Winter Braids Lecture Notes

Anne Pichon

A topological introduction to Lipschitz geometry of complex singularities

Vol. 6 (2019), Course nº III, p. 1-17.

<http://wbln.centre-mersenne.org/item?id=WBLN_2019__6__A3_0>

Winter Braids Lecture Notes
Vol. **6** (2019) 1-17
Course nº III

A topological introduction to Lipschitz geometry of complex singularities

ANNE PICHON

Abstract

These are the notes of the three lectures I gave during the IXth Winterbraids School which took place in Reims from 4 to 7th March 2019. The aim of this course was to introduce researchers working in low-dimensional topology to Lipschitz geometry of complex singularities. In these lectures, I focussed on topological points of view on the objects, avoiding as much as possible technical material from algebraic geometry and singularity theory such as resolution of singularities, Nash transform, generic projections of curves and surfaces, etc.

It starts with an introduction to Lipschitz geometry of singular spaces. It then gives the complete classification of Lipschitz geometry of complex curves and covers the result of [17]. The last part is an introduction to Lipschitz geometry of complex surfaces and states the thick-thin decomposition Theorem of a normal complex surface proved in [6].

1. Introduction to Lipschitz geometry of singular spaces

In the sequel, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$.

Let $(X,0)$ be a germ of analytic space in $\mathbb{K}^n$ which contains the origin. So X is defined by

$$X = \{(x_1,\dots,x_n) \in \mathbb{K}^n \mid f_j(x_1,\dots,x_n) = 0, j = 1,\dots,r\},$$

where the f_j's are r convergent power series $f_j \in \mathbb{K}\{x_1,\dots,x_n\}$ with $f_j(0) = 0$.

Question 1: *how does X look in a small neighbourhood of the origin?*

So we are interested in the geometry of the germ $(X,0)$. There are multiple answers to this vague question depending on the category we work in, i.e., on the chosen equivalence relation between germs.

First, we can consider the topological equivalence relation:

Definition 1.1. Two analytic germs $(X,0)$ and $(X',0)$ are *topologically equivalent* if there exists a germ of homeomorphism $\psi\colon (X,0) \to (X',0)$. The *topological type* of $(X,0)$ is the equivalence class of $(X,0)$ for this equivalence relation.

Two analytic germs $(X,0) \subset (\mathbb{K}^n,0)$ and $(X',0) \subset (\mathbb{K}^n,0)$ are *topologically equisingular* if there exists a germ of homeomorphism $\psi\colon (\mathbb{K}^n,0) \to (\mathbb{K}^n,0)$ such that $\psi(X) = X'$. We call *embedded topological type* of $(X,0)$ the equivalence class of $(X,0)$ for this equivalence relation.

The embedded topological type of $(X,0) \subset (\mathbb{R}^n,0)$ is completely determined by the embedded topology of its link as stated in the following famous Conical Structure Theorem:

Theorem 1.2. *(Conical Structure Theorem). Let B^n_ϵ be the sphere with radius $\epsilon > 0$ centered at the origin of $\mathbb{R}^n$ and let S^{n-1}_ϵ be its boundary.*

Let $(X,0)\subset(\mathbb{R}^n,0)$ be an analytic germ. For $\epsilon>0$, set $X^{(\epsilon)}=S^{n-1}_\epsilon\cap X$. There exists $\epsilon_0>0$ such that for every $\epsilon>0$ with $0<\epsilon\le\epsilon_0$, the pair $(B^n_\epsilon, X\cap B^n_\epsilon)$ is homeomorphic to the pair $(B^n_\epsilon, Cone(X^{(\epsilon)}))$, where $Cone(X^{(\epsilon)})$ means the cone over $X^{(\epsilon)}$, i.e., the union of the half-lines $[0,x)$ joining the origin to a point $x\in X^{(\epsilon)}$.

In other words, the homeomorphism class of the pair $(S^{n-1}_\epsilon, X^{(\epsilon)})$ does not depend on ϵ when $\epsilon>0$ is sufficiently small and it determines completely the embedded topological type of $(X,0)$.

Definition 1.3. When $0<\epsilon\le\epsilon_0$, the intersection $X^{(\epsilon)}$ is called the *link* of $(X,0)$.

Example 1.4.
1. Assume that X is the real cusp in $\mathbb{R}^2$ with equation $x^3-y^2=0$. Then its link at 0 consists of two points in the circle S^1_ϵ.
2. If X is the complex cusp in $\mathbb{C}^2$ with equation $x^3-y^2=0$, its link at 0 is the trefoil knot in the 3-sphere S^3_ϵ.
3. If X is the complex surface E_8 in $\mathbb{C}^3$ with equation $x^2+y^3+z^5=0$, its (non embedded) link at 0 is a Seifert manifold, i.e., i.e., a 3 manifold which admits a action of the circle group $\mathbb{S}^1$ on it with no fixed points.

The Conical Structure Theorem gives a complete answer to Question 1 in the topological category, but it completely ignores the geometric properties of the set $(X,0)$. In particular, a very interesting question is:

Question 2: *How does the link $X^{(\epsilon)}$ evolve metrically as ϵ tends to 0?*

In other words, is $X\cap B_\epsilon$ bilipschitz homeomorphic to the straight cone $Cone(X^{(\epsilon)})$? Or are there some parts of $X^{(\epsilon)}$ which shrink faster than linearly when ϵ tends to 0?

Question 2 can be studied from different points of view depending on the choice of the metric. If $(X,0)\subset(\mathbb{R}^n,0)$ is the germ of a real analytic space, there are two natural metrics on $(X,0)$ which are defined from the Euclidean metric of the ambient space $\mathbb{R}^n$:

Definition 1.5. The *outer metric d_o* on X is the metric induced by the ambient Euclidean metric, i.e., for all $x,y\in X$, $d_o(x,y)=\|x-y\|_{\mathbb{R}^n}$.

The *inner metric d_i* on X is the length metric defined for all $x,y\in X$ by: $d_i(x,y)=\inf length(\gamma)$ where $\gamma\colon[0,1]\to X$ varies among rectifyable arcs on X such that $\gamma(0)=x$ and $\gamma(1)=y$.

Definition 1.6. Let (M,d) and (M',d') be two metric spaces. A map $f\colon M\to M'$ is a *bilipschitz homeomorphism* if f is a bijection and there exists a real constant $K\ge1$ such that for all $x,y\in M$,

$$\frac{1}{K}d(x,y)\le d'(f(x),f(y))\le Kd(x,y).$$

Definition 1.7. Two real analytic germs $(X,0)\subset(\mathbb{R}^n,0)$ and $(X',0)\subset(\mathbb{R}^m,0)$ are *inner Lipschitz equivalent* (resp. *outer Lipschitz equivalent* if there exists a germ of bilipschitz homeomorphism $\psi\colon(X,0)\to(X',0)$ with respect to the inner (resp. outer) metrics.

The equivalence classes of the germ $(X,0)\subset(\mathbb{R}^n,0)$ for these equivalence relations are called respectively the *inner Lipschitz geometry* and the *outer Lipschitz geometry* of $(X,0)$.

Definition 1.8. Throughout these notes, we will use the “big-Theta" asymptotic notations of Bachmann-Landau in the following form: given two function germs $f,g\colon([0,\infty),0)\to([0,\infty),0)$, we say f is *big-Theta* of g and we write $f(t)=\Theta(g(t))$ if there exist real numbers $\eta>0$ and $K>0$ such that for all t such that $f(t)\le\eta$, $\frac{1}{K}g(t)\le f(t)\le Kg(t)$.

Example 1.9. Consider the real cusp C with equation $y^2-x^3=0$ in $\mathbb{R}^2$. For a real number $t>0$, consider the two points $p_1(t)=(t,t^{3/2})$ and $p_2(t)=(t,-t^{3/2})$ on C. Then $d_o(p_1(t),p_2(t))=\Theta(t^{3/2})$ for this notation), while the inner distance is obtained by taking infimum of lengths of paths on C between the two points $p_1(t)$ and $p_2(t)$. The shortest length is obtained by taking a path going through the origin, and we have $d_i(p_1(t), p_2(t)) = \Theta(t)$. Therefore, taking the limit of the quotient as t tends to 0, we obtain:

$$\frac{d_o(p_1(t), p_2(t))}{d_i(p_1(t), p_2(t))} = \Theta(t^{1/2}) \to 0.$$

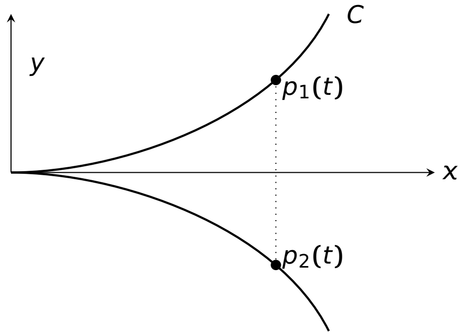


Figure 1.1: The real cusp $y^2 - x^3 = 0$

Using this, you are ready to resolve the following exercise.

Exercise 1.10.
1. *Prove that there is no bilipschitz homeomorphism between the outer and inner metrics on the real cusp C with equation $y^2 - x^3 = 0$ in $\mathbb{R}^2$.*

2. *Prove that $(C, 0)$ equipped with the inner metric is metrically conical, i.e. bilipschitz equivalent to the cone over its link.*

Example 1.11. Consider the real surface S in $\mathbb{R}^3$ with equation $x^2 + y^2 - z^3 = 0$ in $\mathbb{R}^2$. For a real number $t > 0$, consider the two points $p_1(t) = (t^{3/2}, 0, t)$ and $p_2(t) = (-t^{3/2}, 0, t)$ on S. Then $d_o(p_1(t), p_2(t)) = \Theta(t^{3/2})$. We also have $d_i(p_1(t), p_2(t)) = \Theta(t^{3/2})$ since $d_i(p_1(t), p_2(t))$ is the length of a half-circle joining $p_1(t)$ and $p_2(t)$ on the circle $\{z = t\} \cap S$.

Exercise 1.12. *Consider the real surface S of Example 1.11.*

1. *Prove that the identity map is a bilipschitz homeomorphism between the outer and inner metrics on $(S, 0)$.*

2. *Prove that $(S, 0)$ equipped with the inner metric is not metrically conical, that is it is not inner Lipschitz homeomorphic to the straight cone over its link.*

If $(X, 0)$ is a germ of a real analytic space, the two metrics d_o and d_i defined above obviously depend on the choice of an embedding $(X, 0) \subset (\mathbb{R}^n, 0)$ since they are defined by using the Euclidean metric of the ambient $\mathbb{R}^n$. We now prove of one of the main results which motivates the study of Lipschitz geometry of singularities:

Proposition 1.13. *The Lipschitz geometries of $(X, 0)$ for the outer and inner metrics are independent of the embedding $(X, 0) \subset (\mathbb{R}^n, 0)$.*

In other words, bilipschitz classes of $(X, 0)$ just depend on the analytic type of $(X, 0)$. Before proving this result, let us give some consequences which motivate the study of Lipschitz geometry of germs of singular spaces.

The outer Lipschitz geometry determines the inner Lipschitz geometry since the inner metric is determined by the outer one through integration along paths. Moreover, the inner Lipschitz geometry obviously determines the topological type of $(X, 0)$. Therefore, an important consequence of Proposition 1.13 is that the Lipschitz geometries give two intermediate classifications between the analytical type and the topological type.

A very small amount of analytic invariants are determined by the topological type of an analytic germ (even if one considers the embedded topological type). In particular, a natural question is to ask whether the Lipschitz classification is sufficiently rigid to catch analytic invariants:

Question 3: *Which analytical invariants are in fact Lipschitz invariants?*

Recent results show that in the case of a complex surface singularity, a large amount of analytic invariants are determined by the outer Lipschitz geometry. For example, the multiplicity of a complex surface singularity is an outer Lipschitz invariant ([18] for a normal surface, [10] for a hypersurface in $\mathbb{C}^3$ and [9] for the general case). However it is now known that multiplicity is not a Lipschitz invariant in higher dimensions ([5]). In [18] it is shown that many other data are in fact Lipschitz invariants in the case of surface singularities, such as the geometry of hyperplane sections and the geometry of polar curves and discriminant curves of generic projections; higher dimensions remain almost unexplored. This shows that the outer Lipschitz class contains potentially a lot of information on the singularity and that outer Lipschitz geometry of singularities is a very promising area to explore.

Here is another motivation. Analytic types of singular space germs contain continuous moduli, and this is why it is difficult to describe a complete analytic classification. For example, consider the family of curves germs $(X_t,0)_{t\in\mathbb{C}}$ where X_t is the union of four transversal lines with equation $xy(x-y)(x-ty)=0$. For every pair (t,t') with $t\neq t'$, $(X_t,0)$ is not analytically equivalent to $(X_{t'},0)$. On the contrary, it is known since the works of T. Mostowki in the complex case ([16]), and Parusinski in the real case ([19] and [20]), that the outer Lipschitz classification of germs of singular spaces is *tame*, which means that it admits a discrete complete invariant. Then a complete classification of Lipschitz geometry of singular spaces seems to be a more reachable goal.

Proof of Proposition 1.13. Let $(f_1,\dots,f_n)$ and $(g_1,\dots,g_m)$ be two systems of generators of the maximal idea $\mathcal{M}$ of $(X,0)$. We will first prove that the outer metrics d_I and d_J for the embeddings

$$I=(f_1,\dots,f_n)\colon(X,0)\to(\mathbb{R}^n,0)\quad\text{and}\quad J=(g_1,\dots,g_m)\colon(X,0)\to(\mathbb{R}^m,0)$$

are bilipschitz equivalent. It suffices to prove that the outer metric for the embedding

$$(f_1,\dots,f_n,g_1,\dots,g_m)\colon(X,0)\to(\mathbb{R}^{n+m},0)$$

is bilipschitz equivalent to the metric d_I. By induction, we just have to prove that for any $g\in\mathcal{M}$, the metric $d_{I'}$ associated with the embedding $I'=(f_1,\dots,f_n,g)\colon(X,0)\to(\mathbb{R}^{n+1},0)$ is bilipschitz equivalent to d_I.

Since g is in the ideal $\mathcal{M}$, it is a function $g(f_1,\dots,f_n)$ of the generators $f_1,\dots,f_n$. Let Γ be the graph of the function $g(x_1,\dots,x_n)$ in $(\mathbb{R}^n,0)\times\mathbb{R}$. It is defined over a neighbourhood of 0 in $\mathbb{R}^n$. The projection $\pi\colon\Gamma\to\mathbb{R}^n$ is bilipschitz over any compact neighbourhood of 0 in $\mathbb{R}^n$ on which it is defined. We have $I'(X,0)\subset\Gamma\subset\mathbb{R}^n\times\mathbb{R}$, so $\pi|_{I'(X,0)}\colon I'(X,0)\to I(X,0)$ is bilipschitz for the outer metrics $d_{I'}$ and d_I. $\square$

2. Inner Lipschitz geometry of complex curve singularities

We start with an example.

Let $X\subset\mathbb{C}^2$ be the complex cusp with equation $y^2-x^3=0$. Let $t\in\mathbb{R}$ and consider the two points $p_1(t)=(t,t^{3/2})$ and $p_2(t)=(t,-t^{3/2})$ on X. Since these two points are on two distinct strands of the braid $X\cap(S^1_{|t|}\times\mathbb{C})$, it is easy to see that the shortest path in X from $p_1(t)$ to $p_2(t)$ passes through the origin and that $d_i(p_1(t),p_2(t))=\Theta(t)$. This suggests that $(X,0)$ is locally inner Lipschitz homeomorphic to the cone over its link. This means that the inner Lipschitz geometry tells one no more than the topological type, i.e., the number of connected components of the link (which are circles), and is therefore uninteresting. We will prove that this is the same for any complex curve.

Definition 2.1. An analytic germ $(X,0)$ is *metrically conical* if it is inner Lipschitz homeomorphic to the straight cone over its link.

Proposition 2.2. *Any complex space curve germ $(C,0)\subset(\mathbb{C}^N,0)$ is metrically conical.*

Proof. Take a linear projection $p\colon \mathbb{C}^N\to\mathbb{C}$ which is generic for the curve $(C,0)$ (*i.e.*, its kernel contains no tangent line of C at 0) and let $\pi := p|_C$, which is a branched cover of germs. Let $D_\epsilon=\{z\in\mathbb{C} : |z|\le\epsilon\}$ with ϵ small, and let C_ϵ be the part of C which branched covers D_ϵ. Since π is holomorphic away from 0 we have a local Lipschitz constant $K(x)$ at each point $x\in C\smallsetminus\{0\}$ given by absolute value of the derivative map of π at x. On each branch γ of C this $K(x)$ extends continuously over 0 by taking for $K(0)$ the absolute value of the restriction $p\,|_{T_0\gamma}\colon T_0\gamma\to\mathbb{C}$ where T_0X denotes the tangent cone to γ at 0. So the infimum and supremum K^- and K^+ of $K(x)$ on $C_\epsilon\smallsetminus\{0\}$ are defined and positive. For any arc γ in C_ϵ which is smooth except where it passes through 0 we have $K^-\ell(\gamma)\le\ell'(\gamma)\le K^+\ell(\gamma)$, where ℓ respectively ℓ' represent arc length using inner metric on $\mathbb{C}_\epsilon$ respectively the metric lifted from B_ϵ. Since C_ϵ with the latter metric is strictly conical, we are done. □

3. Outer Lipschitz classification of complex curve germs

3.1. Reduction to the plane curve case

Let $\mathbf{G}(n-2,\mathbb{C}^n)$ be the Grassmanian of $(n-2)$-planes in $\mathbb{C}^n$.

Let $\mathcal{D}\in\mathbf{G}(n-2,\mathbb{C}^n)$ and let $\ell_{\mathcal{D}}\colon\mathbb{C}^n\to\mathbb{C}^2$ be the linear projection with kernel $\mathcal{D}$. Suppose $(C,0)\subset(\mathbb{C}^n,0)$ is a complex curve germ. There exists an open dense subset Ω_C of $\mathbf{G}(n-2,\mathbb{C}^n)$ such that for $\mathcal{D}\in\Omega_C$, $\mathcal{D}$ contains no limit of secant lines to the curve C ([23]).

Definition 3.1. The projection $\ell_{\mathcal{D}}$ is said to be *generic for C* if $\mathcal{D}\in\Omega_C$.

In the sequel, we will use extensively the following result

Theorem 3.2 ([23, pp. 352-354])**.** *If $\ell_{\mathcal{D}}$ is a generic projection for C, then the restriction $\ell_{\mathcal{D}}|_C\colon C\to\ell_{\mathcal{D}}(C)$ is a bilipschitz homeomorphism for the outer metric.*

As a consequence of Theorem 3.2, in order to understand Lipschitz geometry of curves germs, it suffices to understand Lipschitz geometry of plane curves germs.

3.2. Lipschitz classification Theorem for the outer metric

We start again with an example.

Example 3.3. Consider the plane curve germ $(C,0)$ with two branches C_1 and C_2 having Puiseux expansions

$$C_1:\quad y=x^{3/2}+x^{13/6},\quad C_2:\quad y=x^{5/2}\,.$$

Its topological type is completely described by the sets of characteristic exponents of the branches: $\{3/2,13/6\}$ and $\{5/2\}$ and by the contact exponents between the two branches: 3/2. This data is summarized in the Eggers tree of the curve germ (see [24]), or equivalently, in what we will call the *carrousel tree* (Figure 3.1), which is exactly the Kuo-Lu tree defined in [13] but with the horizontal bars contracted to points.

Now, for small $t\in\mathbb{R}^+$, consider the intersection $C\cap\{x=t\}$. This gives 8 points $p_i(t), i=1\ldots,8$ and then, varying t, this gives 8 real semi-analytic arcs $p_i\colon[0,1)\to X$ such that $p_i(0)=0$ and $\|p_i(t)\|=\Theta(t)$.

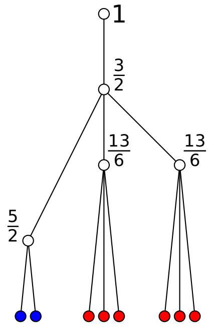


Figure 3.1: The carrousel tree

Figure 3.2 gives pictures of sections of C with complex lines $x = 0.1$, 0.05, 0.025 and 0. The central blue two-points set corresponds to the branch $y = x^{5/2}$ while the two lateral red three-points sets correspond to the other branch.

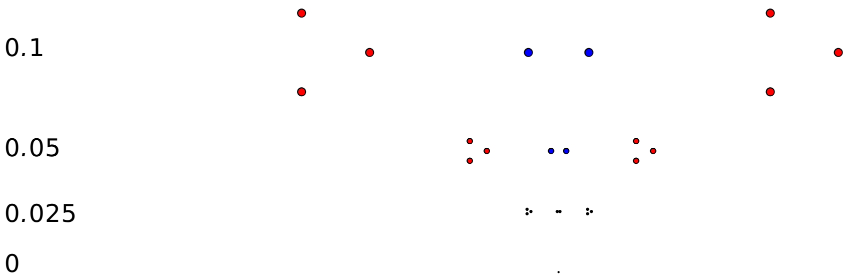


Figure 3.2: Sections of C

It is easy to see on this example that for each pair i, j with $i \neq j$, we have $d_o(p_i(t), p_j(t)) = \Theta(t^{q(i,j)})$ where $q(i,j) \in \mathbb{Q}^+$ and that the set of such $q(i,j)$'s is exactly the set of essential exponents $\{3/2, 13/6, 5/2\}$. This shows that one can recover the essential exponents by measuring the outer distance between points of C.

More generally, we will show that we can actually recover the carrousel tree by measuring outer distances on X even after a bilipschitz change of the metric. Conversely, the outer Lipschitz geometry of a plane curve is determined by its embedded topological type. This gives the complete classification of the outer geometry of complex plane curve germs:

Theorem 3.4. *Let $(C_1, 0) \subset (\mathbb{C}^2, 0)$ and $(C_2, 0) \subset (\mathbb{C}^2, 0)$ be two germs of complex curves. The following are equivalent:*

1. *$(C_1, 0)$ and $(C_2, 0)$ have same outer Lipschitz geometry.*
2. *there is a meromorphic germ $\phi\colon (C_1, 0) \to (C_2, 0)$ which is a bilipschitz homeomorphism for the outer metric;*
3. *$(C_1, 0)$ and $(C_2, 0)$ have the same embedded topological type;*
4. *there is a bilipschitz homeomorphism of germs $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with $h(C_1) = C_2$.*

As a corollary of Theorem 3.2 and Theorem 3.4, we obtain:

Corollary 3.5. *The outer Lipschitz geometry of a curve germ $(C, 0) \subset (\mathbb{C}^N, 0)$ determines and is determined by the embedded topological type of any generic linear projection $(\ell(C), 0) \subset (\mathbb{C}^2, 0)$.*

The equivalence of (1), (3) and (4) of Theorem 3.4 is proved in [17]. The equivalence of (2) and (3) was first proved by Pham and Teissier ([21] or its english transation [22, 10] by developing the theory of Lipschitz saturation and revisited by Fernandes in [8]. In the present lecture notes, we will give the proof of (1) ⇒ (3), since it is based on the so-called *bubble trick* argument which can be considered as a prototype for exploring Lipschitz geometry of singular spaces in various settings. In the next section, we give an alternative version that we call *bubble trick with jumps* which is easier to adapt to more sophisticated situations. For example, bubble tricks with jumps are used in [18] to study outer Lipschitz geometry of surface germs and in [11, 12] which introduce adapted theories of homology and homotopy , called moderately discontinuous homology and homotopy, providing powerful new Lipschitz invariants.

Proof of (1) ⇒ (3) *of Theorem 3.4.* We want to prove that the embedded topological type of a plane curve germ $(C,0) \subset (\mathbb{C}^2,0)$ is determined by the outer Lipschitz geometry of $(C,0)$.

We first prove this using the analytic structure and the outer metric on $(C,0)$. The proof is close to Fernandes' approach in [8]. We then modify the proof to make it purely topological and to allow a bilipschitz change of the metric.

The tangent cone to C at 0 is a union of lines $L^{(j)}$, $j = 1,\dots,m$, and by choosing our coordinates we can assume they are all transverse to the y-axis.

There is $\epsilon_0 > 0$ such that for any $\epsilon \le \epsilon_0$ the curve C meets transversely the set

$$T_\epsilon := \{(x,y) \in \mathbb{C}^2 : |x| = \epsilon\}\,.$$

Let M be the multiplicity of C. The lines $x = t$ for $t \in (0,\epsilon_0]$ intersect C in M points $p_1(t),\dots,p_M(t)$ which depend continuously on t. Denote by $[M]$ the set $\{1,2,\dots,M\}$. For each $j,k \in [M]$ with $j<k$, the distance $d(p_j(t),p_k(t))$ has the form $O(t^{q(j,k)})$, where $q(j,k) = q(k,j)$ is either a characteristic Puiseux exponent for a branch of the plane curve C or a coincidence exponent between two branches of C in the sense of e.g., [14]. We call such exponents *essential*. For $j \in [M]$ define $q(j,j) = \infty$.

Lemma 3.6. *The map $q\colon [M]\times[M] \to \mathbb{Q}\cup\{\infty\}$, $(j,k)\mapsto q(j,k)$, determines the embedded topology of C.*

Proof. To prove the lemma we will construct from q the so-called carrousel tree, which is a tree carrying equivalent data as the Eggers tree, so it determines the embedded topology of C.

The $q(j,k)$ have the property that $q(j,l) \ge min(q(j,k),q(k,l))$ for any triple j,k,l. So for any $q \in \mathbb{Q}\cup\{\infty\}, q>0$, the relation on the set $[M]$ given by $j \sim_q k \Leftrightarrow q(j,k) \ge q$ is an equivalence relation.

Name the elements of the set $q([M]\times[M])\cup\{1\}$ in decreasing order of size: $\infty = q_0 > q_1 > q_2 > \cdots > q_s = 1$. For each $i = 0,\dots,s$ let $G_{i,1},\dots,G_{i,M_i}$ be the equivalence classes for the relation $\sim_{q_i}$. So $M_0 = M$ and the sets $G_{0,j}$ are singletons while $M_s = 1$ and $G_{s,1} = [M]$. We form a tree with these equivalence classes $G_{i,j}$ as vertices, and edges given by inclusion relations: the singleton sets $G_{0,j}$ are the leaves and there is an edge between $G_{i,j}$ and $G_{i+1,k}$ if $G_{i,j} \subseteq G_{i+1,k}$. The vertex $G_{s,1}$ is the root of this tree. We weight each vertex with its corresponding q_i.

The *carrousel tree* is the tree obtained from this tree by suppressing valence 2 vertices (i.e., vertices with exactly two incident edges): we remove each such vertex and amalgamate its two adjacent edges into one edge. We follow the computer science convention of drawing the tree with its root vertex at the top, descending to its leaves at the bottom.

At any non-leaf vertex v of the carrousel tree we have a weight q_v, $1 \le q_v \le q_1$, which is one of the q_i's. We write it as m_v/n_v, where n_v is the lcm of the denominators of the q-weights at the vertices on the path from v up to the root vertex. If v' is the adjacent vertex above v along this path, we put $r_v = n_v/n_{v'}$ and $s_v = n_v(q_v - q_{v'})$. At each vertex v the subtrees cut off below v consist of groups of r_v isomorphic trees, with possibly one additional tree. We label the top of the edge connecting to this additional tree at v, if it exists, with the number r_v, and then delete all but one from each group of r_v isomorphic trees below v. We do this for each non-leaf vertex of the carrousel tree. The resulting tree, with the q_v labels at vertices and the extra label on a downward edge at some vertices is easily recognized as a mild modification of the Eggers tree: there is a natural action of the Galois group whose quotient is the Eggers-tree. □

As already noted, this reconstruction of the embedded topology involved the complex structure and the outer metric. We must show that we can reconstruct it without using the complex structure, even after applying a bilipschitz change to the outer metric. We will use what we call a *bubble trick*.

Recall that the tangent cone of C is a union of lines $L^{(j)}$. We denote by $C^{(j)}$ the part of C tangent to the line $L^{(j)}$. It suffices to recover the topology of each $C^{(j)}$ independently, since the $C^{(j)}$'s are distinguished by the fact that the distance between any two of them outside a ball of radius ϵ around 0 is $\Theta(\epsilon)$, even after bilipschitz change of the metric. We therefore assume from now on that the tangent cone of C is a single complex line.

We now arrive at a crucial moment of the proof:

The bubble trick. The points $p_1(t), \dots, p_M(t)$ we used to find the numbers $q(j,k)$ were obtained by intersecting C with the line $x = t$. The arc $p_1(t)$, $t \in [0, \epsilon_0]$ satisfies $d(0, p_1(t)) = \Theta(t)$. Moreover, the other points $p_2(t), \dots, p_M(t)$ are in the transverse disk of radius rt centered at $p_1(t)$ in the plane $x = t$. Here r can be as small as we like, so long as ϵ_0 is then chosen sufficiently small.

Instead of a transverse disk of radius rt, we can use a ball $B(p_1(t), rt)$ of radius rt centered at $p_1(t)$. This $B(p_1(t), rt)$ intersects C in M disks $D_1(t), \dots, D_M(t)$, and we have $d(D_j(t), D_k(t)) = \Theta(t^{q(j,k)})$, so we still recover the numbers $q(j,k)$. In fact, the ball in the outer metric on C of radius rt around $p_1(t)$ is $B_C(p_1(t), rt) := C \cap B(p_1(t), rt)$, which consists of these M disks $D_1(t), \dots, D_M(t)$.

We now replace the arc $p_1(t)$ by any continuous arc $p'_1(t)$ on C with the property that $d(0, p'_1(t)) = \Theta(t)$. If r is sufficiently small it is still true that $B_C(p'_1(t), rt)$ consists of M disks $D'_1(t), \dots, D'_M(t)$ with $d\big(D'_j(t), D'_k(t)\big) = \Theta(t^{q(j,k)})$. So at this point, we have gotten rid of the dependence on analytic structure in discovering the topology, but not yet dependence on the outer geometry.

Let now d' be a metric on C such that the identity map is a K-bilipschitz homeomorphism in a neighbourhood of the origin. We work inside this neighbourhood, taking t, ϵ_0 and r sufficiently small. $B'(p, \eta)$ will denote the ball in C for the metric d' centered at $p \in C$ with radius $\eta \geq 0$.

The bilipschitz change of the metric may disintegrate the balls in many connected components, as sketched on Figure 3.3. So if we try to perform the same argument as before using the balls $B'(p'_1(t), rt)$ instead of $B_C(p'_1(t), rt)$, we get a problem since $B'(p'_1(t), rt)$ may have many irrelevant components and we can no longer simply use distance between components. To resolve this, we consider the two balls $B'_1(t) = B'(p'_1(t), \frac{rt}{K^3})$ and $B'_2(t) = B'(p'_1(t), \frac{rt}{K})$, so we have the inclusions:

$$B_C\big(p'_1(t), \frac{rt}{K^4}\big) \;\subset\; B'_1(t) \;\subset\; B_C\big(p'_1(t), \frac{rt}{K^2}\big) \;\subset\; B'_2(t) \;\subset\; B_C\big(p'_1(t), rt\big)$$

Using these inclusions, we obtain that only M components of $B'_1(t)$ intersect $B'_2(t)$ and that naming these components $D'_1(t), \dots, D'_M(t)$ again, we still have $d(D'_j(t), D'_k(t)) = \Theta(t^{q(j,k)})$ so the $q(j,k)$ are determined as before (prove this as an exercise). See Figure 3.3 for a schematic picture of the situation (for clarity of the picture, we draw the balls $B'_1(t)$ and $B'_2(t)$ as if the distance d' were induced by an ambient metric, but it is not the case in general). □

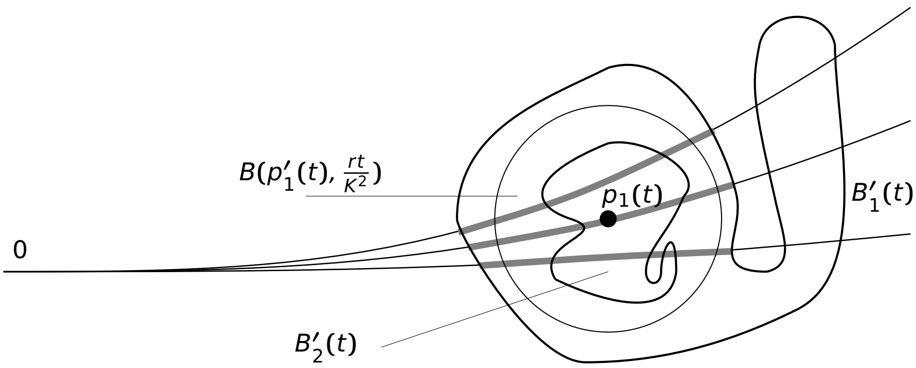


Figure 3.3:

4. The bubble trick with jumps

As we have seen in the above proof, the “bubble trick" is a powerful tool to capture invariants of Lipschitz geometry such as a number of connected components and distance order between them. However, this first version is not well adapted to explore the outer Lipschitz geometry of singular spaces of dimensions ≥ 2 for the following reason. In the case of a plane curve, the method used in the latter proof consisted in considering a projection $\ell\colon (C,0) \to (\mathbb{C},0)$ and to prove that the orders of distances between points of $\ell^{-1}(t) \cap C$ with respect to t are Lipschitz invariants. Now, assume that $(X,0)$ is a complex surface germ with multiplicity $m \geq 2$ (so it has a singularity at 0), and consider a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$. Then the critical locus ℓ is a curve germ $\Pi_\ell, 0) \subset (X,0)$ called the polar curve, and its image $\Delta = \ell(\Pi)$ is a curve germ $(\Delta,0) \subset (\mathbb{C}^2,0)$ called the *discriminant curve* of ℓ. The number of points in $\ell^{-1}(t) \cap C$ depends on $t \in \mathbb{C} \setminus \{0\}$: it equals $m-1$, where the multiplicity of $(X,0)$ if $t \in \Pi_\ell$ and m otherwise. Moreover, considering a continuous real arc $p(t)$ on $(\mathbb{C}^2,0)$, the distance order between the points $p_1(t),\ldots,p_m(t)$ will depend on the position of $p(t)$ with respect to the discriminant curve $\Delta = \ell(\Pi)$ of ℓ. So the situation is much more complicated, even in dimension 2.

In [18], the authors used an adapted version of the bubble trick which enables them to explore the outer Lipschitz geometry of a complex surface $(X,0)$. Roughly speaking, it consists in using horns

$$\mathcal{H}(p(t), r|t|^q) = \bigcup_{t\in[0,1)} B((p(t), a|t|^q)$$

where $\tilde{p}(t)$ is a real arc on $(X,0)$ such that $||p(t)|| = \Theta(t)$ and $r \in]0,+\infty[$ and in exploring “jumps" in the topology of $\mathcal{H}(p(t), a|t|^q)$ when q varies from $+\infty$ to 1, for example, jumps of the number of connected components.

In order to give a flavour of this bubble trick with jumps, we will perform it on a plane curve germ, giving an alternative proof of (1) ⇒ (3) of Theorem 3.4.

The bubble trick with jumps.

We use again the notations of the version of the bubble trick presented in the previous section. Let $(C,0)$ be a plane curve germ with s branches $C_1,\ldots,C_s$ and let $p_1'(t)$ be a continuous arc on C_1 with the property that $d(0,p_1'(t)) = \Theta(t)$. Let us order the numbers $q(1,k), k=2,\ldots,M$ in decreasing order:

$$1 \leq q(1,M) < q(1,M-1) < \cdots < q(1,2) < q(1,1) = \infty.$$

Let us consider the horns $\mathcal{H}_{q,r,} = \mathcal{H}(p_1'(t), r|t|^q)$ with $q \in [1,+\infty[$.

For $q >> 1$ and small $\epsilon > 0$, the number of connected components of $B(0,\epsilon) \cap \big(\mathcal{H}_{q,r} \setminus \{0\}\big)$ equals 1. Now, let us decrease q. For every small $\eta > 0$, the number of connected components of $\mathcal{H}_{q_{1,2}+\eta} \setminus \{0\}$ equals 1, while the number of connected components of $\mathcal{H}_{q_{1,2}-\eta} \setminus \{0\}$ is > 2.

Decreasing q, we have a jump in the number of connected components exactly when passing one of the rational numbers $q(1,k)$. So this enables one to recover all the characteristic exponents of C_1 and its contact exponents with the other branches of C. We can do the same for a real arc $p'_i(t)$ in each branch C_i of $(C,0)$ and this which will recover the integers $q(i,k)$ for $k = 1,\dots,M$. We then reconstruct the function $q\colon [M] \times [M] \to \mathbb{Q}_{\geq 1}$ which characterizes the embedded topology of $(C,0)$, or equivalently the carrousel tree of $(C,0)$.

Moreover, the same jumps appear when using horns

$$\mathcal{H}'(p'(t), r|t|^q) = \bigcup_{t\in[0,1)} B'((p'(t), r|t|^q),$$

where B' means balls with respect to a metric d' which is bilipschitz equivalent to the initial outer metric. Indeed, if K is the bilipschitz constant of such a biblipschitz change, then we have the inclusions

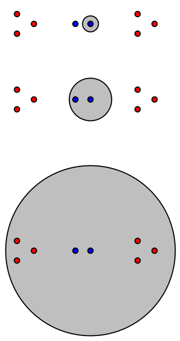
$$\mathcal{H}\big(p'(t), \frac{rt}{K^4}\big) \subset \mathcal{H}'\big(p'(t), \frac{rt}{K^3}\big) \subset \mathcal{H}\big(p'(t), \frac{rt}{K^2}\big) \subset \mathcal{H}'\big(p'(t), \frac{rt}{K^3}\big) \subset \mathcal{H}\big(p'(t), rt\big),$$

the same argument as in the version 1 of the bubble trick shows that for q fixed and different from $q(1,k), k = 2,\dots,M$, the numbers of connected components of $B(0,\epsilon)\cap\big(\mathcal{H}_{q,r} \setminus \{0\}\big)$ and $B(0,\epsilon)\cap\big(\mathcal{H}'_{q,r} \setminus \{0\}\big)$ are equal.

Example 4.1. Consider again the plane curve with two branches of Example 3.3 given by the Puiseux expansions:

$$C_1: \ \ y = x^{3/2} + x^{13/6}, \quad C_2: \ \ y = x^{5/2}\,.$$

Consider first an arc $p'_1(t)$ arc inside C_1 parametrized by $x = t \in [0,1)$. Then $p(t)$ corresponds to one of the two extremities of the carrousel tree of Figure 3.1 whose neighbour vertex is weighted by 5/2. Figure 4.1 represents the intersection of the horn $\mathcal{H}_{q,r,} = \mathcal{H}(p(t), r|t|^q)$ with $q \in [1,+\infty[$ with $\{x = t\}$ for $t \in \mathbb{C}\setminus\{0\}$ with $|t|$ sufficiently small for different values of q. This shows two jumps: a first jump at $q = 5/2$, which says that 5/2 is a characteristic exponent of a branch since $p'(t)$ and the new point appearing in the intersection belong to the same connected component C_1 of $C \setminus \{0\}$ (they are both blue!), while the second jump at 3/2 says that 3/2 is the contact exponent of C_1 with the other component since the new points appearing at $q = 3/2-\eta$ belong to C_2.



$q > 5/2$

$3/2 < q < 5/2$

$1 < q < 3/2$

Figure 4.1: Sections of C

This first exploration enables one to construct the left part of the carrousel tree of C shown on Figure 4.2, i.e., the one corresponding to the carrousel tree of C_1.

To complete the picture, we now consider an arc $p'_2(t)$ inside C_2 corresponding to a component of $C_2 \cap \{x = t\}$. This means that $p'_2(t)$ corresponds to one of the 6 extremities of the

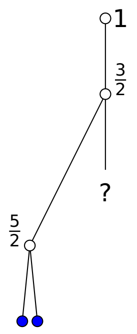


Figure 4.2:

carrousel tree of Figure 3.1 whose neighbour vertex is weighted by 13/6. Figure 4.3 represents the jumps for the horns $\mathcal{H}_{q,r,}$ centered on $p_2'(t)$. This shows two jumps: a first jump at $q = 13/6$, which says that 13/6 is a characteristic exponent of C_2, then a second jump at 3/2 corresponding to the contact exponent of C_1 and C_2.

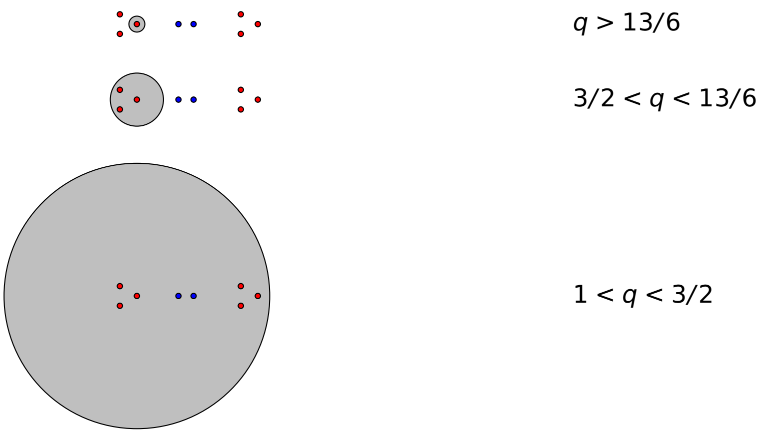


Figure 4.3: Sections of C

This exploration of C_2 enables one to construct the right part of the carrousel tree of C shown on Figure 4.4, i.e., the one corresponding to the carrousel tree of C_2.

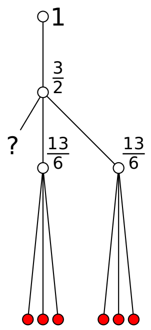


Figure 4.4:

Merging the two above partial carrousel trees, we obtain the carrousel tree of Figure 3.1, recovering the embedded topology of $(C,0)$.

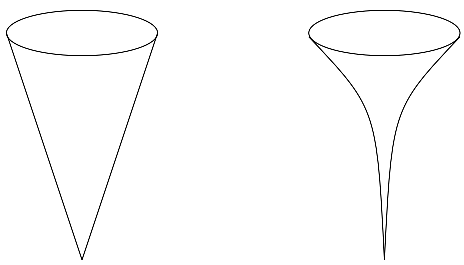
5. The thick-thin decomposition of a normal surface singularity

We know that any curve germ $(C,0) \subset (\mathbb{C}^N,0)$ is metrically conical for the inner geometry (Proposition 2.2). This is no longer true in higher dimensions. The first example of a non-metrically-conical (X,0) appeared in [1]: for $k \geq 2$, the singularity $A_k : x^2+y^2+z^{k+1}=0$ is not metrically conical for the inner metric. The examples in [2, 3, 4] then suggested that failure of metric conicalness is common. For example, among ADE singularities of surfaces, only A_1 and D_4 are metrically conical. In [4] it is also shown that the inner Lipschitz geometry of a singularity may not be determined by its topological type.

A complete classification of the Lipschitz inner geometry of normal complex surfaces is presented in [6]. It is built on the existence of the so-called thick-thin decomposition of the surface into two semi-algebraic sets.

The obstruction to the metric conicalness of a germ $(X,0)$ is the existence of *fast loops*. Let p and q be two pairwise coprime positive integers such that $p \geq q$. Set $\beta = \frac{p}{q}$. The prototype of a fast loop is the *β-horn*.

$$\mathcal{H}_\beta = \{(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}^+ : (x^2+y^2)^q = (z^2)^p\}.$$



$\beta = 1$ $\qquad$ $\beta > 1$

Figure 5.1: The β-horns $\mathcal{H}_\beta$

Exercise 5.1. *Show that $\mathcal{H}_\beta$ is inner bilipschitz homeomorphic to $\mathcal{H}_{\beta'}$ if and only if $\beta = \beta'$.*

$\mathcal{H}_1$ is a straight cone, so it is metrically conical. As a consequence of Exercise 5.1, we obtain that for $\beta > 1$, $\mathcal{H}_\beta$ is not metrically conical. For $t > 0$, set $\gamma_t = \mathcal{H}_\beta \cap \{z = t\}$. When $\beta > 1$, the family of curves $(\gamma_t)_{t>0}$ is a *fast loop* inside $\mathcal{H}_\beta$. More generally:

Definition 5.2. Let $(X,0) \subset (\mathbb{C}^n,0)$ be a semianalytic germ. A *fast loop* in $(X,0)$ is a continuous family of loops $\{\gamma_\epsilon : S^1 \to X^{(\epsilon)}\}_{0<\epsilon\leq\epsilon_0}$ such that:

1. γ_ϵ is essential (i.e., homotopically non trivial) in the link $X^{(\epsilon)} = X \cap S_\epsilon$;

2. there exists $q > 1$ such that
$$\lim_{\epsilon \to 0} \frac{\text{length}(\gamma_\epsilon)}{\epsilon^q} = 0.$$

In the next section, we will define what we call the thick-thin decomposition of a normal surface germ $(X,0)$. It consists in decomposing $(X,0)$ as a union of two semi-algebraic sets $(X,0) = (Y,0)\bigcup(Z,0)$ where $(Z,0)$ is *thin* (Definition 6.2) and where $(Y,0)$ is *thick* (Definition 6.7). The thin part $(Z,0)$ will concentrate all the fast loops of $(X,0)$ inside a Milnor ball with radius ϵ_0. The thick part $(Y,0)$ is the closure of the complement of the thin part and has the property that it contains a maximal metrically conical set. This enables one to characterize the germs $(X,0)$ which are metrically conical:

Theorem 5.3. [6, Theorem 7.5, Corollary 1.8] *Let $(X,0)$ be a normal complex surface and let*

$$(X,0) = (X_{thick},0)\bigcup(X_{thin},0)$$

be its thick-thin decomposition.

Then $(X,0)$ is metrically conical if and only if $X_{thin} = \emptyset$, so $(X,0) = (X_{thick},0)$.

6. Thick-thin decomposition

Definition 6.1. Let $(Z,0) \subset (\mathbb{R}^n,0)$ be a semi-algebraic germ. The *tangent cone* of $(Z,0)$ is the set T_0Z of vectors $v \in \mathbb{R}^n$ such that there exist a sequence of points (x_k) in $Z \setminus \{0\}$ tending to 0 and a sequence of positive real numbers (t_k) such that

$$\lim_{k\to\infty} \frac{1}{t_k} x_k = v.$$

Definition 6.2. A semi-algebraic germ $(Z,0) \subset (\mathbb{R}^n,0)$ of pure dimension is *thin* if the dimension of its tangent cone T_0X at 0 satisfies $\dim(T_0Z) < \dim(Z)$.

Example 6.3. For every $\beta > 1$, the β-horn $\mathcal{H}_\beta$ is thin since $\dim(\mathcal{H}_\beta) = 2$ while $T_0\mathcal{H}_\beta$ is a half-line. On the other hand, $\mathcal{H}_1$ is not thin.

Example 6.4. Let $\lambda \in \mathbb{C}^*$ and denote by $\mathcal{C}_\lambda$ the plane curve with Puiseux parametrization $y = \lambda x^{5/3}$. Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Consider the semi-algebraic germ $(Z,0) \subset (\mathbb{C}^2,0)$ defined by $Z = \bigcup_{a\le|\lambda|\le b} \mathcal{C}_\lambda$. The tangent cone T_0Z is the complex line $y = 0$, while Z is 4-dimensional, so $(Z,0)$ is thin.

Let $Z^{(\epsilon)}$ be the intersection of Z with the boundary of the polydisc $\{|x| \le \epsilon\} \times \{|y| \le \epsilon\}$ (By [7], one obtains, up to homeomorphism (or diffeomorphism in a stratified sense), the same link $Z^{(\epsilon)}$ as when intersecting with a round sphere). When $\epsilon > 0$ is small enough, $Z^{(\epsilon)} \subset \{|x| = \epsilon\} \times \{|y| \le \epsilon\}$ and the projection $Z^{(\epsilon)} \to S^1_\epsilon$ defined by $(x,y) \to x$ is a locally trivial fibration whose fibers are the flat annuli $A_t = Z \cap \{x = t\}$, $|t| = \epsilon$, and the lengths of the boundary components of A_t are $\Theta(\epsilon^{5/3})$.

Notice that Z can be described through a resolution as follows. Let $\sigma\colon Y \to \mathbb{C}^2$ be the minimal embedded resolution of the curve $C_1\colon y = x^{5/3}$. It consists of four successive blow-ups of points. Denote $E_1,\dots,E_4$ the corresponding components of the exceptional divisor $\sigma^{-1}(0)$ indexed by their order of occurence. Then σ is a simultaneous resolution of the curves $\mathcal{C}_\lambda$. Therefore, the strict transform of Z by σ is a neighbourhood of E_4 minus neighbourhoods of the intersecting points $E_4 \cap E_2$ and $E_4 \cap E_3$ as pictured in Figure 6.1. The tree T on the left is the dual tree of σ. Its vertices are weighted by the self-intersections E_i^2 and the arrow represents the strict transform of $\mathcal{C}_1$.

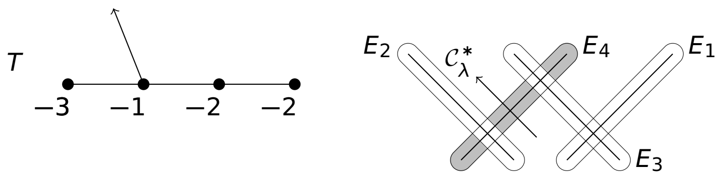


Figure 6.1: The strict transform of Z by σ

Definition 6.5. Let $1 < q \in \mathbb{Q}$. A *q-horn neighbourhood* of a semi-algebraic germ $(A,0) \subset (\mathbb{R}^N,0)$ is a set of the form $\{x \in \mathbb{R}^n \cap B_\epsilon : d(x,A) \le c|x|^q\}$ for some $c > 0$, where d denotes the Euclidean metric.

The following proposition helps picture “thinness”

Proposition 6.6. [6, Proposition 1.3] *Any thin semi-algebraic germ* $(Z,0) \subset (\mathbb{R}^N,0)$ *is contained in some q-horn neighbourhood of its tangent cone* T_0Z.

We will now define thick semi-algebraic sets. The definition is built on the following observation. Let $(X,0) \subset (\mathbb{C}^n,0)$ be a complex surface germ; we would like to decompose $(X,0)$ into two semialgebraic sets $(A,0)$ and $(B,0)$ glued along their boundary germs, where $(A,0)$ is thin and $(B,0)$ is metrically conical. But try to glue a thin germ $(A,0)$ with a metrically conical germ $(B,0)$ so that they intersect only along their boundary germs.... It is not possible! There would be a hole between them (see Figure 6.2). So we have to replace $(B,0)$ by something else than conical.

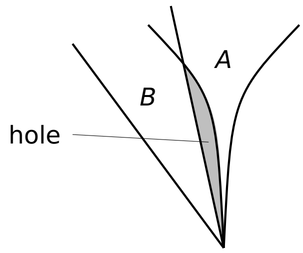


Figure 6.2: Trying to glue a thin germ with a metrically conical germ

"Thick" is a generalization of "metrically conical." Roughly speaking, a thick algebraic set is obtained by slightly inflating a metrically conical set in order that it can interface along its boundary with thin parts. The precise definition is as follows:

Definition 6.7 (Thick)**.** Let $B_\epsilon \subset \mathbb{R}^N$ denote the ball of radius ϵ centered at the origin, and S_ϵ its boundary. A semi-algebraic germ $(Y,0) \subset (\mathbb{R}^N,0)$ is *thick* if there exists $\epsilon_0 > 0$ and $K \geq 1$ such that $Y \cap B_{\epsilon_0}$ is the union of subsets Y_ϵ, $\epsilon \leq \epsilon_0$ which are metrically conical with bilipschitz constant K and satisfy the following properties (see Fig. 1.1):

1. $Y_\epsilon \subset B_\epsilon$, $Y_\epsilon \cap S_\epsilon = Y \cap S_\epsilon$ and Y_ϵ is metrically conical;

2. For $\epsilon_1 < \epsilon_2$ we have $Y_{\epsilon_2} \cap B_{\epsilon_1} \subset Y_{\epsilon_1}$ and this embedding respects the conical structures. Moreover, the difference $(Y_{\epsilon_1} \cap S_{\epsilon_1}) \setminus (Y_{\epsilon_2} \cap S_{\epsilon_1})$ of the links of these cones is homeomorphic to $\partial(Y_{\epsilon_1} \cap S_{\epsilon_1}) \times [0,1)$.

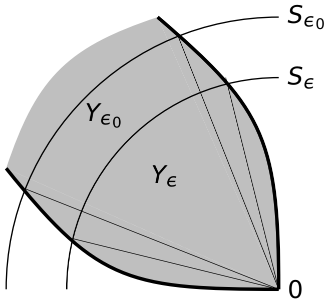


Figure 6.3: Thick germ

Clearly, a semi-algebraic germ cannot be both thick and thin.

Example 6.8. The set $Z = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^3\}$ gives a thin germ at 0 since it is a 3-dimensional germ whose tangent cone is half the z-axis. The intersection $Z \cap B_\epsilon$ is contained in a closed 3/2-horn neighbourhood of the z-axis. The complement in $\mathbb{R}^3$ of this thin set is thick.

Example 6.9. Consider again the thin germ $(Z,0) \subset (\mathbb{C}^2,0)$ of Example 6.4. Fix $\eta > 0$. In $\mathbb{C}^2$, consider the conical set W defined as the union of the complex lines $y = \alpha x$ for $|\alpha| \geq \eta$. Then the germ $(Y,0)$ defined by $Y = \overline{\mathbb{C}^2 \setminus Z}$ is a thick germ which is an inflation of W. Notice that the strict transform of Y by the resolution σ introduced in Example 6.4 is a neighbourhood of the union of curves $E_1 \cup E_3$.

For any semi-algebraic germ $(A,0)$ of $(\mathbb{R}^N,0)$, we write $A^{(\epsilon)} := A \cap S_\epsilon \subset S_\epsilon$. When ϵ is sufficiently small, $A^{(\epsilon)}$ is the ϵ-link of $(A,0)$.

Definition 6.10 (Thick-thin decomposition)**.** A *thick-thin decomposition* of the normal complex surface germ $(X,0)$ is a decomposition of it as a union of semi-algebraic germs of pure dimension 4:

$$(X,0) = \bigcup_{i=1}^{r} (Y_i,0) \cup \bigcup_{j=1}^{s} (Z_j,0), \tag{6.1}$$

such that the $Y_i \setminus \{0\}$ and $Z_j \setminus \{0\}$ are connected and:

1. Each Y_i is thick and each Z_j is thin;
2. The $Y_i \setminus \{0\}$ are pairwise disjoint and the $Z_j \setminus \{0\}$ are pairwise disjoint;
3. If ϵ_0 is chosen small enough that S_ϵ is transverse to each of the germs $(Y_i,0)$ and $(Z_j,0)$ for $\epsilon \leq \epsilon_0$, then $X^{(\epsilon)} = \bigcup_{i=1}^r Y_i^{(\epsilon)} \cup \bigcup_{j=1}^s Z_j^{(\epsilon)}$ decomposes the 3-manifold $X^{(\epsilon)} \subset S_\epsilon$ into connected submanifolds with boundary, glued along their boundary components.

Definition 6.11. A thick-thin decomposition is *minimal* if

1. the tangent cone of its thin part $\bigcup_{j=1}^s Z_j$ is contained in the tangent cone of the thin part of any other thick-thin decomposition and
2. the number s of its thin pieces is minimal among thick-thin decompositions satisfying (1).

The following theorem expresses the existence and uniqueness of a minimal thick-thin decomposition of a normal complex surface singularity.

Theorem 6.12. [6, Section 8] *Let* $(X,0)$ *be a normal complex surface germ. Then a minimal thick-thin decomposition of* $(X,0)$ *exists. For any two minimal thick-thin decompositions of* $(X,0)$*, there exists* $q > 1$ *and a homeomorphism of the germ* $(X,0)$ *to itself which takes the one decomposition to the other and moves each* $x \in X$ *by a distance at most* $\|x\|^q$*.*

The homeomorphism in the above theorem is not necessarily bilipschitz, but the bilipschitz classification described in the further Classification Theorem 6.16 leads to a "best" minimal thick-thin decomposition, which is unique up to bilipschitz homeomorphism.

It well known that the link of a normal surface singularity is a graph manifold in the sense of Waldhausen. Before stating the classification Theorem, let us state a result which shows that the minimal thick-thin decomposition of a normal surface germ induces a JSJ decomposition of the link whose pieces are graph manifolds:

Theorem 6.13. *(*[6, Theorem 1.7]*) A minimal thick-thin decomposition of* $(X,0)$ *as in equation* (6.1) *satisfies* $r \geq 1$, $s \geq 0$ *and has the following properties for* $0 < \epsilon \leq \epsilon_0$*:*

1. *Each thick zone* $Y_i^{(\epsilon)}$ *is a Seifert fibered manifold.*
2. *Each thin zone* $Z_j^{(\epsilon)}$ *is a graph manifold (union of Seifert manifolds glued along boundary components) and not a solid torus.*

3. *There exist constants $c_j > 0$ and $q_j > 1$ and fibrations $\zeta_j^{(\epsilon)}\colon Z_j^{(\epsilon)} \to S^1$ depending smoothly on $\epsilon \le \epsilon_0$ such that the fibers $\zeta_j^{-1}(t)$ have diameter at most $c_j\epsilon^{q_j}$ (we call these fibers the* Milnor fibers *of $Z_j^{(\epsilon)}$).*

In [6], the thick-thin decomposition is constructed by using a resolution of the singularity $(X,0)$. Another way of constructing it is as follows. A line L tangent to X at 0 is *exceptional* if the limit at 0 of tangent planes to X along a curve in X tangent to L at 0 depends on the choice of this curve (see [15]). Just finitely many tangent lines to X at 0 are exceptional. To obtain the thin part one intersects $X \setminus \{0\}$ with a q-horn disk-bundle neighborhood of each exceptional tangent line L for $q > 1$ sufficiently small and then discards any "trivial" components of these intersections (those whose closures are locally just cones on solid tori; such trivial components arise also in our resolution approach, and showing that they can be absorbed into the thick part takes some effort ([6, section 10]).

In [2] a *fast loop* is defined as a family of closed curves in the links $X^{(\epsilon)} := X \cap S_\epsilon$, $0 < \epsilon \le \epsilon_0$, depending continuously on ϵ, which are not homotopically trivial in $X^{(\epsilon)}$ but whose lengths are proportional to ϵ^k for some $k > 1$, and it is shown that fast loops are obstructions to metric conicalness.

Theorem 6.14. [6, 7.5] *Each thin piece Z_j contains fast loops. In fact, each boundary component of its Milnor fiber gives a fast loop.*

Corollary 6.15. *The following are equivalent, and each implies that the link of $(X,0)$ is Seifert fibered:*

1. *$(X,0)$ is metrically conical;*

2. *$(X,0)$ has no fast loops;*

3. *$(X,0)$ has no thin piece (so it consists of a single thick piece).*

Bilipschitz classification. Finally, we will state a complete classification of the inner geometry of $(X,0)$ up to bilipschitz equivalence, based on a refinement of the thick-thin decomposition. We describe this refinement in terms of the decomposition of the link $X^{(\epsilon)}$.

We first refine the decomposition $X^{(\epsilon)} = \bigcup_{i=1}^r Y_i^{(\epsilon)} \cup \bigcup_{j=1}^s Z_j^{(\epsilon)}$ by decomposing each thin zone $Z_j^{(\epsilon)}$ into its JSJ decomposition (minimal decomposition into Seifert fibered manifolds glued along their boundaries), while leaving the thick zones $Y_i^{(\epsilon)}$ as they are. We then thicken some of the gluing tori of this refined decomposition to collars $T^2 \times I$, to add some extra "annular" pieces (the choice where to do this is described in [6, Section 10]). At this point we have $X^{(\epsilon)}$ glued together from various Seifert fibered manifolds (in general not the minimal such decomposition).

Let Γ_0 be the decomposition graph for this, with a vertex for each piece and edge for each gluing torus, so we can write this decomposition as

$$X^{(\epsilon)} = \bigcup_{\nu \in V(\Gamma_0)} M_\nu^{(\epsilon)}, \tag{6.2}$$

where $V(\Gamma_0)$ is the vertex set of Γ_0.

Theorem 6.16 (Classification Theorem)**.** *The bilipschitz geometry of $(X,0)$ with respect to the inner metric determines and is uniquely determined by the following data:*

1. *The decomposition of $X^{(\epsilon)}$ into Seifert fibered manifolds as described above, refining the thick-thin decomposition;*

2. *for each thin zone $Z_j^{(\epsilon)}$, the homotopy class of the foliation by fibers of the fibration $\zeta_j^{(\epsilon)}\colon Z_j^{(\epsilon)} \to S^1$;*

3. *for each vertex $\nu \in V(\Gamma_0)$, a rational weight $q_\nu \geq 1$ with $q_\nu = 1$ if and only if $M_\nu^{(\epsilon)}$ is a $Y_i^{(\epsilon)}$ (i.e., a thick zone) and with $q_\nu \neq q_{\nu'}$ if ν and ν' are adjacent vertices.*

In item (2) we ask for the foliation by fibers rather than the fibration itself since we do not want to distinguish fibrations $\zeta \colon Z \to S^1$ which become equivalent after composing each with a covering maps $S^1 \to S^1$. Note that item (2) describes discrete data, since the foliation is determined up to homotopy by a primitive element of $H^1(Z_j^{(\epsilon)}; \mathbb{Z})$ up to sign.

References

[1] Lev Birbrair, Alexandre Fernandes, Inner metric geometry of complex algebraic surfaces with isolated singularities, Comm. Pure Appl. Math. 61 (2008), 1483–1494.

[2] Lev Birbrair, Alexandre Fernandez, Walter D. Neumann, Bi-Lipschitz geometry of weighted homogeneous surface singularities. Math. Ann. 342 (2008), 139–144.

[3] Lev Birbrair, Alexandre Fernandez, Walter D. Neumann, Bi-Lipschitz geometry of complex surface singularities, Geometriae Dedicata 129 (2009), 259–267.

[4] Lev Birbrair, A. Fernandez, Walter D. Neumann, Separating sets, metric tangent cone and applications for complex algebraic germs, Selecta Math. 16 (2010), 377-391.

[5] Lev Birbrair, Alexandre Fernandes, Jose Edson Sampaio, Misha Verbitsky, Multiplicity of singularities is not a bi-Lipschitz invariant, Math. Ann. 377 (2020), no. 1-2, 115-121.

[6] Lev Birbrair, Walter D Neumann and Anne Pichon, The thick-thin decomposition and the bilipschitz classification of normal surface singularities, Acta Math 212 (2014), 199–254.

[7] Alan Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc. 276 (1983), 517–530.

[8] Alexandre Fernandes, Topological equivalence of complex curves and bi-Lipschitz maps, Michigan Math. J. 51 (2003), 593–606.

[9] Alexandre Fernandes, Javier Fernandes de Bobadilla and Jose Edson Sampaio, Multiplicity and degree as bi-Lipschitz invariants for complex sets, (2017) arXiv:1706.06614.

[10] Alexandre Fernandes, Jose Edson Sampaio, Multiplicity of analytic hypersurface singularities under bi-Lipschitz homeomorphisms. J. Topol. 9 (2016), no. 3, 927–933.

[11] Javier Fernandez de Bobadilla, Sonja Heinze, Maria Pe Pereira, Jose Edson Sampaio, Moderately Discontinuous Homology, (2020), arXiv:1910.12552

[12] Javier Fernandez de Bobadilla, Sonja Heinze, Maria Pe Pereira, Moderately Discontinuous Homotopy, (2020), arXiv:2007.01538

[13] Tzee-Char Kuo and Yung-Chen Lu, On analytic function germs of two complex variables, Topology 16 (1977), 299–301.

[14] Lê Dũng Tráng, Françoise Michel, Claude Weber, Sur le comportement des polaires associées aux germes de courbes planes. Compositio Mathematica 72 (1989), 87-113.

[15] Lê Dũng Tráng, Bernard Teissier, Sur la géométrie des surfaces complexes I. Tangentes exceptionnelles, in Amer. J. Math. 101, (1979), 420-452.

[16] Tadeusz Mostowski, Lipschitz equisingularity, Dissertationes Math. (Rozprawy Mat.) **243** (1985), 46pp.

[17] Walter D Neumann and Anne Pichon, Lipschitz geometry of complex curves, Journal of Singularities 10 (2014), 225-234.

[18] Walter D Neumann and Anne Pichon, Lipschitz geometry of complex surfaces: analytic invariants and equisingularity, arXiv:1211.4897v1.

[19] Adam Parusiński, Lipschitz properties of semi-analytic sets, Université de Grenoble, Annales de l'Institut Fourier, 38 (1988) 189–213.

[20] Adam Parusiński, Lipschitz stratification of subanalytic sets, Ann. Sci. Ec. Norm. Sup. (4) 27 (1994), 661–696.

[21] Frédéric Pham and Bernard Teissier, Fractions Lipschitziennes d'une algèbre analytique complexe et saturation de Zariski. Prépublications Ecole Polytechnique No. M17.0669 (1969). Available at http://hal.archives-ouvertes.fr/hal-00384928/fr/.

[22] Pham F., Teissier B. (2020) Lipschitz Fractions of a Complex Analytic Algebra and Zariski Saturation. In: Neumann W., Pichon A. (eds) Introduction to Lipschitz Geometry of Singularities. Lecture Notes in Mathematics, vol 2280. Springer (2020).

[23] Bernard Teissier, Variétés polaires, II. Multiplicités polaires, sections planes, et conditions de Whitney, *Algebraic geometry (La Ràbida, 1981)* 314–491, Lecture Notes in Math. 961 (Springer, Berlin, 1982).

[24] C.T.C. Wall, *Singular points of plane curves*, Cambridge University Press (2004).

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France •
anne.pichon@univ-amu.fr