let $\varphi(x,y) = 1 - x$.

Obviously φ satisfies the condition $\mathscr{A}_1$ and we have here the fact $\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$. Moreover, for each γ, $0 < \gamma < 1$, $\varphi(x,y) > \gamma$ on Ω_γ where $\Omega_\gamma =]0,1\text{-}\gamma[\ \times\ ]0,1[$.

The limit problem is here given by :

$$\mathscr{P} \begin{cases} u' + b\,\dfrac{\partial u}{\partial x} + \sin u = f \\ u(x,y,0) = u_0 \\ u(x,y,t) = 0 \text{ on } \Sigma_- \end{cases}$$

$\mathscr{P}$ has a unique solution such that $u \in L^\infty(0,T\,;L^2(\Omega))$ and $L_1 u \in L^2(Q)$.

Then, with the use of the function φ the results of convergence are

(i) For $u_o \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(Q)$, the solution u_ϵ converges to u in $L^\infty(0,T\,;L^2(\Omega))$ weak-star and in $L^q(Q)$, $\forall q < 2$. Moreover u_ϵ converges to u in $L^\infty(0,T\,;L^2(\Omega_\gamma))$ and L_1u_ϵ converges to L_1u in $L^2(0,T\,;L^2(\Omega_\gamma))$, $\forall \gamma \in]0,1[$.

Besides for u'_ϵ we have : u'_ϵ converges to u' in $L^\infty(0,T\,;L^2(\Omega))$ weak star

(ii) If we take $u_o \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in L^2(0,T\,;H^1(\Omega))$ such that $f(0,y,t) = f(1,y,t) = 0$ and $f' \in L^2(Q)$ we can state that u_ϵ converges to u in $H^1(Q_\gamma)$ where $Q_\gamma = \Omega_\gamma \times]0,T[$, and we have the estimation :

$|u_\epsilon - u|_{L^2(\Omega_\gamma)} \leqslant K_\gamma\ \epsilon^{1/2}$ where the constant K_γ can be written $K_\gamma = C\,\gamma^{-3}$ with C constant independant of ϵ and γ.

Let now, $\varphi(x,y) = (1-x)(1-y)y$.

This new function φ satisfies the condition $\mathscr{A}_1$ and is such that :

$\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$ and for each γ, $0 < \gamma < 1$; or each γ, $0 < \gamma < 1$;

$\varphi(x,y) > \gamma^2(1-\gamma)$ on the open subset of Ω : $]0,1-\gamma[\ \times\ ]\gamma,1-\gamma[$.

Moreover this function has the supplementary property $\varphi(x,y) = 0$ on Γ_o.

Then by the use of this more particular function, we obtain the result of convergence and the estimation of the point (ii), without the condition $f(x,y,t) = 0$ on Σ_o, but with Ω_γ replaced by $]0,1-\gamma[\ \times\ ]\gamma,1-\gamma[$, Q_γ by $]0,1-\gamma[\ \times\ ]\gamma,1-\gamma[\ \times\ ]0,T[$.

EXEMPLE 2. We take the same problems $\mathscr{P}_\epsilon$ and $\mathscr{P}$ but we consider here the open set $\Omega = \{(x,y) \in R^2\ ;\ (x-1)^2 + y^2 < 1\}$ (fig. 2) .

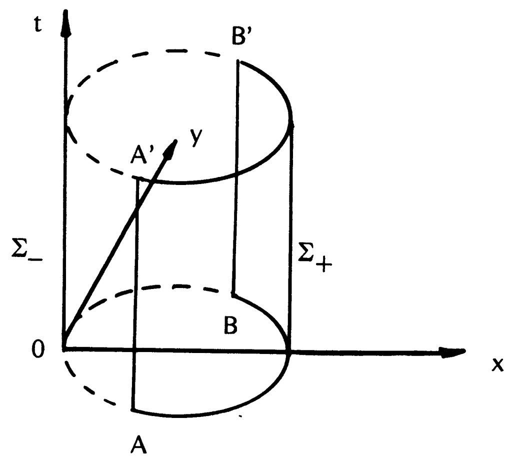

Fig. 2

Then, $\Gamma_- = \{(x,y)\ ;x = 1-(1-y^2)^{1/2}, -1 < y < 1\}$, $\Sigma_- = \Gamma_- \times [0,T]$

$\Gamma_+ = \{(x,y)\ ;x = 1+(1-y^2)^{1/2}, -1 < y < 1\}$, $\Sigma_+ = \Gamma_+ \times [0,T]$

and Σ_0 is composed of the two generating lines AA' and BB'.

The weight function we will use, is :

$$\varphi(x,y) = \begin{cases} 1-y^2 & \text{if } x \leqslant 1 \\ 1-(x-1)^2-y^2 & \text{if } 1 \leqslant x \leqslant 2 \end{cases}$$

It is such that : $\varphi(x,y) = 0$ on Γ_0, $\varphi(x,y) > 0$ on $\Omega \cup \Gamma_-$ and for each γ, $0 < \gamma < 1$, $\varphi(x,y) \geqslant \frac{\gamma^2}{4}$ on Ω_γ where $\Omega_\gamma = \Omega \cap \{(x,y)\ ;(x-1+\gamma)^2+y^2 < 1\}$ (fig. 3)

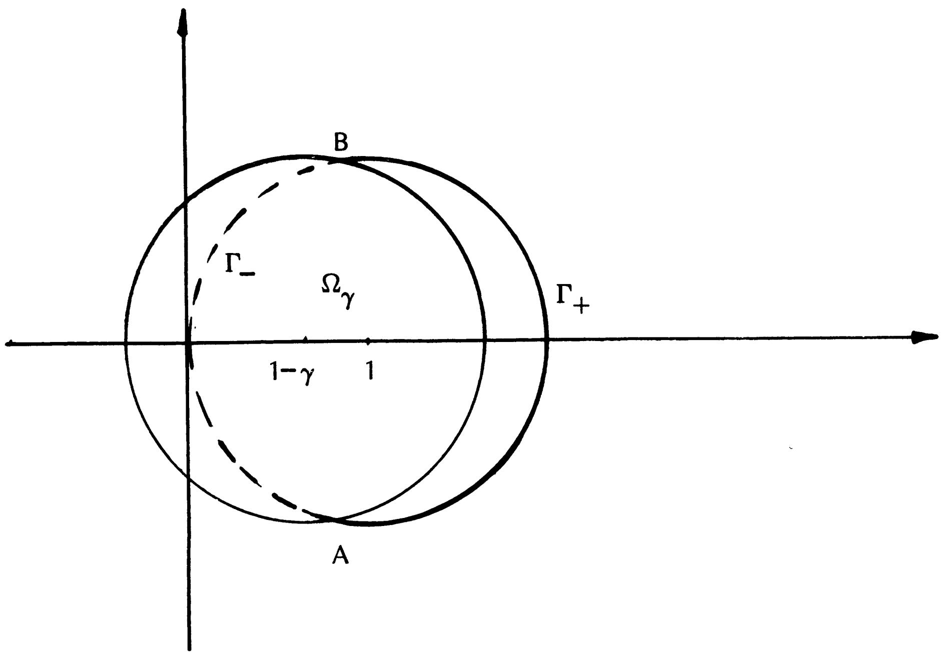

Fig. 3

We have the same results as in example 1, point (i). Because of the fact : $\varphi(x,y) = 0$ on Γ_0, we have

if we take $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u_1 \in H^1_0(\Omega)$, $f \in L^2(0,T\ ;H^1(\Omega))$ and $f' \in L^2(Q)$, u_ϵ converges to u in $H^1(Q_\gamma)$, $\forall\ \gamma \in]0,1[$, where $Q_\gamma = \Omega_\gamma \times\ ]0,T[$.

Moreover the use of the interpolation theory can improve the results in the following way :

if $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0,T\ ;H^s(\Omega))$, $0 \leqslant s \leqslant 1$, we have the estimation

$|u_\epsilon - u|_{L^2(\Omega_\gamma)} \leqslant K_\gamma \quad \epsilon^{s/2}$ for each $\gamma \in]0,1[$, where the constant K_γ can be written $K_\gamma = C\,\gamma^{-3}$ with C constant independent of ϵ and γ.

To avoid a too long text the remarks about the use of function φ such that $\varphi = 0$ on $\Sigma_o \cup \Sigma_+$ will not be detailed in the general case.

Throughout this paper, C_j and k_j will denote positive constants which are independent of f , u_o , u_1 and ϵ.

2. CONVERGENCE OF u_ϵ AND $L_1\, u_\epsilon$

In this section we obtain under the hypothesis H_1' and the condition (A) the convergence of the solution u_ϵ of the problem $\mathscr{P}_\epsilon$ to u solution of the problem.

$$P \quad \begin{cases} L_1\, u + G(u) = f \\ u \in L^\infty(0,T\,;L^2(\Omega))\;;u(x,0) = u_o\;;u\,|_{\Sigma_-} = 0 \end{cases}$$

as also the convergence of $L_1\, u_\epsilon$ to $L_1\, u$ in a local space. We use a method of regularization and monotonicity-compactness arguments. The first subsection is devoted to the study of a priori estimates and the second one to the convergence.

2.1 - A PRIORI ESTIMATES

THEOREM 2.1.1. *We assume condition* (A) ; *then* $\exists\, \epsilon_o > 0$ *such that* : $\forall \epsilon < \epsilon_o$ *the solution* u_ϵ *of the problem* $\mathscr{P}_\epsilon$ *satisfies :*

$$|u_\epsilon|_2 + \sqrt{\epsilon}\;|u_\epsilon'|_2 + \sqrt{\epsilon}\,\|u_\epsilon\|_2 \leqslant C_1\, K_1(f,u_o,u_1,\epsilon)$$

with $$K_1^2(f,u_o,u_1,\epsilon) = \|f\|_2^2 + |u_o|_2^2 + \epsilon^2\|u_o\|_2^2 + \epsilon^2\,|u_1|_2^2 + (G(0))^2$$

Preuve. We take off the method used in [4] theorem 2.1.

With assumption H_2 :

Then we can make $v = u_\epsilon + 2\epsilon\, a^{-1}\, u_\epsilon'$ in (1.4). With the same transformations as in [4] for the linear terms and taking into account that the nonlinear terms are bounded as follows

$$|(G(u_\epsilon), u_\epsilon)| \leqslant (\ell+1)\,|u_\epsilon|_2^2 + |G(0)|^2 \text{ mes}(\Omega)$$

$$\left|\epsilon\int_0^t (a^{-1}G(u_\epsilon),u'_\epsilon)ds\right| \leq \frac{\epsilon\omega}{16}\int_0^t |u'_\epsilon|_2^2 ds + \epsilon k_1 \ell^2 \int_0^t |u_\epsilon|_2^2 ds + \epsilon k_2 (G(0))^2 \text{ mes }(\Omega)$$

we obtain the statement.

With assumption H_1 :

We use a method of approximation. We consider a family $(f_\mu \,; u_{o,\mu} \,; u_{1,\mu})$ satisfying hypothesis H_2, such that

$$(f_\mu \,; u_{o,\mu} \,; u_{1,\mu}) \to (f,u_o,u_1) \text{ in } L^2(Q) \times H_o^1(\Omega) \times L^2(\Omega)$$

Then $| u_{\epsilon,\mu} |_2 + \sqrt{\epsilon} \| u_{\epsilon,\mu} \|_2 + \sqrt{\epsilon} | u'_{\epsilon,\mu} |_2 \leq K_{1,\mu}$ and $K_{1,\mu}$ is bounded independently of μ. So we can extract a subsequence still noted $u_{\epsilon,\mu}$ such that :

$u_{\epsilon,\mu}$ converges to v_ϵ in $L^\infty(0,T\,; H_o^1(\Omega))$ weak star, $u'_{\epsilon,\mu}$ converges weakly to v'_ϵ in $L^2(Q)$, $u''_{\epsilon,\mu}$ converges weakly to v''_ϵ in $L^2(0,T\,;H^{-1}(\Omega))$.

As $u_{\epsilon,\mu}$ converges to v_ϵ in $L^2(Q)$, $G(u_{\epsilon,\mu})$ converges to $G(v_\epsilon)$ in $L^2(Q)$.

Hence we can take the limit with respect to μ in the equation satisfied by $u_{\epsilon,\mu}$ and in boundary conditions and initial datas.
We deduce that $v_\epsilon = u_\epsilon$ which gives us the estimates of the theorem.

The estimates on the derivatives of u_ϵ are not sufficient to conclude about the behavior of u_ϵ as $\epsilon \to 0_+$. Under the assumptions of this section, they may be improved by an estimate of $\sqrt{\varphi}\, L_1 u_\epsilon$ independent of ϵ, the weight function φ beeing introduced in order to compensate the behavior of the derivatives of u_ϵ, in a neighborhood of the surface defining the boundary layer.

THEOREM 2.1.2. *Under assumption H'_1 and condition (A), for each function φ satisfying $\mathscr{A}_1$ (i), the solution u_ϵ of problem $\mathscr{P}_\epsilon$ verifies :*

$$\forall \epsilon \in]0,\epsilon_o[\,, | \sqrt{\varphi}\, L_1 u_\epsilon |_2 + \sqrt{\epsilon} | \sqrt{\varphi}\, u'_\epsilon |_2 \leq C_2 K_1 (f,u_o,u_1,\sqrt{\epsilon})$$

Proof : One can easily check as for theorem 2.1.1 that it is sufficient to show theorem 2.1.2. under hypothesis H_2.
Then we take the inner product of the two members of (1.1) with $\varphi\, L_1 u_\epsilon$.
We transform the linear terms as in [4] theorem 2.3, the nonlinear term is bounded by :

$$\int_0^t (G(u_\epsilon),\varphi\, L_1 u_\epsilon)\, ds \leq \int_0^t \ell |\sqrt{\varphi} u_\epsilon|_2 |\sqrt{\varphi}\, L_1 u_\epsilon|_2 ds + \int_0^t |G(0)|_2 |\sqrt{\varphi}\, L_1 u_\epsilon|_2\, ds$$
$$\leq k_3 K_1^2(f,u_o,u_1,\epsilon) + \frac{1}{4}\int_0^t |\sqrt{\varphi}\, L_1 u_\epsilon|_2^2\, ds$$

and theorem 2.1.2 follows.

At last, with the additional hypothesis H_2, we can obtain an estimate of u'_ϵ in $L^\infty(0,T\,;L^2(\Omega))$ which is independent of ϵ, by the method of differential ratios.

THEOREM 2.1.3. With assumptions H'_1, H_2, condition (A) and the coefficients b_k independent of t ; for each ϵ, $0<\epsilon<\epsilon_o$, the solution u_ϵ of $\mathcal{P}_\epsilon$ verifies

$$|u'_\epsilon|_2 + \sqrt{\epsilon}\,\|u'_\epsilon\|_2 + \sqrt{\epsilon}\,|u''_\epsilon|_2 \leqslant C_3 K_3(f,u_o,u_1,\epsilon)$$

where $$K_3^2(f,u_o,u_1,\epsilon) = |f'|_2^2 + \|u_1\|_2^2 + \epsilon^2\|u_o\|^2_{H^2(\Omega)} + \|u_o\|_2^2 + (G(0))^2 + |f(0)|_2^2.$$

Proof. We use a method of differential ratios. We consider equality (1.4) with $v\in H^1_o(\Omega)$, at time s and s + h (h > 0).

We set $w_{\epsilon,h}(s) = \frac{1}{h}\left[u_\epsilon(s+h) - u_\epsilon(s)\right]$ and throughout the proof the constants k_j are moreover independent of h.

By subtracting the two equalities, we have :

$$\epsilon(w''_{\epsilon,h},v) + \epsilon\,\alpha(w_{\epsilon,h},v) + (a(s+h)\,w'_{\epsilon,h},v) + \left(\frac{a(s+h)-a(s)}{h}\,u'_\epsilon(s),v\right) + \sum_{k=1}^{n}\left(b_k\,\frac{\partial w_{\epsilon,h}}{\partial x_k},v\right)$$
$$+\frac{1}{h}\,(G(u_\epsilon(s+h)) - G(u_\epsilon(s)),v) = \frac{1}{h}\,(f(s+h) - f(s),v)$$

By taking $v = w_{\epsilon,h} + 2\epsilon a^{-1}(s+h)w'_{\epsilon,h}$ and integrating from 0 to t, we obtain as in the first part of theorem 2.1.1 :

$$\frac{\epsilon^2}{4}\,\delta_o\,|w'_{\epsilon,h}|_2^2 + \epsilon^2\delta_o\,\alpha(w_{\epsilon,h},w_{\epsilon,h}) + \frac{\epsilon\omega}{8}\int_o^t(|w'_{\epsilon,h}|_2^2 + \alpha(w_{\epsilon,h},w_{\epsilon,h}))ds + \frac{\delta}{6}|w_{\epsilon,h}|_2^2$$
$$\leqslant K_\epsilon(h) + k_o\int_o^t|u'_\epsilon(s)|_2^2\,ds + k_1\int_o^t|w_{\epsilon,h}|_2^2\,ds \qquad (2.1)$$

where $$K_\epsilon(h) = \epsilon(w'_{\epsilon,h}(0),w_{\epsilon,h}(0)) + \frac{1}{2}|\sqrt{a(x,0)}\,w_{\epsilon,h}(0)|_2^2 + \epsilon^2|\sqrt{a^{-1}(x,0)}\,w'_{\epsilon,h}(0)|_2^2$$
$$+\epsilon^2\,|\sqrt{a^{-1}(x,0)}\;|\,\vec{\text{grad}}\,w_{\epsilon,h}(0)|\,|_2^2 + \int_o^T\left|\frac{f(s+h)-f(s)}{h}\right|_2^2 ds.$$

$$\delta_o = \inf_{\overline{Q}} a^{-1}(x,t)$$

and where the nonlinear term has been bounded as follows :

$$\left|\frac{1}{h}\int_o^t(G(u_\epsilon(s+h)) - G(u_\epsilon(s)),v)ds\right| \leqslant k_2\int_o^t|w_{\epsilon,h}(s)|_2^2\,ds + k_3\epsilon^2\int_o^t|w'_{\epsilon,h}(s)|_2^2\,ds$$

Thanks to (1.1), one can see that $\epsilon\, u''_\epsilon(0)$ is bounded in $L^2(\Omega)$ independently of ϵ and so that : $K_\epsilon(h) \leq k_4\ K_3^2(f,u_o,u_1,\epsilon)$ for small h.

Then (2.1) implies :

$$\frac{\delta}{6}|w_{\epsilon,h}|_2^2 \leq k_4 K_3^2(f,u_o,u_1,\epsilon) + k_o \int_o^T |u'_\epsilon(s)|_2^2\,ds + k_1 \int_o^t |w_{\epsilon,h}|_2^2\,ds$$

from which we deduce, by Gronwall's lemma :

$$\int_o^t |w_{\epsilon,h}|_2^2\,ds \leq k_5\Big(K_3^2(f,u_o,u_1,\epsilon) + \int_o^T |u'_\epsilon(s)|_2^2\,ds\Big). \tag{2.2}$$

It results from (2.1) and (2.2) that a subsequence of $w_{\epsilon,h}$ is such that :

$w_{\epsilon,h}$ converges to u'_ϵ in $L^\infty(0,T\,;L^2(\Omega))$ weak star, weakly in $L^2(0,T\,;H_0^1(\Omega))$ and strongly in $L^2(Q)$,

$w'_{\epsilon,h}$ converges to u''_ϵ weakly in $L^2(Q)$ and consequently by taking the limit with respect to h in (2.1), we have :

$$\frac{\epsilon\omega}{8}\int_o^t (|u''_\epsilon|_2^2 + \|u'_\epsilon\|_2^2)ds + \frac{\delta}{6}|u'_\epsilon|_2^2 \leq k_6\, K_3^2(f,u_o,u_1,\epsilon) + k_7 \int_o^t |u'_\epsilon(s)|_2^2\,ds$$

Theorem 2.1.3 follows thanks to Gronwall's lemma :

2.2 - CONVERGENCE

2.2.1 - FIRST RESULTS OF CONVERGENCE :

We assume in all this subsection hypotheses H'_1, H_2, conditions (A) et (B). The solution u_ϵ of $\mathcal{P}_\epsilon$ satisfies the estimates of theorems 2.1.1, 2.1.2. Moreover we deduce from (1.1) that for each functions φ satisfying conditions $\mathcal{A}_1$ (i) and for $\epsilon < \epsilon_o$:

$$\epsilon\, |\sqrt{\varphi}\, L_2 u_\epsilon|_2 \leq k_o\, K_1(f,u_o,u_1,\sqrt{\epsilon})$$

Then, we can extract a subsequence, still written u_ϵ, such that :

$$\left.\begin{array}{ll} u_\epsilon \rightharpoonup u & \text{in } L^\infty(0,T\,;L^2(\Omega)) \text{ weak-star} \\ \sqrt{\varphi}\,L_1u_\epsilon \rightharpoonup \sqrt{\varphi}\,L_1 u & \text{weakly in } L^2(Q) \\ \epsilon\sqrt{\varphi}\,L_2u_\epsilon \rightharpoonup 0 & \text{weakly in } L^2(Q) \\ G(u_\epsilon) \rightharpoonup \chi & \text{in } L^\infty(0,T\,;L^2(\Omega)) \text{ weak-star} \end{array}\right\} \tag{2.3}$$

Where u verifies ([4], section 3)

$$\begin{cases} L_1 u + \chi = f \\ u(x,0) = u_o \ , \ u\,|_{\Sigma_-} = 0. \end{cases}$$

Its remains to prove that $\mathcal{X} = G(u)$, which can be established by a monotonicity method ([4], section 3), by noting that we can write $G(u) = -(\ell+1)u + Mu$ where M is a strictly monotone and hemicontinuous operator (the monotonicity method is used in [4] when the operator $L_1 - (\ell+1)I$ is positive ; we are brought back to this case by the change of variable $U_\epsilon = u_\epsilon e^{\lambda t}$, the constant λ beeing choosen such that the new first order linear operator is positive. We remark that the new nonlinear function is defined by $\tilde{G}(U_\epsilon) = e^{-\lambda t} G(U_\epsilon e^{\lambda t})$ and verifies

$$|\tilde{G}(U) - \tilde{G}(V)| \leqslant \ell\,|U - V|$$

We can then apply the monotonicity method to the function U_ϵ which satisfies the same properties of regularity and the same estimates as u_ϵ, because :

$$U_\epsilon = u_\epsilon e^{\lambda t}$$

So u is solution of the problem

$$P \quad \begin{cases} L_1 u + G(u) = f \\ u \in L^\infty(0,T\,;L^2(\Omega)) \\ u(x,0) = u_o \ , \ u|_{\Sigma_-} = 0 \end{cases} \tag{2.4}$$

Remark. It results from (2.4) that $L_1 u \in L^\infty(0,T\,;\,L^2(\Omega))$. Moreover it is easy to see that u is unique, thanks to Green's formula (1.7).

Hence, we have the

LEMMA 2.2.1. *(weak convergence) With assumptions H_1', H_2, conditions (A), (B), the solution u_ϵ of $\mathcal{P}_\epsilon$ converges to the solution of problem P in $L^\infty(0,T\,;L^2(\Omega))$ weak star.*

Moreover $\sqrt{\varphi}\,L_1 u_\epsilon$ converges to $\sqrt{\varphi}\,L_1 u$ weakly in $L^2(Q)$ and if $b'_k = 0$ $(k=1,2,...,n)$ u'_ϵ converges to u' in $L^\infty(0,T\,;L^2(\Omega))$ weak star.

Our aim is now to obtain some results of strong convergence.

LEMMA 2.2.2. *With the hypotheses of lemma 2.2.1, the solution u_ϵ of problem $\mathcal{P}_\epsilon$ verifie $\sqrt{\varphi}\,u_\epsilon$ converges to $\sqrt{\varphi}\,u$ in $L^\infty(0,T\,;L^2(\Omega))$ for each function φ satisfying $\mathcal{A}_1$.*

Proof. We consider $w_\epsilon = u_\epsilon - u$; w_ϵ satisfies

$$\begin{cases} \epsilon\, L_2 u_\epsilon + L_1 w_\epsilon + G(u_\epsilon) - G(u) = 0 & \text{a.e. in } L^2(\Omega) \\ \varphi\, w_\epsilon \big|_\Sigma = 0 \,,\; w_\epsilon(x,0) = 0. \end{cases} \tag{2.5}$$

We can take the inner product of the two members of (2.5) with $\varphi\, w_\epsilon \in L^\infty(0,T;L^2(\Omega))$. After integration from 0 to t, it comes :

$$\epsilon \int_0^t \{(u''_\epsilon, \varphi\, u_\epsilon) + \alpha(u_\epsilon, \varphi u_\epsilon)\}\, ds + \frac{1}{2}\,|\sqrt{a\varphi}\; w_\epsilon|_2^2 - \frac{1}{2}\int_{Q_t} (L_1\,\varphi)\, w_\epsilon^2\, dx\, ds$$

$$= \int_0^t (\epsilon\sqrt{\varphi}\, L_2 u_\epsilon, \sqrt{\varphi}\, u)\, ds + \frac{1}{2}\int_{Q_t} (a' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k})\varphi\, w_\epsilon^2\, dx\, ds - \int_0^t (G(u_\epsilon) - G(u), \varphi\, w_\epsilon)\, ds$$

from where, we deduce, by integrating by parts the term

$\int_0^t \{(u''_\epsilon, \varphi u_\epsilon) + \alpha(u_\epsilon, \varphi u_\epsilon)\}\, ds$, and taking into account (1.6) with $\theta = \sqrt{\varphi}$, $L_1\,\varphi \leqslant 0$, $|G(u_\epsilon) - G(u)| \leqslant \ell\, |w_\epsilon|$:

$$\epsilon\, \frac{3\omega}{4}\int_0^t |\sqrt{\varphi}\,|\operatorname{grad} u_\epsilon||_2^2\, ds + \frac{\delta}{2}|\sqrt{\varphi}\;\; w_\epsilon|_2^2 \leqslant H_\epsilon(t) + k_1 \int_0^t |\sqrt{\varphi}\;\; w_\epsilon|_2^2\, ds \tag{2.6}$$

$$\text{with } H_\epsilon(t) \leqslant \Big|\int_0^t (\epsilon\sqrt{\varphi}\;\; L_2\, u_\epsilon, \sqrt{\varphi}\;\; u)\, ds\Big| + \sqrt{\epsilon}\Big\{|\sqrt{\epsilon}\sqrt{\varphi}\;\; u'_\epsilon|_2\; |\sqrt{\varphi}\;\; u_\epsilon|_2$$

$$+ k_2(\sqrt{\epsilon}\;|u'_\epsilon|_2 + \sqrt{\epsilon}\;\|u_\epsilon\|_2)|u_\epsilon|_2\Big\} + \epsilon\Big\{(u_1, \varphi\,(x,0)u_0) + \omega_1|\sqrt{\varphi}\;\; L_1\, u_\epsilon|_2^2\Big\}$$

$$\leqslant \Big|\int_0^t (\epsilon\sqrt{\varphi}\;\; L_2\, u_\epsilon, \sqrt{\varphi}\;\; u)\, ds\Big| + \sqrt{\epsilon}\;\; k_3\, K_1^2\,(f, u_0, u_1, \sqrt{\epsilon})$$

thanks to the theorems 2.1.1, 2.1.2.

From (2.6) we deduce that :

$$\int_0^t |\sqrt{\varphi}\;\; w_\epsilon|_2^2\, ds \leqslant k_4 \int_0^t H_\epsilon(s)\, ds.$$

As (2.3) implies : $H_\epsilon(s)$ bounded by a constant independent of ϵ and $\lim_{\epsilon \to 0} H_\epsilon(s) = 0$, lemma 2.2.2 follows thanks to Lebesgue's theorem.

The following lemma gives a result of convergence for $\varphi\, L_1\, u_\epsilon$.

LEMMA 2.2.3. *We assume hypotheses H'_1, H_2 and conditions (A), (B). Then, for each function φ satisfying condition $\mathscr{A}_1$, the solution u_ϵ of $\mathscr{P}_\epsilon$ verifies :*

$$\varphi\; L_1 u_\epsilon \;\to\; \varphi\; L_1 u \quad \text{in } L^2(Q)$$

Proof. We consider the inner product of the two members of (2.5) with $\varphi^2 L_1\, w_\epsilon \in L^2(Q)$ and we integrate from 0 to t. We obtain :

$$\epsilon \int_0^t (L_2 u_\epsilon , \varphi^2 L_1 u_\epsilon) ds + \int_0^t |\varphi L_1 w_\epsilon|_2^2 ds + \int_0^t (G(u_\epsilon) - G(u), \varphi^2 L_1 w_\epsilon) ds$$
$$= \int_0^t (\epsilon L_2 u_\epsilon , \varphi^2 L_1 u) ds \quad (2.7)$$

By integrating by parts the term $\epsilon \int_0^t (L_2 u_\epsilon , \varphi^2 L_1 u_\epsilon) ds$, then using inequality (1.5), and inequality (1.6) with $\theta^2 = -\varphi L_1 \varphi$, we show the minoration :

$$\epsilon \int_0^t (L_2 u_\epsilon , \varphi^2 L_1 u_\epsilon) ds \geq \epsilon \frac{\delta\omega}{4} \{ |\varphi u'_\epsilon|_2^2 + \| |\varphi| \; \vec{\text{grad}}\, u_\epsilon \|_2^2 \} - m_1 \sqrt{\epsilon}$$
$$- m_2 \epsilon \int_0^t \{ |\varphi u'_\epsilon|_2^2 + \| |\varphi| \; \vec{\text{grad}}\, u_\epsilon \|_2^2 \} ds$$

where m_i, $(i = 1,2)$ is a constant independent of ϵ.

Then, it results from (2.7) that :

$$\epsilon \frac{\delta\omega}{4} \{ |\varphi u'_\epsilon|_2^2 + \| |\varphi| \vec{\text{grad}}\, u_\epsilon \|_2^2 \} + \frac{1}{2} \int_0^t |\varphi L_1 w_\epsilon|_2^2 ds \leq M_\epsilon (t) + \epsilon m_2 \int_0^t \{ |\varphi u'_\epsilon|_2^2$$
$$+ \| |\varphi| \; \vec{\text{grad}}\, u_\epsilon \|_2^2 \} ds$$

where $M_\epsilon (t) = \frac{1}{2} | \varphi [G(u_\epsilon) - G(u)] |_2^2 + \left| \int_0^t (\epsilon L_2 u_\epsilon , \varphi^2 L_1 u) ds \right| + m_1 \sqrt{\epsilon}$.

As $M_\epsilon(t) \to 0$ when $\epsilon \to 0+$, and $M_\epsilon (t)$ is bounded independently of ϵ thanks to (2.3), we conclude thanks to Lebesgue's theorem that $|\varphi L_1 w_\epsilon|_2 \to 0$ and the lemma follows.

Remark. The proof of the lemma also shows that $\sqrt{\epsilon}\, \varphi u'_\epsilon \to 0$ and $\sqrt{\epsilon}\, \varphi \, |\vec{\text{grad}}\, u_\epsilon| \to 0$ in $L^\infty(0,T;L^2(\Omega))$.

2.2.2 - CONVERGENCE OF u_ϵ and $L_1 u_\epsilon$:

The results of the subsection 2.2.1 may be improved as follows.

THEOREM 2.2.4. *With hypothesis H'_1, conditions (A) and (B), the solution u_ϵ of problem $\mathscr{P}_\epsilon$ verifies :*

(i) *u_ϵ converges to u in $L^\infty(0,T;L^2(\Omega))$ weak star and in $L^q(Q)$, $\forall q < 2$ where u is the solution of the problem P.*

u_ϵ converges to u and $L_1 u_\epsilon$ to $L_1 u$ in $L^2(Q')$ where Q' is an open set of Q such that $\overline{Q}' \cap \mathscr{V}(\Sigma_+) = \phi$, where $\mathscr{V}(\Sigma_+)$ is a neighborhood of Σ_+ in Σ.

(ii) *for each function φ satisfying conditions $\mathcal{A}_1$:*

$$\sqrt{\varphi}\ u_\epsilon \to \sqrt{\varphi}\ u \ \text{ in } \ L^\infty(0,T;L^2(\Omega))$$

$$\sqrt{\varphi}\ L_1 u_\epsilon \rightharpoonup \sqrt{\varphi}\ L_1 u \ \text{ weakly in } L^2(Q) \text{ and } \ \varphi\ L_1 u_\epsilon \to \varphi\ L_1 u \text{ strongly in } L^2(Q)$$

(iii) *If we also assume hypothesis H_2 and that the coefficients b_k are independent of t, $u'_\epsilon \rightharpoonup u'$ in $L^\infty(0,T;L^2(\Omega))$ weak-star.*

Proof. We remark that existence and uniqueness of u solution of the problem P is insured under the single hypotheses H_1 and H'_1 (C. Bardos [1] p. 199, by using the transformation $Gu = -(\ell+1)u+Mu$).

To prove points (i) and (ii), we use a method of regularization as in the proof of theorem 2.1.1. We approximate the triplet (f,u_o,u_1) by a sequence $(f;u_{o,\mu};u_{1,\mu})$ satisfying H_2 such that :

$$(f_\mu\ ;u_{o,\mu};u_{1,\mu}) \to (f,u_o,u_1) \text{ in } L^2(Q) \times H^1_o(\Omega) \times L^2(\Omega) \tag{2.8}$$

Let $w_{\epsilon,\mu} = u_{\epsilon,\mu} - u_\epsilon$ and $w_\mu = u_\mu - u$. We have :

(1) $$\epsilon\ L_2 w_{\epsilon,\mu} + L_1 w_{\epsilon,\mu} + G(u_{\epsilon,\mu}) - G(u_\epsilon) = f_\mu - f$$

and it results from theorems 2.1.1 and 2.1.2, since $|G(u_{\epsilon,\mu}) - G(u_\epsilon)| \leqslant \ell\,|w_{\epsilon,\mu}|$, that

$$|w_{\epsilon,\mu}|_2^2 + |\sqrt{\varphi}\ L_1 w_{\epsilon,\mu}|_2^2 \leqslant k_o\ (|f_\mu - f|_2^2 + \|u_{o,\mu} - u_o\|_2^2 + |u_{1,\mu} - u_1|_2^2) \tag{2.9}$$

where k_o is a positive constant independent of μ and ϵ

(2) $$L_1 w_\mu + G(u_\mu) - G(u) = f_\mu - f \tag{2.10}$$

from where we deduce by taking the inner product of (2.10) with w_μ, using Green's formula (1.7) and at last by integrating from 0 to t

$$\frac{\delta}{2}\ |w_\mu|_2^2 + \frac{1}{2}\int_{\Sigma_t} (\sum_{k=1}^n b_k\ \nu_k) w_\mu^2\ d\sigma \leqslant k_1\,|u_{o,\mu} - u_o|_2^2 + \frac{1}{2}|f_\mu - f|_2^2 + k_2 \int_o^t |w_\mu|_2^2\ ds.$$

Then, Gronwall's lemma implies :

$$|w_\mu|_2^2 \leqslant k_3\,(|f_\mu - f|_2^2 + |u_{o,\mu} - u_o|_2^2),\ k_3 \text{ positive constant independent of } \mu \tag{2.11}$$

Now by taking the inner product of (2.10) with $L_1 w_\mu$, we obtain :

$$|L_1\ w_\mu|_2^2 \leqslant k_4\,(|f_\mu - f|_2^2 + |u_{o,\mu} - u_o|_2^2),\ k_4 \text{ positive constant independent of } \mu \tag{2.12}$$

At last, by using the results of subsection 2.2.1, for each fixed μ

$$\left.\begin{array}{l} u_{\epsilon,\mu} \rightharpoonup u_\mu \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star} \\ \sqrt{\varphi}\, u_{\epsilon,\mu} \rightarrow \sqrt{\varphi}\, u_\mu \text{ in } L^\infty(0,T;L^2(\Omega)) \\ \sqrt{\varphi}\, L_1 u_{\epsilon,\mu} \rightharpoonup \sqrt{\varphi}\, L_1 u_\mu \text{ weakly in } L^2(Q) \\ \varphi\, L_1 u_{\epsilon,\mu} \rightarrow \varphi\, L_1 u_\mu \text{ in } L^2(Q) \end{array}\right\} \tag{2.13}$$

as $\epsilon \rightarrow 0_+$.

As $u_\epsilon - u = - w_{\epsilon,\mu} + u_{\epsilon,\mu} - u_\mu + w_\mu$, one can easily check thanks to (2.8), (2.9), (2.11), (2.12), (2.13) that :

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star}$$

$$\sqrt{\varphi}\; u_\epsilon \longrightarrow \sqrt{\varphi}\, u \text{ in } L^\infty(0,T;L^2(\Omega))$$

$$\sqrt{\varphi}\; L_1 u_\epsilon \rightharpoonup \sqrt{\varphi}\; L_1 u \text{ weakly in } L^2(Q) \text{ and } \varphi\; L_1 u_\epsilon \rightarrow \varphi\; L_1 u \text{ in } L^2(Q)$$

and the point (ii) follows. To achieve the proof of the point (i) we remark that the convergence of u_ϵ and $L_1 u_\epsilon$ in $L^2(Q')$ results from the properties of the functions φ . These properties also imply that $u_\epsilon \rightarrow u$ a.e. in Q. As $|u_\epsilon - u|^q$ is bounded in $L^{2/q}(Q)$, $\forall q \leqslant 2$, there is a subsequence of u_ϵ such that $|u_\epsilon - u|^q \rightharpoonup 0$ weakly in $L^{2/q}(Q)$ $\forall q < 2$, from where u_ϵ converges to u strongly in $L^q(Q)$, $\forall q < 2$.
The point (iii) results from lemma 2.2.1.

3. CONVERGENCE OF THE DERIVATIVES OF u_ϵ

In this section, we improve the results of convergence. We aim at obtaining, on the one hand, the strong convergence of the derivatives of u_ϵ in local spaces, on the other hand, the rate of convergence in ϵ of $\varphi^{3/2}(u_\epsilon - u)$ in the space $L^\infty(0,T;L^2(\Omega))$. This kind of results needs hypotheses of regularity on f, because of the non-regularity of u under the only assumptions : f, f' $\in L^2(Q)$, the derivatives of the function u generally having poles on the part Σ_0 of Σ.
So, we impose on f the hypothesis

$$H_3 \quad \begin{cases} f \in L^2(0,T;H^1(\Omega)) \\ f = 0 \text{ on } \mathcal{V} = \mathcal{V}(\Sigma_0 \cup \Lambda) \cap \Sigma\text{- where } \mathcal{V}(\Sigma_0 \cup \Lambda) \text{ is a neighborhood of } \Sigma_0 \cup \Lambda \text{ in } \Sigma. \end{cases}$$

Then, there exists $\lambda > 0$, such that $\sum_{k=1}^{n} b_k \nu_k \leqslant -\lambda$ on $(\Sigma\text{-}) - \mathcal{V}$. (3.1)

With the hypothesis H_3, we first establish additional a priori estimates which allow us to obtain by compactness artuments the convergence of u_ϵ to u solution of the problem P.

3.1 - A PRIORI ESTIMATES

THEOREM 3.1.1. *We suppose hypotheses H_1', H_2, H_3, conditions (A), (B) and $G(0) = 0$.*
Then for ϵ, $0 < \epsilon < \epsilon_o$, the solution u_ϵ satisfies the estimates of theorems 2.1.1, 2.1.2, 2.1.3 and moreover verifies :

for each function φ satisfying $\mathcal{A}_1$

$$|\varphi^{3/2} u_\epsilon'|_2 + \|\varphi^{3/2} u_\epsilon\|_2 + \sqrt{\epsilon}\,|\varphi^{3/2} \Delta u_\epsilon|_2 \leqslant C_4 K_4 (f,u_o,u_1,\epsilon).$$

where

$$K_4^2(f,u_o,u_1,\epsilon) = \frac{1}{\lambda} \|f\|^2_{L^2(\Sigma)} + \|f\|_2^2 + |f'|_2^2 + \|u_1\|_2^2 + \epsilon^2 \|u_o\|^2_{H^2(\Omega)} + \|u_o\|_2^2 + |f(0)|_2^2.$$

Proof. The smoothness properties of u_ϵ, under hypothesis H_2, allow us to take the inner product of two members of (1.1) with $-\varphi^3 \Delta u_\epsilon$. It comes :

$$-\epsilon(u_\epsilon'', \varphi^3\Delta u_\epsilon) + \epsilon|\varphi^{3/2} \Delta u_\epsilon|_2^2 - (L_1 u_\epsilon, \varphi^3 \Delta u_\epsilon) - (G(u_\epsilon), \varphi^3\Delta u_\epsilon) = -(f, \varphi^3\Delta u_\epsilon) \tag{3.2}$$

Green's formula gives the following transformations :

$$-(L_1 u_\epsilon, \varphi^3\Delta u_\epsilon) = \frac{1}{2}\frac{d}{dt}|\sqrt{a\varphi^3}\,|\overrightarrow{\text{grad}}\, u_\epsilon|\,|_2^2 - \frac{1}{2}\int_\Gamma (\sum_{k=1}^n b_k \nu_k)\varphi^3 |\overrightarrow{\text{grad}}\, u_\epsilon|^2 d\Gamma \tag{3.3}$$
$$-\frac{3}{2}\int_\Omega \varphi^2 (L_1\varphi)|\overrightarrow{\text{grad}}\, u_\epsilon|^2 dx + R(u_\epsilon)$$

where $R(u_\epsilon) = 3\int_\Omega \varphi^2 L_1 u_\epsilon (\overrightarrow{\text{grad}}\,\varphi \,.\, \overrightarrow{\text{grad}}\, u_\epsilon)\,dx + \int_\Omega \varphi^3 u_\epsilon' (\overrightarrow{\text{grad}}\, a \,.\, \overrightarrow{\text{grad}}\, u_\epsilon)\,dx$

$$+ \sum_{k=1}^n \int_\Omega \varphi^3 \frac{\partial u_\epsilon}{\partial x_k} (\overrightarrow{\text{grad}}\, b_k \,.\, \overrightarrow{\text{grad}}\, u_\epsilon)dx - \frac{1}{2}\int_\Omega (\bar{a}' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k})\varphi^3 |\overrightarrow{\text{grad}}\, u_\epsilon|^2 dx,$$

$$-(G(u_\epsilon), \varphi^3\Delta u_\epsilon) = 3\int_\Omega \varphi^2 G(u_\epsilon)(\overrightarrow{\text{grad}}\,\varphi \,.\, \text{grad}\, u_\epsilon)\,dx + \int_\Omega G'(u_\epsilon)\varphi^3 |\overrightarrow{\text{grad}}\, u_\epsilon|^2 dx \tag{3.4}$$

as $G(u_\epsilon) \in L^2(0,T;H_0^1(\Omega))$, thanks to lemma 1.1,

$$-(f, \varphi^3\Delta u_\epsilon) = \alpha(\varphi^3 f, u_\epsilon) - \int_\Gamma \varphi^3 f \frac{\partial u_\epsilon}{\partial \nu}\, d\Gamma \tag{3.5}$$

Then we have :

by taking into account theorem 2.1.2 :

$$\left|\int_0^t R(u_\epsilon)ds\right| \leqslant K_1^2(f,u_o,u_1,\sqrt{\epsilon}) + k_1 \int_0^t |\varphi^{3/2} u_\epsilon'|_2^2 ds + k_2 \int_0^t |\varphi^{3/2}|\overrightarrow{\text{grad}}\, u_\epsilon|\,\|_2^2 ds \tag{3.6}$$

thanks to theorem 2.1.1. and lemma 1.1 :

$$\left|\int_0^t (G(u_\epsilon), \varphi^3 \Delta u_\epsilon) ds\right| \leq k_3 K_1^2(f,u_o,u_1,\epsilon) + k_4 \int_0^t \left|\varphi^{3/2}\right| \overrightarrow{\text{grad}}\, u_\epsilon \|_2^2 ds. \tag{3.7}$$

and at last :

$$\left|\int_0^t \epsilon(u''_\epsilon, \varphi^3 \Delta u_\epsilon) ds\right| \leq \frac{\epsilon}{2}\int_0^t \left|\varphi^{3/2} u''_\epsilon\right|^2 ds + \frac{\epsilon}{2}\int_0^t \left|\varphi^{3/2} \Delta u_\epsilon\right|_2^2 ds \tag{3.8}$$

$$\left|\int_{\Sigma_t} f\varphi^3 \frac{\partial u_\epsilon}{\partial \nu} d\Gamma\right| \leq \frac{k_5}{\lambda} \|f\|^2_{L^2(\Sigma)} - \frac{1}{4}\int_{\Sigma_t} \left(\sum_{k=1}^n b_k \nu_k\right)\varphi^3 \left|\overrightarrow{\text{grad}}\, u_\epsilon\right|^2 d\Gamma \tag{3.9}$$

So, taking into account results (3.3) to (3.9), $L_1\varphi \leq 0$ on Q, $\varphi\left(\sum_{k=1}^n b_k \nu_k\right) \leq 0$ on Σ and the properties of the coefficients, equality (3.2) gives :

$$\frac{\epsilon}{2}\int_0^t \left|\varphi^{3/2} \Delta u_\epsilon\right|_2^2 ds + \frac{\delta}{2}\left|\varphi^{3/2}\right| \overrightarrow{\text{grad}}\, u_\epsilon \|_2^2 \leq k_6 \|f\|_2^2 + \frac{k_5}{\lambda}\|f\|_{L^2(\Sigma)} + k_7 K_4^2(f,u_o,u_1,\epsilon)$$

$$+ \frac{\epsilon}{2}\int_0^t \left|\varphi^{3/2} u''_\epsilon\right|_2^2 ds + k_1 \int_0^t \left|\varphi^{3/2} u'_\epsilon\right|_2^2 ds + k_8 \int_0^t \left|\varphi^{3/2}\right| \overrightarrow{\text{grad}}\, u_\epsilon \|_2^2 ds \tag{3.10}$$

Now, we consider the method of the differential ratios when the coefficents b_k depend on t. We have with the same notation as in the proof of the theorem 2.1.3 :

$$\epsilon(w''_{\epsilon,h}, v) + \epsilon\alpha(w_{\epsilon,h}, v) + (a(s+h)w'_{\epsilon,h}, v) + \sum_{k=1}^n \left(b_k(s+h)\frac{\partial w_{\epsilon,h}}{\partial x_k}, v\right)$$

$$+ \left(\frac{a(s+h)-a(s)}{h} u'_\epsilon(s), v\right) + \sum_{k=1}^n \left(\frac{b_k(s+h)-b_k(s)}{h}\frac{\partial u_\epsilon(s)}{\partial x_k}, v\right)$$

$$+ \frac{1}{h}(G(u_\epsilon(s+h)) - G(u_\epsilon(s)), v) = \frac{1}{h}(f(s+h) - f(s), v) \tag{3.11}$$

We first obtain by taking $v = \epsilon(w_{\epsilon,h} + 2\epsilon a^{-1}(s+h)w'_{\epsilon,h})$ in (3.11) as in the proof of the theorem 2.1.3

$$\epsilon^2 \left|w'_{\epsilon,h}\right|_2^2 + \epsilon^2 \|w_{\epsilon,h}\|^2 \leq k_9 K_3^2(f,u_o,u_1,\epsilon) \tag{3.12}$$

and then by putting $v = \varphi^3 (w_{\epsilon,h} + 2\epsilon a^{-1}(s+h) w'_{\epsilon,h})$ in (3.11) and taking into account (3.12), it comes :

$$\frac{\delta}{6}\left|\varphi^{3/2} u'_\epsilon\right|_2^2 + \epsilon\int_0^t \left(\left|\varphi^{3/2} u''_\epsilon\right|_2^2 + \left|\varphi^{3/2}\right| \overrightarrow{\text{grad}}\, u'_\epsilon \|_2^2\right) ds,$$

$$\leq k_{10}(K_3^2(f,u_o,u_1,\epsilon) + \int_0^t \left\{\left|\varphi^{3/2} u'_\epsilon\right|_2^2 + \left|\varphi^{3/2}\right| \text{grad}\, u_\epsilon \|_2^2\right\} ds \tag{3.13}$$

It results from (3.10) and (3.13) :

$$\frac{\delta}{6}\left|\varphi^{3/2} u'_\epsilon\right|_2^2 + \frac{\delta}{2}\left|\varphi^{3/2}\right| \text{grad}\, u_\epsilon\|_2^2 \leq k_{11}(K_4^2(f,u_o,u_1,\epsilon) + \int_0^t \left\{\left|\varphi^{3/2} u'_\epsilon\right|_2^2 + \left|\varphi^{3/2}\right| \overrightarrow{\text{grad}}\, u_\epsilon \|_2^2\right\} ds$$

and theorem 3.1.1 follows.

3.2 - CONVERGENCE

With hypotheses H'_1, H_2, H_3, conditions (A), (B) and $G(0) = 0$, the following a priori estimates

$$|u_\epsilon|_2 \leqslant C_1 K_1(f,u_o,u_1,\epsilon), \left|\sqrt{\varphi}\; L_1 u_\epsilon\right|_2 \leqslant C_2 K_1(f,u_o,u_1,\sqrt{\epsilon})$$

$$|\varphi^{3/2} u'_\epsilon|_2 + \|\varphi^{3/2} u_\epsilon\|_2 \leqslant C_4 K_4(f,u_o,u_1,\epsilon)$$

allow us to extract a subsequence still denoted by u_ϵ such that :

$u_\epsilon \rightharpoonup u$ in $L^\infty(0,T;L^2(\Omega))$ weak star and $\varphi^{3/2} u_\epsilon \rightharpoonup \varphi^{3/2} u$ in $L^\infty(0,T;H^1_o(\Omega))$ weak star, $\varphi^{3/2} u'_\epsilon \rightharpoonup \varphi^{3/2} u'$ in $L^\infty(0,T;L^2(\Omega))$ weak star, $\sqrt{\varphi}\, L_1 u_\epsilon \rightharpoonup \sqrt{\varphi}\, L_1 u$ weakly in $L^2(Q)$.

So, in particular, we have : $\varphi^{3/2} u_\epsilon$ converges to $\varphi^{3/2} u$ weakly in $H^1(Q)$ and strongly in $L^2(Q)$. The properties of functions φ imply the existence of a new subsequence such that u_ϵ converges to u a.e. on Q and so $G(u_\epsilon)$ converges to $G(u)$ a.e. on Q. As $|G(u_\epsilon)|_2 \leqslant \ell K_1(f,u_o,u_1,\epsilon)$ we finally have :

$$G(u_\epsilon) \rightharpoonup G(u) \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star.}$$

One can check that u is the solution of problem P (for which we also have shown smoothness properties). Hence we have the

THEOREM 3.2.1. *(weak-convergence). Under hypotheses H'_1, H_2, H_3, conditions (A), (B) and $G(0) = 0$, the solution u_ϵ of $\mathcal{P}_\epsilon$ verifies :*

i) $u_\epsilon \rightharpoonup u$ *in* $L^\infty(0,T;L^2(\Omega))$ *weak star.*

ii) for each function φ satisfying $\mathcal{A}_1$:

$\sqrt{\varphi}\, L_1 u_\epsilon \rightharpoonup \sqrt{\varphi}\, L_1 u$ *weakly in* $L^2(Q)$, $\varphi^{3/2} u_\epsilon \rightharpoonup \varphi^{3/2} u$ *in* $L^\infty(0,T;H^1_o(\Omega))$ *weak-star.* $\varphi^{3/2} u'_\epsilon \rightharpoonup \varphi^{3/2} u'$ *in* $L^\infty(0,T;L^2(\Omega))$ *weak-star,*

where u is the solution of the problem P.

Of course, the results of strong convergence of theorem 2.2.4 are still valid in the frame of this section. We are now interested by the rate of convergence in ϵ of $\varphi^{3/2}(u_\epsilon - u)$ in $L^\infty(0,T;L^2(\Omega))$. For this, we first improve the estimates satisfied by u in $L^\infty(0,T;H^1_o(\Omega))$ and u' in $L^\infty(0,T;L^2(\Omega))$ which result from the theorem 3.2.1. We obtain the

LEMMA 3.2.2. *With the same hypotheses as in theorem 3.2.1, we have*

$$\|\varphi^{3/2} u\|_2 + |\varphi^{3/2} u'|_2 \leqslant K_5 \text{ where } K_5^2 = C_5\left(\frac{1}{\lambda}\|f\|^2_{L^2(\Sigma)} + \|f\|_2^2 + \|u_o\|_2^2 + |u_1|_2^2\right)$$

Proof. We consider the equality

$$(L_1 u, \varphi^3 (u' - \Delta u)) + (G(u), \varphi^3(u' - \Delta u)) = (f, \varphi^3(u' - \Delta u)).$$

Thanks to (3.3), (3.4), (3.5), (3.7), (3.9) where u_ϵ is replaced by u, inequality

$$\left|\int_o^t R(u)ds\right| \leqslant k_1 K_1^2(f,u_o,u_1,\sqrt{\epsilon}) + \frac{\delta}{8}\int_o^t |\varphi^{3/2} u'|_2^2 ds + k_2 \int_o^t |\varphi^{3/2}| \overrightarrow{\text{grad}}\, u\|_2^2 ds,$$

the fact that $L_1 \varphi \leqslant 0$ on Q, $\varphi(\sum_{k=1}^n b_k \nu_k) \leqslant 0$ on Σ, and the properties of the coefficients, it comes :

$$\frac{\delta}{2}|\varphi^{3/2}| \overrightarrow{\text{grad}}\, u\|_2^2 + \frac{\delta}{4}\int_o^t |\varphi^{3/2} u'|_2^2 ds \leqslant k_3\left(\frac{1}{\lambda}\|f\|^2_{L^2(\Sigma)} + \|f\|_2^2 + \|u_o\|_2^2 + |u_1|_2^2\right)$$
$$+ k_4 \int_o^t |\varphi^{3/2}| \overrightarrow{\text{grad}}\, u\|_2^2 ds$$

And Gronwall's lemma gives the estimates. (When u is not smooth enough, the lemma results from the study of the solution of the regularized problem $\left(\begin{cases} -\eta\Delta v + L_1 v + G(v) = F \;,\; \eta > o \\ v_{\Sigma} = 0 \;,\; v(x,0) = u_o \end{cases}\right).$

Now, we may prove the

THEOREM 3.2.3. *(rate of convergence). With hypotheses of theorem 3.2.1, for each* ϵ, $0 < \epsilon < \epsilon_o$, *we have :*

$$|\varphi^{3/2}(u_\epsilon - u)|_2 \leqslant K_5\sqrt{\epsilon} \text{ (for each } \varphi \text{ satisfying } \mathcal{A}_1)$$

$$|u_\epsilon - u|_{L^2(Q')} \leqslant K_5'\sqrt{\epsilon}$$

where Q' is an open set of Q such that $\overline{Q}' \cap \mathcal{V}(\Sigma_+) = \phi$.

Proof. We set $w_\epsilon = u_\epsilon - u$ and we take the inner product of $\epsilon L_2 u_\epsilon + L_1 w_\epsilon + G(u_\epsilon) - G(u) = 0$ with $\varphi^3 w_\epsilon \in L^\infty(0,T;H_o^1(\Omega))$. It comes :

$$\epsilon\frac{3\omega}{4}\int_o^t |\varphi^{3/2}| \overrightarrow{\text{grad}}\, w_\epsilon\|_2^2 ds + \frac{\delta}{2}|\varphi^{3/2} w_\epsilon|_2^2 \leqslant \epsilon A_\epsilon(t) + k_1 \int_o^t |\varphi^{3/2} w_\epsilon|_2^2 ds \tag{3.14}$$

where $A_\epsilon(t) = -(\varphi^{3/2} u_\epsilon', \varphi^{3/2} w_\epsilon) + \omega_1 \int_o^t |\varphi^{3/2} L_1 w_\epsilon|_2^2 ds + \int_o^t (\varphi^{3/2} u', \varphi^{3/2} w_\epsilon')ds$

$$-\int_{Q_t} \varphi^3 (\overrightarrow{\text{grad}}\, u \,.\, \overrightarrow{\text{grad}}\, w_\epsilon)\, dx\, ds + 3\int_{Q_t} \varphi^2 w_\epsilon \,(\varphi' u_\epsilon' + \overrightarrow{\text{grad}}\, \varphi \,.\, \overrightarrow{\text{grad}}\, u_\epsilon)\, dx\, ds.$$

As $\epsilon|A_\epsilon(t)| \leqslant \epsilon k_2 K_5^2 + \frac{\delta}{6}|\varphi^{3/2} w_\epsilon|_2^2 + \epsilon\frac{3\omega}{8}\int_o^t |\varphi^{3/2}| \overrightarrow{\text{grad}}\, w_\epsilon\|_2^2 ds + k_3 \int_o^t |\varphi^{3/2} w_\epsilon|_2^2 ds$

thanks to the theorems 2.1.1, 2.1.2 and lemma 3.2.2, the statement follows by application of Gronwall's lemma.

Remark. We also have shown that $|\varphi^{3/2}| \overrightarrow{\text{grad}}\, u_\epsilon \|_2 \leqslant K_5$ and $|\varphi^{3/2} u'_\epsilon|_2 \leqslant K_5$.

The following theorem gives results of strong convergence for the derivatives of u_ϵ .

THEOREM 3.2.4. *(strong convergence of the derivatives). With hypotheses of theorem 3.2.1, we have :*

(i) $\varphi^{3/2} u_\epsilon \to \varphi^{3/2} u$ *in* $L^2(0,T;H_0^1(\Omega))$

$\varphi^{3/2} u'_\epsilon \to \varphi^{3/2} u'$ *in* $L^2(Q)$ *for each function* φ *satisfying* $\mathscr{A}_1$.

(ii) $u_\epsilon \to u$ *in* $H^1(Q')$ *where Q' is an open set of Q with* $\overline{Q}' \cap \mathscr{V}(\Sigma_+) = \phi$.

Proof. We consider again (3.14).

As $\lim_{\epsilon \to 0_+} A_\epsilon(t) = 0$ because $\sqrt{\varphi}\, w_\epsilon \to 0$ in $L^\infty(0,T;L^2(\Omega))$ (theorem 2.2.4)

$\varphi^{3/2} u'_\epsilon \rightharpoonup \varphi^{3/2} u'$ in $L^\infty(0,T;L^2(\Omega))$ weak star (theorem 3.2.1)

$\varphi L_1 w_\epsilon \to 0$ in $L^2(Q)$ (theorem 2.2.4)

$\varphi^{3/2}\, \overrightarrow{\text{grad}}\, u_\epsilon \rightharpoonup \varphi^{3/2}\, \overrightarrow{\text{grad}}\, u$ in $L^\infty(0,T;L^2(\Omega))$ weak star (theorem 3.2.1)

Gronwall's lemma and Lebesgue's theorem allow us to conclude because $|A_\epsilon(t)|$ is bounded. (ii) results from properties of functions φ.

Remark 3.2.5. With hypothesis H'_1, conditions (A),(B) and G(0)=0, the results of theorem 3.2.3 are still valid if $f \in L^2(0,T;H_0^1(\Omega))$. It is once more enough to approximate the triplet $(f;u_0;u_1)$ in $L^2(0,T;H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ by a sequence $(f_\mu;u_{0,\mu};u_{1,\mu})$ satisfying hypothesis H_2.

4. APPLICATION OF NON LINEAR INTERPOLATION

The application of non linear interpolation theory (L. Tartar [11]) allows us to explicite $|\varphi^{3/2}(u_\epsilon - u)|_2$ in ϵ, with less assumptions than in section 3, in particular without condition on f over $\mathscr{V}(\Sigma_0 \cup \Lambda)$.

We first recall the theorem of non linear interpolation of [11] which will be then applied to our problem. The useful result is the following :

Let $A_0 \subset A_1$, $B_0 \subset B_1$ Banach spaces and T a map such that $T(A_1) \subset B_1$, $T(A_0) \subset$ Bo and :

$$\exists\, \alpha, \beta\ ;\ 0 < \alpha \leqslant 1\ ,\ 0 < \beta \text{ such that}$$

$$\|Ta - Tb\|_{B_1} \leqslant f(\|a\|_{A_1}, \|b\|_{A_1})\ \|a-b\|_{A_1}^{\alpha}\ , \quad \forall a,b \in A_1$$

$$\|T_a\|_{B_0} \leqslant g(\|a\|_{A_1}) \|a\|_{A_0}^{\beta}\ , \quad \forall a \in A_0$$

f continuous on $\overline{\mathbb{R}}_+^2$, g continuous on $\overline{\mathbb{R}}_+$.

Then, if $0 < \theta < 1, 1 \leqslant p \leqslant \infty$, we have :

$$\|Ta\|_{(B_o,B_1)_{\eta,q}} \leqslant C\, h(\|a\|_{A_1})\, \|a\|_{(A_o,A_1)_{\theta,p}}^{\frac{(1-\eta)}{\beta+\eta\alpha}}$$

where $\frac{1-\eta}{\eta} = \frac{1-\theta}{\theta}\,\frac{\alpha}{\beta}$

$$q = \max\left(1, \left(\frac{1-\theta}{\beta} + \frac{\theta}{\alpha}\right)p\right)$$

$$h(r) = g(2r)^{1-\eta} \quad f(r,2r)^{\eta}$$

the space $(A_o,A_1)_{\theta,p}$ beeing defined by :

$$(A_o,A_1)_{\theta,p} = \left\{ a \in A_o + A_1 \;\middle|\; t^{-\theta} \quad K(t,a) \in L^p(0,\infty;\tfrac{dt}{t}) \right\} \text{ with the norm}$$

$$\|a\|_{(A_o,A_1)_{\theta,p}} = \|t^{-\theta} \quad K(t,a)\|_{L^p(0,\infty;\frac{dt}{t})}$$

with $K(t,a) = \mathrm{Inf}\left\{ a_o \in A_o, a_1 \in A_1 ; a_o + a_1 = a \;\middle|\; \|a\|_{A_o} + t\|a\|_{A_1} \right\}$

This result applied to our problem gives the :

THEOREM 4.1. *We suppose hypothesis H_1', conditions (A), (B) and $G(0) = 0$,*

(i) Let θ, $0 \leqslant \theta \leqslant 1$, if $f \in L^2(0,T;[H_o^1(\Omega) ; L^2(\Omega)]_\theta)$, for each $\epsilon < \epsilon_o$, we have :

$$|\varphi^{3/2}(u_\epsilon - u)|_2 \leqslant K_6\ \epsilon^{\frac{1-\theta}{2}} \quad \text{where } K_6^2 = C_6\left(\|f\|^2_{L^2(0,T;[H_o^1(\Omega);L^2(\Omega)]\theta)} + \|u_o\|_2^2 + |u_1|_2^2\right)$$

for each function φ satisfying $\mathscr{A}_1$, and :

$$|u_\epsilon - u|_{L^2(Q')} \leqslant K_6'\ \epsilon^{\frac{1-\theta}{2}} \quad \text{where } Q' \text{ is an open set of } Q \text{ such that } \overline{Q}' \cap \mathscr{V}(\Sigma_+) = \phi.$$

(ii) In particular, if $f \in L^2(0,T;H^s(\Omega))$, $0 \leqslant s < \frac{1}{2}$, Ω regular, we have :

$$|\varphi^{3/2}(u_\epsilon - u)|_2 \leqslant K_6\ \epsilon^{s/2}$$

$$|u_\epsilon - u|_{L^2(Q')} \leqslant K_6'\ \epsilon^{s/2}$$

Proof. We consider $A_o = L^2(0,T;H_o^1(\Omega)) \times H_o^1(\Omega) \times L^2(\Omega)$, $A_1 = L^2(Q) \times H_o^1(\Omega) \times L^2(\Omega)$

$$B_o = B_1 = L^\infty(0,T;L^2(\Omega))$$

$$T = T_\epsilon \ : (f,u_o,u_1) \to \varphi^{3/2}(u_\epsilon - u)$$

It results from theorems 1.2 and 2.2.4 that T_ϵ maps A_1 into B_1 and also A_o into B_o.

(a) We first consider $T_\epsilon : A_1 \to B_1$. Let $(f,u_o,u_1) \in A_1$ and $(g,v_o,v_1) \in A_1$, $T_\epsilon(f,u_o,u_1) = \varphi^{3/2}(u_\epsilon - u)$ and $T_\epsilon(g,v_o,v_1) = \varphi^{3/2}(v_\epsilon - v)$.

If we put $w_\epsilon = u_\epsilon - v_\epsilon$ and $w = u - v$, then :

$$\epsilon L_2 w_\epsilon + L_1 w_\epsilon + G(u_\epsilon) - G(v_\epsilon) = f - g$$

from where by recalling the proof of theorem 2.1.1 and taking into account $|G(u_\epsilon) - G(v_\epsilon)| \leq \ell |w_\epsilon|$, we deduce :

$$|w_\epsilon|_2^2 \leq k_1(|f-g|_2^2 + \|u_o - v_o\|_2^2 + |u_1 - v_1|_2^2) \tag{4.1}$$

$$L_1 w + G(u) - G(v) = f - g \tag{4.2}$$

We take the inner product of two members of (4.2) with w and we integrate from 0 to t. Green's formula (1.7) gives :

$$\frac{\delta}{2}|w|_2^2 + \frac{1}{2}\int_{\Sigma_t} (\sum_{k=1}^{n} b_k \nu_k) w^2 d\Gamma \leq k_1 |u_o - v_o|_2^2 + \frac{1}{2}|f-g|_2^2 + k_2 \int_o^t |w|_2^2 ds$$

and Gronwall's lemma then implies that :

$$|w|_2^2 \leq k_3 (|f-g|_2^2 + |u_o - v_o|_2^2) \tag{4.3}$$

at last, (4.1) and (4.3) give the inequality :

$$|T_\epsilon(f,u_o,u_1) - T_\epsilon(g,v_o,v_1)|_2 \leq k_4(|f-g|_2^2 + \|u_o - v_o\|_2^2 + |u_1 - v_1|_2^2)^{1/2}$$

$$\forall (f,u_o,u_1) \in A_1 , \quad \forall (g,v_o,v_1) \in A_1$$

(b) If we consider $T_\epsilon : A_o \to B_o$, it results from remark 3.2.5 that :

$$\forall (f,u_o,u_1) \in A_o \qquad |T_\epsilon(f,u_o,u_1)|_2 \leq K_5 \, \epsilon^{1/2}$$

(c) The hypotheses of the theorem of non linear interpolation are satisfied thanks to (a) and (b), with $\alpha = 1$, $\beta = 1$, $f(r,s) = k_4$, $g(r) = (C_5\epsilon)^{1/2}$, $p = 2$
and the application of this theorem allows us to assert that

if $(f,u_o,u_1) \in [A_o,A_1]_\theta$, then $T_\epsilon(f,u_o,u_1) \in L^\infty(0,T;L^2(\Omega))$ and

$$|T_\epsilon(f,u_o,u_1)|_2 \leq C_6 \, \epsilon^{\frac{1-\theta}{2}} \|(f,u_o,u_1)\|_{[A_o,A_1]_\theta}$$

and point (i) follows as $[A_o,A_1]_\theta = L^2(0,T;[H^1_o(\Omega);L^2(\Omega)]\theta) \times H^1_o(\Omega) \times L^2(\Omega)$.

The result (ii) is an application of point (i) since, for $0 \leqslant s < \frac{1}{2}$,

$$[H^1_o(\Omega)\,;L^2(\Omega)]_{1-s} = H^s(\Omega).$$

5. REMARK ABOUT CORRECTORS

We can define under hypothesis H_1 and condition (A) correctors in the sense of J.L. Lions [6].

Let $g_\epsilon \in L^2(Q)$ given and θ_ϵ defined by

$$\begin{cases} \epsilon\,((\theta_\epsilon+u)'',v) + \epsilon\,\alpha(\theta_\epsilon+u,v) + (L_1(\theta_\epsilon+u),v) + (G(\theta_\epsilon+u),v) = (f,v) + \epsilon^{1/2}(g_\epsilon,v) \\ \qquad\qquad\qquad \forall v \in H^1_o(\Omega) \quad \text{a.e. on } t \in \,]0,T[\\ (\theta_\epsilon+u)(x,0) = u_o\,,\ (\theta_\epsilon+u)'(x,0) = u_1. \end{cases}$$

The theorem 1.2 ensures the existence and uniqueness of $\theta_\epsilon + u$ such that :

$$\theta_\epsilon + u \in L^\infty(0,T;H^1_o(\Omega))\,;\,(\theta_\epsilon+u)' \in L^\infty(0,T;L^2(\Omega))$$

Then θ_ϵ is a corrector in the following sense :

THEOREM 5.1. *Under hypothesis H_1, condition (A), if $g_\epsilon \in L^2(Q)$ with $|g_\epsilon|_2$ bounded independently of ϵ, we have :*

$|u_\epsilon - (\theta_\epsilon+u)|_2 \leqslant K\sqrt{\epsilon}$ *where K is a positive constant independent of* ϵ

$u_\epsilon - (\theta_\epsilon+u) \rightharpoonup 0$ *weakly in* $L^2(0,T;H^1_o(\Omega))$

$u'_\epsilon - (\theta_\epsilon + u') \rightharpoonup 0$ *weakly in* $L^2(Q)$.

Proof. We consider $w_\epsilon = u_\epsilon - (\theta_\epsilon+u)$ which verifies

$$\begin{cases} \epsilon\,(w''_\epsilon,v) + \epsilon\,\alpha(w_\epsilon,v) + (L_1w_\epsilon,v) + (G(u_\epsilon) - G(\theta_\epsilon+u),v) = -\,\epsilon^{1/2}(g_\epsilon,v) \\ w_\epsilon(x,0) = 0,\ w'_\epsilon(x,0) = 0 \end{cases}$$

and we follow once more the method of the proof of theorem 2.1.1.

We first suppose that $g'_\epsilon \in L^2(Q)$ and hypothesis H_2.

We obtain by the same arguments, taking into account $|G(u_\epsilon) - G(\theta_\epsilon+u)| \leqslant \ell|w_\epsilon|$ the inequality :

$$|w_\epsilon|_2 + \sqrt{\epsilon}\,|w'_\epsilon|_2 + \sqrt{\epsilon}\,\|w_\epsilon\|_2 \leqslant k_1\,|g_\epsilon|_2\,\sqrt{\epsilon} \tag{4.4}$$

It is enough then to approximative in $L^2(Q) \times L^2(Q) \times H^1_0(\Omega) \times L^2(\Omega)$ $(f\,;g_\epsilon\,;u_o\,;u_1)$ by $(f_\mu; g_{\epsilon,\mu}; u_{o,\mu}; u_{1,\mu})$ satisfying hypothesis H_2 with $g'_{\epsilon,\mu} \in L^2(Q)$ to assert that (4.4) is still valid under hypotheses of the theorem which thus is proved.

REFERENCES

[1] C. BARDOS. *«Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels»*, Ann. Sci. Ecole Norm. Sup. 3, (1970), 185-283.

[2] L. GARDING. *«Solution directe du problème de Cauchy pour les équations hyperboliques* in «La Théorie des équations aux dérivées partielles, Nancy 1956, Colloques internationaux du Centre National de la Recherche Scientifique LXXI», pp. 71-90. Centre National de la Recherche Scientifique, Paris 1956.

[3] R. GEEL. *«Singular Perturbations of hyperbolic type»*, Thesis, University of Amsterdam, 1978.

R. GEEL and E.M. DE JAGER. *«Hyperbolic singular perturbation of non linear first order differential equations»*.

[4] J. GENET and M. MADAUNE. *«Singular Perturbations for a class of non linear hyperbolic-hyperbolic problems»*, J. Math. Anal. Appli. 63 (1978).

[5] J. LERAY. *«Lectures on hyperbolic equations with variable coefficients»*, The Institute for Advanced Study, Princeton, 1952.

[6] J.L. LIONS. *«Perturbations singulières dans les problèmes aux limites et en contrôle optimal»*, Lecture Notes in Mathematics, N° 323, Springer-Verlag, Berlin / New-York, 1973.

[7] M. MARCUS and V.J. MIZEL. *«Absolute Continuity on Tracks and Mappings of Sobolev Spaces»*, Archiv. Rational Mech. Anal., 1972, pp. 294-320.

[8] F. MIGNOT and J.P. PUEL. *«Un résultat de perturbations singulières dans les inéquations variationnelles»*, Lecture Notes in Mathematics, N° 594, Singular Perturbations and Boundary Layer Theory, Lyon 1976, Springer-Verlag, Berlin / New-York, 1977.

[9] J. NECAS. *«Les méthodes directes en théorie des équations elliptiques»*, Masson, Paris, 1967.

[10] J.C. SAUT. Thèse de 3ème cycle, Université d'Orsay, 1972 (France).

[11] L. TARTAR. *«Interpolation non linéaire et Régularité»*, J. Funct. Anal., vol. 9, 1972, pp. 469-489.

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