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Congruence subgroups of braid groups

TARA E. BRENDLE

Abstract

These notes are based on a mini-course given at CIRM in February 2018 as part of the workshop Winter Braids VIII.

1. Introduction

Congruence subgroups of matrix groups are usually defined as the kernel of the mod m reduction of a linear group. More precisely, starting with a subgroup $G < \mathrm{GL}(n, \mathbb{Z})$, we define the level-$m$ subgroup $G[m]$ as the kernel of the composition map $G \hookrightarrow \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/m)$. Congruence subgroups play an important role in the theory of arithmetic groups and hence in any closely related groups; as a starting point, see Raghunathan's survey of results on the congruence problem for algebraic groups [49] which includes an overview of contributions from Bass, Margulis, Prasad, Serre, and many others, or see Farb-Margalit [18, Section 6.4] for an introduction specific to the context of mapping class groups.

Similarly, we can define congruence subgroups of any group via a choice of representation into $\mathrm{GL}(n, \mathbb{Z})$. In the case of the braid group, and mapping class groups more generally, we will define a symplectic representation of B_n and use it to define braid congruence groups $\mathrm{B}_n[m]$.

Our viewpoint throughout will be heavily influenced by that of mapping class groups, and indeed, we will define the braid group B_n as the mapping class group of a disk with n marked points. As such, we will use simple closed curves in surfaces as a key mechanism for studying braid groups.

There are many excellent references on braid groups and mapping class groups. These lecture notes will largely follow the notation and terminology of Farb-Margalit's "Primer" [18]. Other references that will be particularly useful as supplements to these lecture notes include Birman's classic text [9] and the recent survey on Birman-Hilden theory by Margalit-Winarski [37].

Overview. In Section 2 below, we cover some basic material on mapping class groups of surfaces, before using these as a vehicle for defining symplectic representations of braid groups, and of mapping class groups more generally, in Section 3, where we also introduce the kernels of these maps, including congruence subgroups. These lecture notes will largely focus on three particular braid congruence groups, corresponding to the choices $m = 0, 2,$ and 4; these will be covered in Sections 5, 4, and 6, respectively; the order here reflects the fact that it turns out the level-2 braid congruence group plays a key role in all three cases and hence we treat it first. In Section 7 we give sample applications of our characterizations of these groups, and we also describe connections between the various congruence groups that make an appearance, of the symplectic group as well as the braid group. Finally in Section 8 we discuss some related work and further directions for future exploration.

Acknowledgments

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2. Mapping class groups

Let S_g^r denote a compact orientable surface of genus g with r boundary components. If $r = 0$, we will simply write S_g. Let D_n denote a disk with n points removed. Since our surfaces are orientable, we will always consider them as coming equipped with a particular orientation. For $S = S_g^r$ or $S = D_n$, we define the *mapping class group* of S, denoted $\mathrm{Mod}(S)$, as follows.

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\text{isotopy rel } \partial S.$$

In other words, a homeomorphism representing an element of $\mathrm{Mod}(S)$ must be orientation-preserving and must fix the boundary pointwise, as must all isotopies. If the surface has n points removed, these n punctures may be permuted by a mapping class, though the punctures cannot move during isotopies. For our purposes, such as in the case $S = D_n$, it is often useful to think of the n removed points as a collection of n marked points in the disk. The mapping class group is also known as the *Teichmüller modular group*, hence the notation $\mathrm{Mod}(S)$.

The extended mapping class group $\mathrm{Mod}^\pm(S)$ is defined similarly to $\mathrm{Mod}(S)$, except that we allow orientation-reversing maps; in other words we take the quotient of the group $\mathrm{Homeo}^\pm(S, \partial S)$ under isotopy rel ∂S. When $\partial S = 0$, the group $\mathrm{Mod}(S)$ is a subgroup of $\mathrm{Mod}^\pm(S)$, and when $\partial S \neq 0$, we have $\mathrm{Mod}(S) = \mathrm{Mod}^\pm(S)$.

Basic cases: sphere and torus. The mapping class group of a sphere $\mathrm{Mod}(S_0)$ is trivial; this follows from the Jordan Curve Theorem. The case of the torus is more interesting: $\mathrm{Mod}(S_1) \cong \mathrm{SL}(2, \mathbb{Z})$. To see this, consider the torus as the unit square with sides identified in the usual way. Note that a map of the plane that preserves the integer lattice is determined by where it sends the basis vectors $(1, 0)$ and $(0, 1)$. In the torus we can picture these vectors as corresponding to standard meridian and longitude curves.

Any simple closed curve on a torus can be uniquely described up to isotopy by a pair of relatively prime integers r and s; we can think of the curve (in homology, say) as r copies of the meridian plus s copies of the longitude. The fact that the meridian and longitude intersect exactly once means that once we know that the image of the meridian, say, is given by relatively prime integers r, s, then the corresponding pair r', s' for the longitude is uniquely determined by the equation $rs' - sr' = 1$. This enables us to write down a matrix in $\mathrm{SL}(2, \mathbb{Z})$ of the form $\begin{pmatrix} r & s \\ r' & s' \end{pmatrix}$ recording the action of the mapping class on our chosen meridian and longitude. In fact, any map of the torus can be understood in this way; for details see [18, Chapter 2].

In general, however, mapping class groups are not isomorphic to familiar groups. And although the example of the torus correctly suggests that we stand to gain a great deal of insight by comparing mapping class groups to linear groups, it is a long-standing open question as to whether mapping class groups are themselves linear in general. (Bigelow-Budney have shown that $\mathrm{Mod}(S_2)$ embeds as a subgroup of $\mathrm{GL}(64, \mathbb{C})$ [7], but for $g \geq 3$ the question remains open.)

Other viewpoints. Before continuing, we note here some useful further viewpoints on $\mathrm{Mod}(S)$ and/or $\mathrm{Mod}^{\pm}(S)$.

- **Group theoretic.** The Dehn-Nielsen-Baer Theorem gives us a purely (combinatorial) group theoretic description of $\mathrm{Mod}^{\pm}(S)$. It states that the action of the extended mapping class group $\mathrm{Mod}^{\pm}(S_g)$ on the surface group $\pi_1(S_g, \star)$ with respect to a basepoint $\star$ induces the following isomorphism:

  $$\mathrm{Mod}^{\pm}(S_g) \cong \mathrm{Out}(\pi_1(S_g, \star))$$

  See, for example, [18, Section 8.1] for a proof and discussion of the contributions of the theorem's three namesakes.

- **Riemann surfaces.** For $g \geq 2$, the mapping class group $\mathrm{Mod}(S_g)$ arises as the orbifold fundamental group of the moduli space of Riemann surfaces $\mathcal{M}(S_g)$, the parameter space of hyperbolic metrics on S_g; see, for example, [18, Chapter 12]. Moreover $\mathrm{Mod}^{\pm}(S_g)$ is isomorphic to the group of isometries of the Teichmüller space $\mathrm{Teich}(S_g)$ when $g \geq 3$, the universal cover of $\mathcal{M}(S_g)$, which is the space of marked surfaces of genus g. A description of this isomorphism and its injectivity is given in [18, Section 12.1], and its surjectivity is a theorem of Royden [51].

- **Classifying spaces.** When $g \geq 2$, the space $\mathrm{BHomeo}^{+}(S_g)$ is a $K(\mathrm{Mod}(S_g), 1)$-space [18, Proposition 5.12]. See also further discussion relating this viewpoint and the previous in [18, Section 12.6].

- **Combinatorial models.** There are many abstract simplicial complexes associated to surfaces that serve as combinatorial models for mapping class groups. The most famous example is the so-called *complex of curves* $\mathcal{C}(S)$, and it is a theorem of Ivanov ([28]; see also [34] and [36]) that the simplicial automorphism group of $\mathcal{C}(S_g)$ is isomorphic to the extended mapping class group $\mathrm{Mod}^{\pm}(S_g)$.

Basic examples. Many examples of mapping classes arise from "nice" embeddings of the surface in 3-space. For example, Figure 2.1 shows two rotations of a surface of genus g arising from two different embeddings of the surface.

Dehn twists. We next define an important type of element in $\mathrm{Mod}(S)$ that is more local in nature and intrinsic to the surface rather than any ambient space. Let c denote a simple closed curve in the surface S. Choose a regular (annular) neighborhood of c, and parametrize this annulus A as follows: $A := \{re^{i\theta} \mid 1 \leq r \leq 2\}$, where c corresponds to the subset of A where $r = \frac{3}{2}$. We define a homeomorphism of A by $re^{i\theta} \mapsto re^{i(\theta + 2\pi r)}$; see Figure 2.2. This map has the following nice properties.

1. Each component of ∂A is fixed pointwise.

2. The core c is fixed setwise.

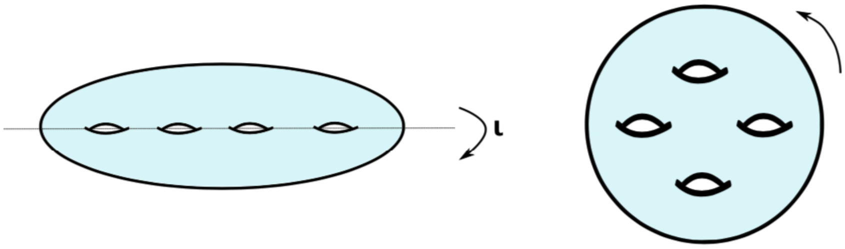

Figure 2.1. Two embeddings of a surface of genus four. *Left:* The hyperelliptic involution ι; any surface admits such an involution. *Right:* A "one-click" counter-clockwise rotation by $\frac{\pi}{2}$ with respect to an axis normal to the plane of the page; a genus g surface admits an analogous rotation by $\frac{2\pi}{g}$.

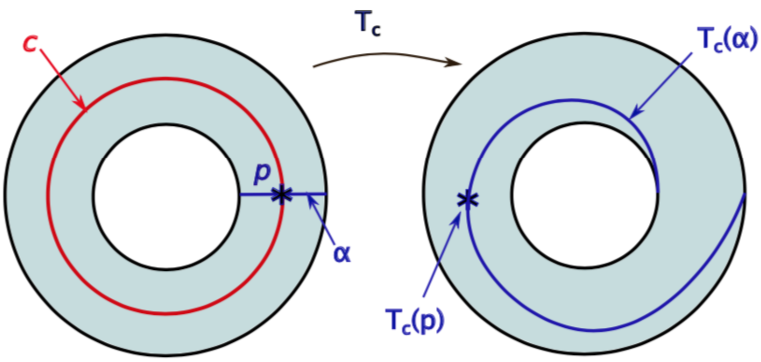

Figure 2.2. The Dehn twist T_c on an annulus with core curve c.

3. The two points on c corresponding to $\theta = 0$ and $\theta = \pi$ are interchanged by this map.

Using Property (1) above, we can define a homeomorphism on all of S, simply by extending by the identity. The corresponding mapping class is called the *(left) Dehn twist about c* and is denoted T_c. The mapping class T_c is independent of the choice of annular neighborhood A and the parametrization of A. Moreover T_c is well defined on the isotopy class of the curve c; for this reason we will often not distinguish between a simple closed curve and its isotopy class. (We leave it to the reader as a somewhat tedious exercise to prove well-definedness carefully.) See Farb-Margalit [18, Chapter 3] or Rolfsen[50, Chapter 2(C)] for further details. It is also important to note that the definition of a Dehn twist does not rely on any orientation of c; our notations of a 'left' twist versus a 'right' twist arise from an orientation of the surface instead.

Dehn first proved that finitely many Dehn twists generate $\mathrm{Mod}(S_g^r)$ [16]. Humphries later showed that the twists about the $2g+1$ simple closed curves shown in Figure 2.3 generate $\mathrm{Mod}(S_g)$ when $g \geq 2$ (when $g = 1$, two of the three indicated curves are isotopic and hence only two twists are required to generate the mapping class group of a torus) and moreover that $\mathrm{Mod}(S_g)$ cannot be generated with fewer Dehn twists [25]. Indeed, the Humphries generators also generate $\mathrm{Mod}(S_g^1)$ if we view S_g^1 has being obtained from S_g as follows: the complement of the curves shown in Figure 2.3 has two components, and we remove the interior of a disk from the component on the right. In general, a total of $2g + r$ Dehn twists are required n order to generate $\mathrm{Mod}(S_g^r)$ when $r \geq 1$ [35, Theorem 3.1].

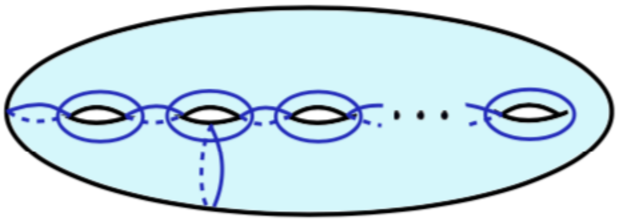

Figure 2.3. The Humphries generating set for $\mathrm{Mod}(S_g)$ consists of $2g+1$ Dehn twists about the simple closed curves pictured above.

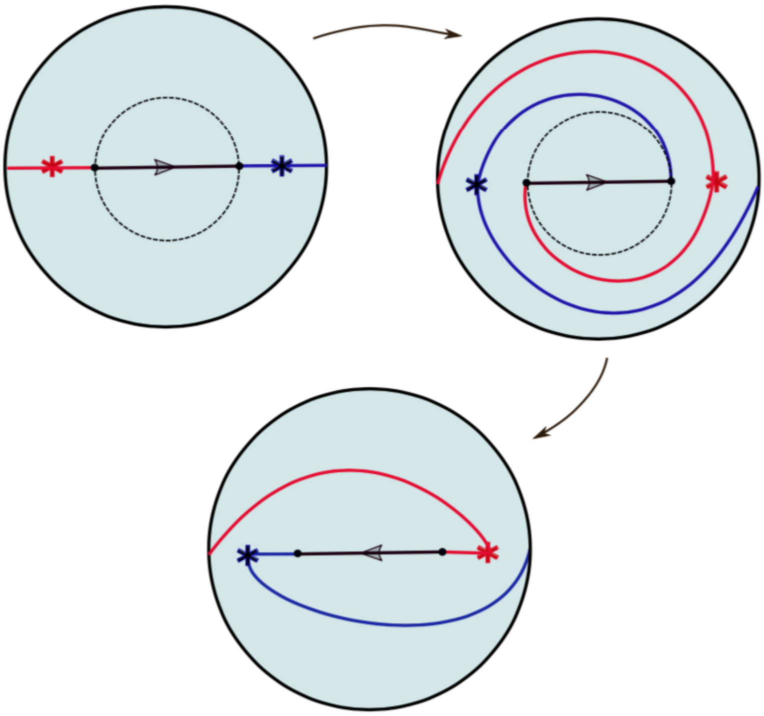

Figure 2.4. A half-twist exchanging two marked points: the intermediate step shows the effect of the map as defined by the formula above, while the final step shows the result after isotopy in which only the boundary and the two points marked by stars must remain fixed.

Half-twists. In the case where we have marked points on the surface S we also define a *half-twist*. In this case, it is easiest to view the punctures of S as marked points. To begin, consider the disk $D = \{re^{i\theta} \mid r \leq 2\}$ in $\mathbb{C}$, with two marked points $p = \frac{3}{2}$ and $q = -\frac{3}{2}$ on the real axis. We define the *half-twist map* exchanging p and q as follows.

$$re^{i\theta} \mapsto \begin{cases} re^{i\theta} & \text{if } 0 \leq r \leq 1 \\ re^{i(\theta+2\pi r)} & \text{if } 1 \leq r \leq 2. \end{cases}$$

In other words, we extend the Dehn twist about the curve $c = \{\frac{3}{2}e^{i\theta}\}$ (defined on our standard annulus $A := \{re^{i\theta} \mid 1 \leq r \leq 2\}$) by the identity across the unit disk. If we did not have the two marked points, this map would be isotopic to the identity. The map itself exchanges the two marked points p and q, with p moving 'in front of' q. The key point is that isotopies are not allowed to move marked points, and so this map is nontrivial in the mapping class group of the disk with two marked points (or punctures). The net effect is to interchange the points

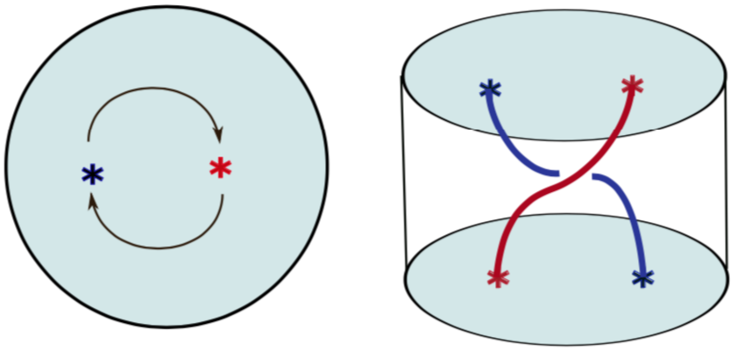

Figure 2.5. A half-twist exchanging two marked points: on the disk D_n (here $n = 2$), and in the cylinder $D_n \times [0,1]$.

p and q, while fixing the boundary of D pointwise. One can imagine putting two fingers on the points p and q and then interchanging the two points by rotating one's hand clockwise while holding the boundary of D_2 fixed. See Figure 2.4.

More generally, we can choose any properly embedded arc α joining two marked points in D_n, and define the *half-twist* h_α as an element of $\mathrm{Mod}(D_n)$ by mapping a regular neighborhood of α to our disk D, taking the pair of marked points to p and q and taking α to the segment of the real axis joining p and q, performing the half-twist map exchanging p and q, and then mapping back to D_n (and extending by the identity outside the chosen disk).

Braid groups as mapping class groups. We define the *n-strand braid group*, denoted B_n, as the mapping class group of the n-punctured disk D_n. We can recover the traditional viewpoint of the braid group B_n in terms of geometric braids by keeping track of the marked points "during" a half-twist; see Figure 2.5. If we perform the half-twist shown in Figure 2.5 in a disk containing the i^{th} and $(i+1)^{st}$ marked points, and extend by the identity to the rest of the disk D_n, then we will denote this map by σ_i.

The following well-known presentation for B_n first appeared in an early paper of Artin [5] :

$$\begin{aligned}\langle \sigma_1, \ldots, \sigma_{n-1} \quad | \quad & \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2;\\ & \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1\rangle\end{aligned}$$

3. Symplectic representations and their kernels

The mapping class group of any surface $S = S_g^r$ acts naturally on the first homology group $H_1(S;\mathbb{Z})$, preserving the algebraic (i.e. signed) intersection pairing $\widehat{i}(\cdot,\cdot)$ on oriented simple closed curves on S. It is easy to check that this intersection pairing gives rise to an alternating bilinear form $\langle\cdot,\cdot\rangle$ on $H_1(S,\mathbb{R})$. When $r \in \{0,1\}$, the form $\langle\cdot,\cdot\rangle$ is also nondegenerate and hence symplectic. This in turn gives rise to a symplectic representation:

$$\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g,\mathbb{Z}).$$

Exercise 1. Explain why the intersection pairing fails to give rise to a symplectic form when $r \geq 2$.

Sample calculations. In the cases $r \in \{0,1\}$, the surface S also admits a *symplectic basis* for $H_1(S;\mathbb{Z})$, that is, a free basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(S;\mathbb{Z})$ (viewed as a $\mathbb{Z}$-module) with the property that $\langle a_i, b_i\rangle = 1$ and $\langle a_i, a_j\rangle = \langle a_i, b_j\rangle = 0$ whenever $i \neq j$. Figure 3.1 shows simple closed curves representing a standard symplectic basis in the case $S = S_g^1$; we can also think of the "same" curves as representing a symplectic basis in the case $S = S_g$ by capping off the boundary component with a disk to obtain a closed surface.

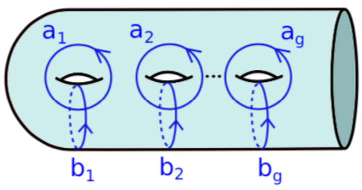

Figure 3.1. A standard symplectic basis for $H_1(S_g^1; \mathbb{Z})$.

As a first example, consider the torus $T = S_1$, with the simple closed curves a and b as shown in Figure 3.2. Then taking the curves a and b as representatives of a symplectic basis

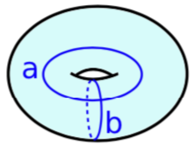

Figure 3.2. A torus with two simple closed curves.

for $H_1(S; \mathbb{Z})$, with appropriately chosen orientations, we have the following

$$\rho(T_\alpha) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \rho(T_\beta) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

For a second example, recall that $\iota \in \mathrm{Mod}(S_g)$ denotes the order-two rotation depicted in Figure 2.1 above. In this case we have that $\rho(\iota) = -\mathrm{Id}_{2g}$, where Id_{2g} denotes the $2g \times 2g$ identity matrix. In other words, ι is a *hyperelliptic* involution, by which we mean an involution that acts by $-\mathrm{Id}$ on $H_1(S_g; \mathbb{Z})$. This example will play a crucial role in what follows.

Torelli groups. Given any symplectic basis for $H_1(S; \mathbb{Z})$, one can find a representative set of simple closed curves realizing this basis; we will call this "homology realization", and refer the reader to Farb-Margalit's development of the Meeks-Petrusky realization method as one example [18, Chapter 6.2]. Using this fact, one can show that the symplectic representation $\rho : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ defined above is surjective in the case where $S = S_g^r$ and $r \in \{0, 1\}$; see, for example, [44, Chapter 2]. In these cases, we define the Torelli group $\mathcal{I}(S)$ of the surface S to be the kernel of the symplectic representation ρ; in general ρ is not faithful. The notation $\mathcal{I}$ stands for "Identity" and is commonly used for the Torelli group.

Putting this all together, we have the following short exact sequence:

$$1 \longrightarrow \mathcal{I}(S) \longrightarrow \mathrm{Mod}(S) \xrightarrow{\rho} \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1. \tag{3.1}$$

It is important to emphasize that the above definition of the Torelli group is only valid in the cases $r \in \{0, 1\}$. When $r \geq 2$, the "right" definition of the Torelli group is less clear, due to the degeneracy of the intersection form; again see [44] or [46]. Having said that, we note for the sake of completeness that the process of capping off boundary components gives rise to a surjection from $\mathrm{Mod}(S_g^r)$ onto $\mathrm{Mod}(S_g)$, and hence there is always a surjection from $\mathrm{Mod}(S_g^r)$ onto $\mathrm{Sp}(2g, \mathbb{Z})$ for all $r \geq 0$.

We note that when $g = 1$, the Torelli group is trivial. Considering the short exact sequence (3.1) above, this is just saying that the mapping class group of a torus is just $\mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$, as described in the previous section.

Basic elements in Torelli groups. In the remainder of this section we will continue to assume that $r \in \{0, 1\}$, and we will also continue to use the same notation for a curve and its isotopy class in the surface S.

We noted above that $\mathrm{Mod}(S_g^r)$ is generated by Dehn twists. Let a and b be oriented simple closed curves in S_g^r. The following formula [18, Proposition 6.3] tells us the effect of any power k of the Dehn twist T_a on the homology class $[b]$.

$$\rho(T_a^k)([b]) = [b] + k \cdot \widehat{i}(a, b)[a] \tag{3.2}$$

The formula makes precise our intuition that, when performing a Dehn twist about a simple closed curve a, the effect on another simple closed curve b is that b "picks up a copy of a" each time b intersects a. Using this formula, we can immediately find some families of elements in $\mathcal{I}(S)$.

Separating twists. It is clear from Formula 3.2 above that if a is a nullhomologous curve, then $T_a \in \mathcal{I}(S)$. Some examples of separating curves appear in Figure 3.3.

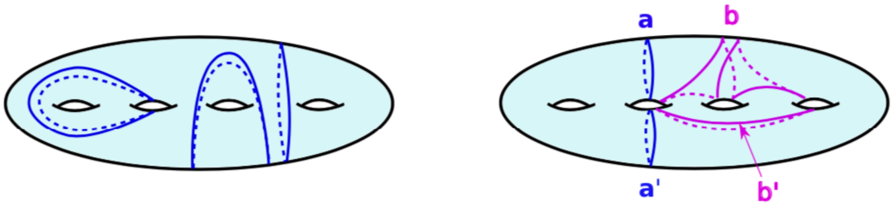

Figure 3.3. *Left:* Three examples of separating curves; *Right:* Two examples of bounding pairs: $a \cup b$ and $a' \cup b'$.

Bounding pairs (BPs). If we perform a Dehn twist about a nonseparating simple closed curve a, then we can see that T_a is not in the Torelli group by applying Formula 3.2 to any curve c with $\widehat{i}(a, c) = 1$. (Exercise: why does such a curve c always exist?) However, we can cancel out the action of T_a by finding a simple closed curve b that is disjoint from a and such that the union $a \cup b$ separates the surface, and then composing the first twist with the inverse of the second. In other words, $T_a T_b^{-1} \in \mathcal{I}(S)$.

To see this, first note that since a and b cobound a subsurface R of S_g^r, we have (with appropriately chosen orientations) $[a] = [b]$ in $H_1(S; \mathbb{Z})$. Further, for every time a curve c crosses a "from the right", say, then c must also either cross a a second time from the left, or else crosses b from the left, so that the net effect on homology is trivial. (Exercise: check this carefully.) We call the curves a, b a *bounding pair* and the composition $T_a T_b^{-1}$ a *bounding pair map*, or BP-map for short. See Figure 3.3.

Fake BPs. In the previous example, we gave a topological explanation for the fact that BPs are elements of $\mathcal{I}(S)$, which used the fact that the two curves involved in the BP-map $T_a T_b^{-1}$ were disjoint. However, we do not need this assumption to prove the same result. We only need the assumption that $[a] = [b]$, again, with appropriate choice of orientations for the curves a and b. In this case, Formula 3.2 tells us immediately that $\rho(T_a) = \rho(T_b)$, and hence $\rho(T_a T_b^{-1}) = \mathrm{Id}_{2g}$; it is important to recall here that the algebraic intersection number $\widehat{i}$ is well defined on homology classes, not just on isotopy classes of curves, as we noted at the start of Section 3.

Therefore, in order to generalise the notion of a BP map, we can simply choose any two simple closed curves a and b such that $[a] = [b]$ and with $\widehat{i}(a, b) = 0$, and Formula 3.2

again shows immediately that $T_a T_b^{-1} \in \mathcal{I}(S)$. If in addition the geometric intersection $i(a,b)$ is nonzero, we say that $T_a T_b^{-1}$ is a *fake BP map*. We will next describe a particularly important type of fake BP maps.

Simply intersecting pairs (SIPs). Consider two curves a, c where $\widehat{i}(a,c) = 0$ and $i(a,c) = 2$; in this case we say that a and c form a *simply intersecting pair*; see Figure 3.4. Using Formula 3.2 again, we see that the condition that $\widehat{i}(a,c) = 0$ implies that $T_c(a)$ is homologous to a. Thus $T_a T_{T_c(a)}^{-1}$ is a fake BP map. Now, one can check easily that for any mapping class $f \in \mathrm{Mod}(S)$ and any Dehn twist T_c, we have that $f T_a f^{-1} = T_{f(a)}$. Hence $T_a T_{T_c(a)}^{-1} = T_a T_c T_a^{-1} T_c^{-1}$. In other words, the commutator $[T_a, T_c]$ of the twists corresponding to a simply intersecting pair lies in the Torelli group. We refer to such an element as an *SIP map*.

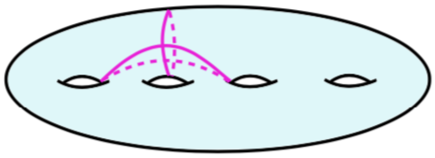

Figure 3.4. The two curves shown here form a simply intersecting pair (SIP), from which we can derive a special case of a fake BP.

In much of the early literature on mapping class groups, separating twists and BP maps played a leading role. For example, combined work of Birman [8] and her student Powell [43] first showed that $\mathcal{I}(S)$ was generated by separating twists and BP maps. A few years later, Johnson showed that finitely many BP maps suffice to generate $\mathcal{I}(S)$ [30] when $g \geq 3$. Mess then showed that $\mathcal{I}(S_2)$ is a free group on an infinite collection of separating twists [40].

As far as we are aware, the type of example now known as a fake BP first appeared in Turaev's Bourbaki survey on linear representations of braid groups as a tool for exploring faithfulness [56]; this serves as our first clue that Torelli groups are a useful tool to inform our study of braid groups. Fake BPs finally gained a more prominent role in the study of Torelli groups through work of Putman [47], who used them to give an infinite presentation of $\mathcal{I}(S)$.

Generalizations: congruence groups. Using the symplectic representation, there is an easy way to find finite index subgroups of $\mathrm{Mod}(S)$, simply by passing from $\mathbb{Z}$ to $\mathbb{Z}/m$ for some integer m. We define a map ρ_m to be the composition of the symplectic representation with the map on corresponding symplectic groups induced by mod m reduction. See Newman's book "Integral Matrices" for a good general discussion of symplectic groups over the ring $\mathbb{Z}/m$ [42, Chapter VII, Section 33].

$$\begin{tikzcd} \mathrm{Mod}(S) \arrow[r, "\rho"] \arrow[rr, bend right, "\rho_m"'] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r, "\mathrm{mod}", "m"'] & \mathrm{Sp}(2g,\mathbb{Z}/m). \end{tikzcd} \tag{3.3}$$

The kernel of the map $\mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/m)$ is known as the *level m principal congruence subgroup* of $\mathrm{Sp}(2g,\mathbb{Z})$ and is denoted $\mathrm{Sp}(2g,\mathbb{Z})[m]$ or sometimes just $\mathrm{Sp}[m]$. The kernel of the map ρ_m is known as the *level m congruence subgroup* of $\mathrm{Mod}(S)$ and is denoted $\mathrm{Mod}(S)[m]$ or simply $\mathrm{Mod}[m]$. We can view congruence subgroups of mapping class groups as generalizations of the Torelli group; indeed, we can consider the Torelli group itself as the level 0 congruence subgroup $\mathrm{Mod}(S_g)[0]$.

Congruence subgroups play an important role in the theory of mapping class groups in a variety of settings. For example, when $m \geq 3$, the congruence subgroup $\mathrm{Mod}(S_g)[m]$ is torsion-free and (by definition) finite-index, which enables us to come to grips with geometric group theory invariants such as virtual cohomological dimension and duality [24]. As another example, in the algebro-geometric setting, the regular cover of $\mathcal{M}(S_g)$ corresponding to $\mathrm{Mod}(S_g)[m]$ is the moduli space of surfaces of genus g equipped with a *full level m structure*, that is, a basis for the m-torsion in their Jacobian; see Fullarton-Putman for an overview of this viewpoint [19]. See Putman's lecture notes [45] for further exposition of this topic.

We record some further notes here about congruence subgroups. For further information, Raghunathan's survey includes a comprehensive list of various further references [49].

- The symplectic group has the congruence property.
- For $m \geq 3$, the congruence group $\mathrm{Mod}(S_g)[m]$ is pure in the sense of Ivanov [27]. Roughly speaking, this means that all elements have a (nearly) canonical factorisation.
- The existence of congruence subgroups is closely related to the property of residual finiteness. For example, see Ivanov's proof [26, Section 11.1] of Grossman's theorem [22] that mapping class groups are residually finite
- There is a more general notion of congruence subgroups of braid groups. Let H be a characteristic, finite index subgroup of the fundamental group $\pi_1(S)$; in our case we consider $S = D_n$. We then get a map from $\mathrm{Mod}(S)$ to $\mathrm{Out}(\pi_1(S)/H)$ (recall the Dehn-Nielsen-Baer Theorem from Section 2 above). The kernel of this map is called a *principal congruence subgroup* in this setting. For all n, the pure braid group of the sphere with n marked points is congruence, that is, every finite index subgroup contains a principal congruence subgroup, a fact first proven by Diaz-Donagi-Harbeter in 1989 [17]. More recently McReynolds recorded an elementary proof of this fact due to Thurston; the introduction of McReynolds' paper also contains many further references to related work and different proofs [38]. In fact, this is essentially the same framework used by Ivanov to establish residual finiteness, and in addition to Ivanov's paper cited above we refer the reader to Farb-Margalit's treatment of Ivanov's approach for more details [18, Section 6.4.3].

Hyperelliptic mapping class groups. Recall the involution ι of S_g given by rotation by π indicated in Figure 2.1 above; this is a *hyperelliptic* involution since it acts as $-\mathrm{Id}$ on $H_1(S_g, \mathbb{Z})$. We can consider the surface S_g^1 as a subsurface of S_g (we can "cap off" the boundary component of S_g^1 to recover S_g), and consider the restriction of ι to S_g^1. In both cases, we define the hyperelliptic mapping class group, denoted $\mathrm{SMod}(S)$, as follows:

$$\mathrm{SMod}(S) := \{f \in \mathrm{Mod}(S) \mid f\iota = \iota f\}$$

We are somewhat abusing notation in the above definition in the case where $S = S_g^1$. This is because ι does not induce a mapping class in $\mathrm{Mod}(S_g^1)$ since it does not fix ∂S_g^1 pointwise. The notation $\mathrm{SMod}(S)$ comes from Birman-Hilden [10], who used the term *symmetric mapping class group* for this subgroup of $\mathrm{Mod}(S)$. Indeed, we say that a simple closed curve is *symmetric* if it is preserved setwise by ι, and we say that an isotopy class α of simple closed curves is *symmetric* if it has a symmetric representative.

For the moment, we will focus on the case of the surface S_g^1. The hyperelliptic involution ι induces a branched double cover of the disk D_n, with the n marked points of D_n corresponding to the $n = 2g + 1$ branch points. Consider Figure 3.5, which shows a collection of curves and a properly embedded arc in D_n, along with their pre-images in the cover S_g^1.

Birman-Hilden proved that this branched double cover induces an isomorphism between $\mathrm{Mod}(D_n)$ and $\mathrm{SMod}(S_g^1)$, giving us a new characterization of the braid group [10, Theorem 1]. If one wants to work with braid groups from this viewpoint, it is worthwhile to work out

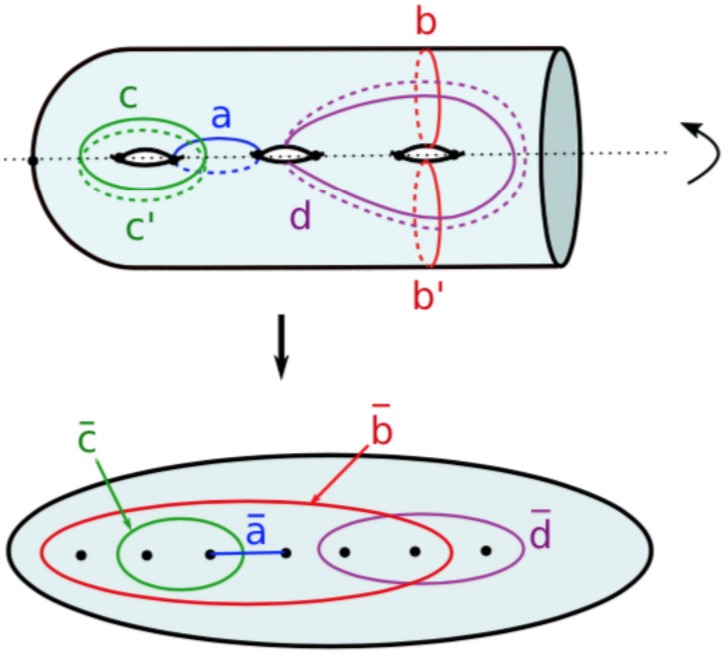

Figure 3.5. The Birman-Hilden dictionary: a collection of simple closed curves and an arc in the disk D_7 together with their pre-images under the hyperelliptic involution.

the details of the "dictionary" between the two groups given in Exercise 2. To that end, it is helpful to distinguish between topological types of simple closed curves in D_n. We say a curve is *even* if it bounds a disk containing an even number of marked points; we similarly say it is *odd* if it bounds a disk containing an odd number of marked points. When we need to be more specific, we will refer to a *k-curve*, meaning a curve that bounds a disk in D_n containing precisely k marked points.

Exercise 2. Work out the dictionary between the groups $\mathrm{Mod}(D_n)$ and $\mathrm{SMod}(S_g^1)$ by demonstrating the following correspondences induced by the hyperelliptic involution indicated in Figure 3.5. The notation here follows the same figure.

1. A half-twist in D_n corresponds to a Dehn twist about a nonseparating symmetric curve in S_g^1:
$$h_{\bar{a}} \longleftrightarrow T_a$$
2. A Dehn twist about an even curve $\bar{b}$ in D_n corresponds to a product of the Dehn twists about the two curves b, b' in its pre-image upstairs in S_g^1:
$$T_{\bar{b}} \longleftrightarrow T_b T_{b'}.$$
It is worth distinguishing a special case here: if $\bar{c}$ is a 2-curve in D_n, then the two components in its pre-image will be isotopic to a symmetric curve c in S_g^1, and we have:
$$T_{\bar{c}} \longleftrightarrow T_c^2.$$
3. The square of a Dehn twist about a $2k+1$-curve $\bar{d}$ in D_n corresponds to a Dehn twist about its pre-image d, where d is a separating curve that bounds a genus k subsurface of S_g^1:
$$T_{\bar{d}}^2 \longleftrightarrow T_d$$

Symplectic representations of braid groups. Again, we focus primarily on the case where $n = 2g + 1$. The Birman-Hilden isomorphism enables us to define symplectic representations of the braid group as follows. In the case $n = 2g + 1$, we can define a map $\beta : \mathrm{B}_n \to \mathrm{Sp}(2g, \mathbb{Z})$ by composing the Birman-Hilden isomorphism with inclusion of the hyperelliptic mapping class group into the full mapping class group, and then finally applying the classical symplectic representation ρ as defined above. For any integer m, we can then pass to $\mathrm{Sp}(2g, \mathbb{Z}/m)$ to obtain a family of representations of the braid group that we will denote β_m, by analogy with the representations ρ_m defined above. The following commutative diagram summarizes the preceding discussion in the case where $n = 2g + 1$.

$$\begin{tikzcd}
\mathrm{B}_n = \mathrm{Mod}(D_n) \arrow[r, "\beta"] \arrow[d, "\text{Birman-Hilden } \cong"'] \arrow[rr, bend left, "\beta_m"] & \mathrm{Sp}(2g, \mathbb{Z}) \arrow[r, "\mathrm{mod}", "m"'] & \mathrm{Sp}(2g, \mathbb{Z}/m) \\
\mathrm{SMod}(S_g^1) \arrow[r, hook] & \mathrm{Mod}(S_g^1) \arrow[u, "\rho"] &
\end{tikzcd}$$

The case where n is even. If we wish to work with the braid group B_n for $n = 2g + 2$, we will need to modify the Birman-Hilden construction given above. In this case, we consider the restriction of the hyperelliptic involution ι to a surface S_g^2 as shown in Figure 3.6. This gives rise to a branched double cover of the disk D_n for $n = 2g + 2$. We can then define $\mathrm{SMod}(S_g^2)$ using this involution, and the Birman-Hilden isomorphism and dictionary go through similarly in this case.

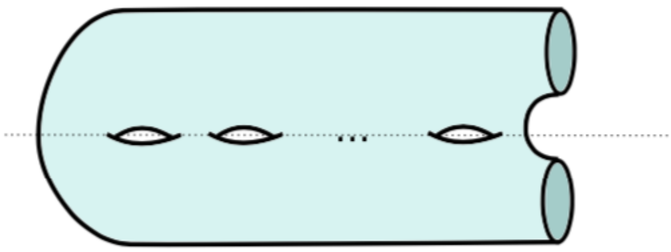

Figure 3.6. The quotient of a surface of genus g with two boundary components under the hyperelliptic involution (given by rotation about the indicated axis) is a disk with $n = 2g + 2$ marked points corresponding to the n branch points.

Defining the symplectic representation of B_n is only slightly more complicated when $n = 2g + 2$. We give a brief summary here and refer the reader to [14] for details. As explained above, the Birman-Hilden correspondence again allows us to consider B_n as the subgroup $\mathrm{SMod}(S_g^2)$ of $\mathrm{Mod}(S_g^2)$. Now, let $\tilde{p} = \{p_1, p_2\}$ in ∂S_g^2 be the pre-image of a basepoint $p \in \partial D_n$. There is a map $\mathrm{Mod}(S_g^2) \to \mathrm{Aut}(H_1(S_g^2, \tilde{p}; \mathbb{Z}))$. The relative homology group $H_1(S_g^2, \tilde{p}; \mathbb{Z})$ admits a symplectic intersection form, with symplectic basis given by $2g$ "standard" basis elements analogous to the basis for $H_1(S_g^1, \mathbb{Z})$ shown in Figure 3.1, together with two further basis elements, one represented by an arc joining p_1 to p_2 and one by a single component of the boundary, with suitable orientations.

In other words, the map $\mathrm{Mod}(S_g^2) \to \mathrm{Aut}(H_1(S_g^2, \tilde{p}; \mathbb{Z}))$ is in fact a map $\mathrm{Mod}(S_g^2) \to \mathrm{Sp}(2g + 2, \mathbb{Z})$. Moreover, all our maps fix boundaries pointwise, so in fact the image of our map lies in the subgroup $\mathrm{Sp}(2g+2,\mathbb{Z})_\partial$ consisting of those elements of $\mathrm{Sp}(2g+2,\mathbb{Z})$ that fix the element of $H_1(S_g^2,\tilde{p};\mathbb{Z})$ corresponding to a component of ∂S_g^2.

We summarize the preceding discussion in the following commutative diagram in the case where $n = 2g+2$:

$$\begin{tikzcd}
\mathrm{B}_n = \mathrm{Mod}(D_n) \arrow[r, "\beta"] \arrow[d, "\cong"] \arrow[rrr, bend left, "\beta_m"] & \mathrm{Sp}(2g+2,\mathbb{Z})_\partial \arrow[r, hook] & \mathrm{Sp}(2g+2,\mathbb{Z}) \arrow[r, "\mathrm{mod}", "m"'] & \mathrm{Sp}(2g+2,\mathbb{Z}/m) \\
\mathrm{SMod}(S_g^2) \arrow[r, hook] & \mathrm{Mod}(S_g^2) \arrow[u, "\rho"] & &
\end{tikzcd}$$

Although it is useful to know that the constructions described in these notes go through for all B_n, regardless of parity, the reader may find it useful to focus initially on the case where $n = 2g+1$ in what follows.

Burau representation. We just defined a symplectic representation of the braid group via a branched double cover of the disk D_n. For those familiar with the Burau representation, this should sound familiar. We will describe a different way to obtain the same representation (up to conjugacy) in this context, largely inspired by Turaev's excellent survey article [56].

Consider D_n^0, the n-punctured disk, with a basepoint p on ∂D_n. The exponent sum map $\pi_1(D_n,p) \to \mathbb{Z}$ corresponds to a regular cover $\widetilde{D_n^0}$ with infinite cyclic deck group; we fix a generating deck transformation t. (Note that this map can also be viewed recording the total winding number of loops around the marked points of D_n.) We can picture $\widetilde{D_n^0}$ by taking copies of D_n^0 indexed by powers of the generator t as shown in Figure 3.7 for the case $n = 3$. We

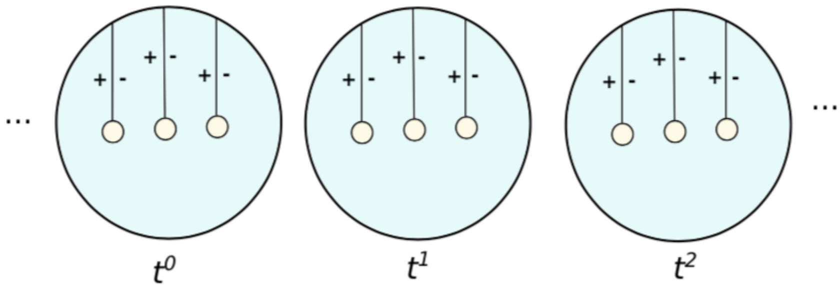

Figure 3.7. The zeroth, first, and second "decks" in the universal cyclic cover of D_3^0.

construct $\widetilde{D_n^0}$, the universal cyclic cover of D_n^0 by cutting along an arc from each puncture to ∂D_n. We then assign each side of the arc a "+" or "-", and glue the + side of a cut-arc in copy t^k to the $-$ side of the corresponding arc in copy t^{k+1}. Thus, whenever a path crosses one of these arcs "downstairs", in the universal cyclic cover the lifted path will move from deck t^k "up" a level to deck t^{k+1} or "down" a level to deck t^{k-1}, depending on the direction of travel.

The first homology group $H_1(\widetilde{D_n^0};\mathbb{Z})$ is a $\mathbb{Z}[t^{\pm 1}]$-module.

Exercise 3. Show that $H_1(\widetilde{D_n^0};\mathbb{Z})$ is free of rank $n-1$ as a $\mathbb{Z}[t^{\pm 1}]$-module, and find generators.

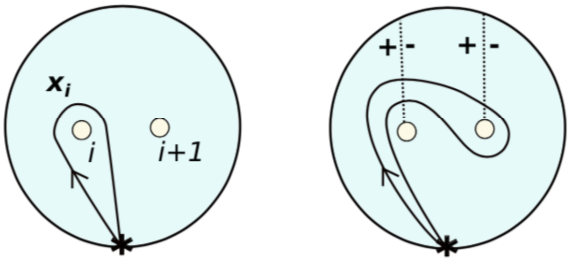

Figure 3.8. *Left:* The loop x_i forms part of a standard generating set for $\pi_1(D_n^0, p)$; *Right:* the result of applying the half-twist σ_i to x_i.

We define the *reduced Burau representation* as follows:

$$\begin{aligned} \mathrm{B}_n &\to \mathrm{Aut}(H_1(\widetilde{D_n^0}; \mathbb{Z})) \\ f &\mapsto \tilde{f}_* \end{aligned}$$

where $\tilde{f}_*$ denotes the unique lift of $f \in \mathrm{B}_n = \mathrm{Mod}(D_n^0)$ to $\widetilde{D_n^0}$ fixing the fiber over the basepoint $p \in \partial D_n$.

Similarly, we define the *unreduced Burau representation* as

$$\begin{aligned} \mathrm{B}_n &\to \mathrm{Aut}(H_1(\widetilde{D_n^0}, \tilde{p}; \mathbb{Z})) \\ f &\mapsto \tilde{f}_* \end{aligned}$$

where $\tilde{p}$ denotes the pre-image of the basepoint p in $\widetilde{D_n^0}$.

Figure 3.8 shows a disk with two punctures which we think of as the i^{th} and $(i+1)^{st}$ punctures for the general. The left-hand side of the figure indicates a standard generating set for $\pi_1(D_n^0, p)$ consisting of loops x_i, for $i \in \{1, \ldots, n\}$, where each loop travels clockwise around the i^{th} puncture. The right-hand side of Figure 3.8 shows the effect of the half-twist σ_i on the loops x_i and x_{i+1}:

$$\begin{aligned} x_i &\mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\mapsto x_i \end{aligned}$$

Considering our "sliced disk model" for the infinite cyclic cover, we see that lifting the loop $x_i x_{i+1} x_i^{-1}$ to $\widetilde{D_n^0}$ corresponds to the following element:

$$(t^0 \cdot x_i) + (t^1 \cdot x_{i+1}) - (t^1 \cdot x_i) = (1-t)x_i + t x_{i+1}$$

On the left-hand side of the equation above, we are emphasizing the fact that the coefficient of the form t^k records the fact that a lift of a given loop x_j is occuring in "level k" of the infinite cyclic cover. If we consider σ_i as acting on the right, this calculation gives us the i^{th} row of the unreduced Burau representation written as an $n \times n$ matrix over $\mathbb{Z}[t^{\pm 1}]$. The $(i+1)^{th}$ row is even simpler to work out, and we obtain:

$$\sigma_i \mapsto \mathrm{Id}_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus \mathrm{Id}_{n-i-1}\,.$$

The notation here indicates that the 2×2 block occurs along the diagonal in the $i, i+1$ position, and the rest of the matrix is the identity. This is perhaps a more familiar definition of the unreduced Burau representation of the braid group B_n.

To recover the double cover viewpoint we had previously utilized, one can simply take two copies of the "sliced disk", corresponding to t^0 and t^1. We begin by glueing the t^0-level to the t^1-level as before. Then, instead of glueing the t^1-level to the t^2-level as in the construction of the infinite cyclic cover, we replace t^2 with t^0. In other words, set $t^2 = 1$. Doing so results in a torus with one boundary component, as shown below in Figure 3.9.

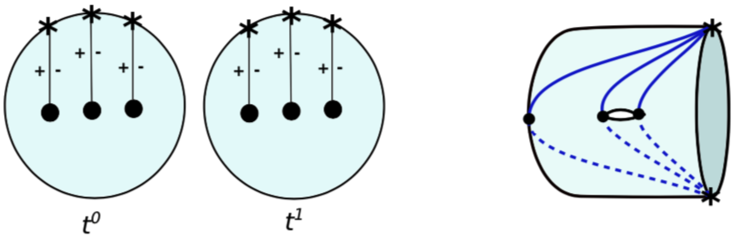

Figure 3.9. *Left:* Two copies of D_3^0, where dots correspond to branch points and $*$ is the basepoint; *Right:* The two copies are now glued together to form a torus.

Exercise 4. Perform this glueing construction carefully. As indicated in Figure 3.9, one needs to “pinch” the endpoints of the slices in each copy of the disk to a single point, so that each arc that we are cutting along in each copy of the disk has a copy of a branch point on one end, and the basepoint $* \in D_n$ at the other.

It turns out that our symplectic representation of B_n is (up to conjugacy) nothing other than the Burau representation with $t=-1$. It is a theorem due to Squier that if we consider t as a complex parameter with $|t|=1$, then the Burau representation is unitary [53].

Exercise 5. Prove the following result from linear algebra: if a matrix is unitary with respect to a non-degenerate hermitian form h, then the imaginary part of h is a symplectic form.

Gambaudo-Ghys work out carefully how to recover the topological version of the Burau representation at $t=-1$, that is, the symplectic representation β defined above, from the algebraic description arising from Squier’s work [21]. We note that they refer to this representation of B_n as the *Burau-Squier* representation.

Braid congruence groups. Our goal in what follows is to study the kernels of these symplectic representations. To that end, we introduce the following notation:

$$\begin{aligned} \mathcal{BI} &:= \ker\beta \\ \mathrm{B}_n[m] &:= \ker\beta_m \end{aligned}$$

The notation $\mathcal{BI}$ indicates “braid Torelli”. By analogy with congruence subgroups of mapping class groups, we will refer to the group $\mathrm{B}_n[m]$ as the *level-m braid congruence group*. We will also understand the “level zero” braid congruence group $\mathrm{B}_n[0]$ to be just the braid Torelli group $\mathcal{BI}$.

Problem 3.1. *Characterize the level-m braid congruence group* $\mathrm{B}_n[m]$*.*

Of course, this problem is somewhat vague as stated. Given that our viewpoint is based on mapping class groups of surfaces, our specific goal will be to find topological characterizations of $\mathrm{B}_n[m]$ that are somehow intrinsic to the disk D_n.

4. Level 2: Pure Braids and Dehn Twists

The case $m=2$ turns out to be the key to understanding all three level-m braid congruence groups that are the focus of these lecture notes. Indeed, it turns out that $\mathrm{B}_n[2]$ is a familiar and well-studied group.

Recall that the *pure braid group* PB_n is just the kernel of the map from the braid group B_n to the symmetric group on n objects that records the permutation of marked points induced by a braid. It is a well-known result of Artin [4] that PB_n is generated by Dehn twists in

D_n. Indeed, he proved much more, including the fact that PB_n is finitely generated by such elements. These elements are commonly denoted A_{ij}, for $1 \le i < j \le n$. As a geometric braid this is usually depicted as the j^{th} strand of a braid crossing in front of the other strands and "hooking around" the i^{th} strand before returning to its original position, as shown in Figure 4.1.

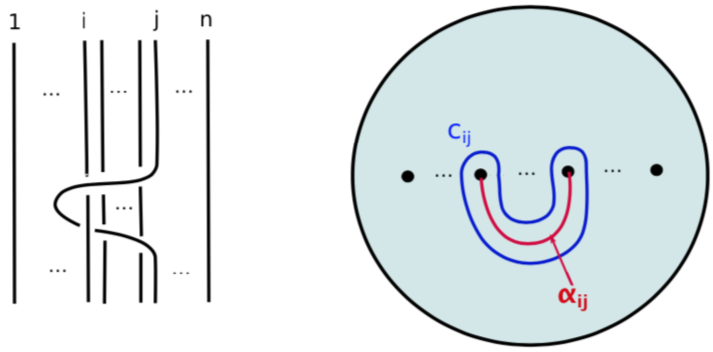

Figure 4.1. *Left:* The pure braid generator A_{ij} as a geometric braid; *Right:* The Dehn twist $T_{c_{ij}}$ and the square of the half-twist $h_{\alpha_{ij}}$ both correspond to the braid A_{ij}.

In the disk model, as an element of $\mathrm{Mod}(D_n)$, this is nothing other than a Dehn twist about the 2-curve c_{ij} as shown. Note that $T_{c_{ij}}$ is just the square of the half-twist corresponding to the arc α_{ij} contained in the interior of the disk bounded by c_{ij}, joining the i^{th} marked point to the j^{th} marked point as shown in Figure 4.1.

Approximately two decades after Artin's description of the pure braid group PB_n appeared, Arnol'd gave a different characterization of PB_n as the level-2 braid congruence group. We briefly sketch his argument here; for full details see his original paper [3], particularly Lemma 1 and subsequent discussion, or see Brendle-Margalit [14, Section 2].

Let X_n denote the surface S_g^1 if $n = 2g + 1$, or the surface S_g^2 if $n = 2g + 2$. In either case, as described in our discussion of the Birman-Hilden correspondence in Section 3, we consider the hyperelliptic involution ι as a branched double cover of X_n over the disk D_n. For our current purposes it will be useful to consider the punctured disk D_n^0 obtained from the marked disk D_n by removing the n marked points.

We define a map $H_1(X_n; \mathbb{Z}/2) \to H_1(D_n^0; \mathbb{Z}/2)$ as follows. Suppose γ is a simple closed curve in X_n that represents a cycle $H_1(X_n; \mathbb{Z}/2)$. We can modify γ by a homotopy to ensure that it avoids branch points, and then project to D_n^0 in order to obtain a representative of an element of $H_1(D_n^0; \mathbb{Z}/2)$. In our Birman-Hilden dictionary (Figure 3.5), γ might be a symmetric representative of the isotopy class of c, so we could replace γ by the curve labeled c (or we could choose instead the curve labeled c', or many other possibilities!), which in turn projects to the 2-curve $\bar{c}$ in D_n^0. Arnol'd proved that this process is independent of the choices involved.

Arnol'd further proved that the image of this map is the subspace of $H_1(D_n^0; \mathbb{Z}/2)$ consisting of those elements that have an even number of nonzero coordinates in a standard homology basis; we denote this subspace by $H_1(D_n^0; \mathbb{Z}/2)^{\mathrm{even}}$. Keeping in mind the Birman-Hilden dictionary of Figure 3.5, the key observation here is that the boundary of a disk containing a branch point in X_n maps to an odd curve, specifically a 1-curve in D_n^0; such a curve is nullhomologous working over $\mathbb{Z}/2$.

Now, the isomorphism $H_1(X_n; \mathbb{Z}/2) \to H_1(D_n^0; \mathbb{Z}/2)^{\mathrm{even}}$ is B_n-equivariant. The braid group B_n acts in the obvious way on $H_1(D_n^0; \mathbb{Z}/2)^{\mathrm{even}}$ via D_n, and the elements of B_n that act trivially on $H_1(D_n^0; \mathbb{Z}/2)^{\mathrm{even}}$ are precisely those that fixed each of the n marked points of D_n, i.e., the pure braid group PB_n. The braid group B_n also acts on $H_1(X_n; \mathbb{Z}/2)$ via the Birman-Hilden

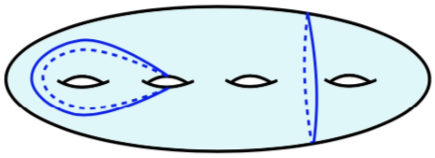

Figure 5.1. Two symmetric separating curves.

correspondence, and by definition the elements of B_n acting trivially are precisely those in the level 2 congruence group $\mathrm{B}_n[2]$. Hence $\mathrm{B}_n[2]$ is nothing other than the pure braid group PB_n.

Putting Arnold's result together with that of Artin, we have the following characterization of the level 2 braid congruence group.

Theorem 4.1 (Artin, Arnol'd)**.** *The braid congruence group* $\mathrm{B}_n[2]$ *is normally generated by Dehn twists. In fact,* $\mathrm{B}_n[2]$ *is normally generated by Dehn twists about 2-curves.*

For a subset R of a group G, we let $\mathcal{N}_G\langle R\rangle$ denote the normal closure of the set R in G, that is, the smallest normal subgroup of G containing R; we will drop the subscript G when this is clear from context. Using this notation, we can rewrite the statement of Theorem 4.1 as follows:

$$\mathrm{B}_n[2] = \mathrm{PB}_n = \mathcal{N}_{\mathrm{B}_n}\langle T_c \mid c \text{ is a simple closed curve in } D_n\rangle.$$

The results we have gathered in the case of the level-2 braid congruence group turn out to be important in understanding levels 0 and 4, as we shall see.

5. Level 0: Braid Torelli

We will next apply the Arnol'd-Artin characterization of the level-2 braid congruence group $\mathrm{B}_n[2]$ to the level-0 braid congruence group, that is, to the braid Torelli group $\mathcal{BI}$. We will exploit the Birman-Hilden viewpoint in order to draw on our knowledge of the Torelli group $\mathcal{I}(S)$. In other words, our starting point will be the hyperelliptic Torelli group $\mathcal{SI}(S)$, that is, the intersection of the Torelli group $\mathcal{I}(S)$ and the hyperelliptic mapping class group $\mathrm{SMod}(S)$.

We first consider how basic elements of the Torelli group can be realized symmetrically with respect to our hyperelliptic involution ι.

Symmetric separating twists. If c is a separating curve in the surface S that is also symmetric with respect to ι, then $T_c \in \mathcal{SI}(S)$. We refer to such an element as a *symmetric separating twist*. Examples of such curves are shown in Figure 5.1.

Symmetric SIP. If a and b are two symmetric curves in S that together form a simply intersecting pair, then the commutator $[T_a, T_b] \in \mathcal{SI}(S)$. We refer to such an element as a *symmetric SIP map*. An example of the curves involved in a symmetric SIP are shown in Figure 5.2.

Symmetrized SIP. Suppose that a and b form a simply intersecting pair, and let a' denote $\iota(a)$ and similarly let b' denote $\iota(b)$. If $a' \cup b'$ is disjoint from $a \cup b$, then the product of the two SIP-maps $[T_a, T_b][T_{a'}, T_{b'}]$ lies in $\mathcal{SI}(S)$. We call this element a *symmetrized SIP map*; see Figure 5.2.

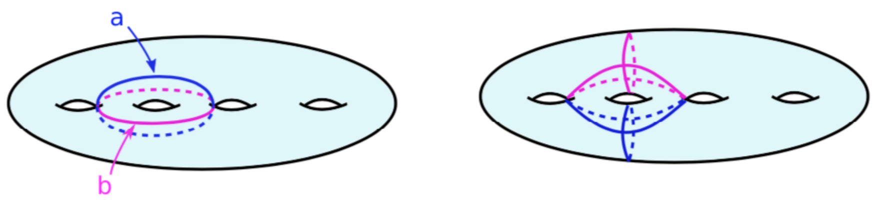

Figure 5.2. *Left:* a symmetric SIP; *Right:* a symmetrized SIP

Non-example: BP maps. Consider the BP map $T_a T_{a'}^{-1}$ indicated in Figure 3.3. Despite the symmetric nature of the two curves, this map is not an element of $\mathcal{SI}(S)$, since the hyperelliptic involution ι conjugates $T_a T_{a'}^{-1}$ to its inverse.

Hain-Morifuji Conjecture. It was a conjecture of Hain [23, Conjecture 1], also implicit in work of Morifuji [41, Section 4], that the group $\mathcal{SI}(S)$ is generated by symmetric separating twists for all $g \geq 0$. This conjecture was proven by Brendle-Margalit-Putman, and we state here a formulation of this result in terms of braid groups.

Theorem 5.1. [11, Theorem C] *For $n \geq 1$, the group $\mathcal{BI}_n$ is normally generated by squares of Dehn twists about odd curves in D_n.*

Using the notation introduced in the previous section, we can reformulate this as follows:

$$\mathrm{B}_n[0] = \mathcal{BI}_n = \mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \mid c \text{ is an odd simple closed curve in } D_n\rangle.$$

In fact, it turns out that for any n, it suffices to include only 3-curves and 5-curves in the normal generating set for $\mathcal{BI}_n$. However, as observed by Fullarton, one cannot pare the list down further due to constraints imposed by abelianization; see the discussion following Theorem C in Brendle-Margalit-Putman [11].

Johnson kernel. The symplectic representation ρ of the mapping class group $\mathrm{Mod}(S_g)$ records the action of a mapping class group on homology $H_1(S_g, \mathbb{Z})$, which is just $\pi_1(S)$ modulo its commutator subgroup. The Johnson homomorphism extends this idea, recording the action on $\pi_1(S_g)$ modulo the second term in its lower central series (up to conjugation). More precisely, Johnson homomorphism is a map from the Torelli group $\mathcal{I}(S_g)$ onto (a quotient of) the triple wedge product of $H_1(S_g, \mathbb{Z})$; for details two good references are Johnson's first paper on this topic [29] or Farb-Margalit's overview of Johnson's work [18, Section 6.6].

Exercise 6. Show that any element of $\mathcal{SI}(S)$ must lie in the kernel of the Johnson homomorphism. *HINT: One can establish this without knowing much about the Johnson homomorphism. The key points are:*

1. *the fact that our hyperelliptic involution, by definition, acts by* $-\mathrm{Id}$ *on homology; and*

2. *the parity of the number of factors in our wedge product target.*

Johnson proved the deep theorem that the kernel of the Johnson homomorphism is precisely the normal subgroup of $\mathrm{Mod}(S_g)$ generated by Dehn twists about separating simple closed curves [31]. Hence we conclude from Exercise 6 above that any element of $\mathcal{SI}(S)$ can be expressed as a product of separating twists. The Hain-Morifuji conjecture further asserts that this can be done symmetrically.

From a purely group theoretic perspective, the conjecture may seem highly implausible: if one knows set of generators R for a group G, one doesn't normally find a generating set for a subgroup H of G simply by selecting those generators from the set R that happen to lie in the subgroup H. For example, $\mathrm{Mod}(S_g)$ is generated by the set of all Dehn twists. But the set of all Dehn twists that happen to lie in the Torelli group is just the set of Dehn twists about all separating curves. This subset generates the Johnson kernel, an infinite index subgroup of the Torelli group.

The starting point for understanding $\mathcal{BI}_n$ (and hence $\mathcal{SI}(S)$) is Arnol'd's result, stated as part of Theorem 4.1 above, that $\mathrm{B}_n[2] = \mathrm{PB}_n$, together with a result due to A'Campo [1, Théorème 1] stating that the image of PB_n under the symplectic representation $\beta : \mathrm{B}_n \to \mathrm{Sp}(2g, \mathbb{Z})$ is precisely the principal congruence group $\mathrm{Sp}[2]$. (Arnold's result tells us only that $\beta(\mathrm{PB}_n)$ is contained in $\mathrm{Sp}[2]$.) Piecing together these results, we obtain the following commutative diagram.

(5.1)
$$\begin{tikzcd}
\mathcal{BI} \arrow[d, hook] \arrow[dr, hook] & & \\
\mathrm{PB}_n \arrow[r, hook] \arrow[d, two heads] & \mathrm{B}_n \arrow[r, two heads] \arrow[d, "\beta"] & \Sigma_n \arrow[d, hook] \\
\mathrm{Sp}[2] \arrow[r, hook] & \mathrm{Sp}(2g, \mathbb{Z}) \arrow[r, two heads] & \mathrm{Sp}(2g, \mathbb{Z}/2)
\end{tikzcd}$$

From this commutative diagram, we have that $\mathrm{PB}_n/\mathcal{BI} \cong \mathrm{Sp}[2]$.

Let θ denote the subgroup of the pure braid group PB_n normally generated by squares of Dehn twists about odd curves. Since $\theta < \mathcal{BI}$, we obtain a quotient map π:

$$\mathrm{PB}_n/\theta \xrightarrow{\pi} \mathrm{PB}_n/\mathcal{BI} \xrightarrow{\cong} \mathrm{Sp}[2]. \tag{5.2}$$

If we had a sufficiently nice presentation of $\mathrm{Sp}[2]$, that is, a presentation that was obviously the right kind of quotient of PB_n, then we'd be done.

However, we don't have this *a priori*, and so instead we use the action of $\mathrm{Sp}[2]$ on the *complex of lax isotropic bases*, an abstract simplicial complex where vertices correspond (roughly) to certain partial symplectic basis for $H_1(S_g; \mathbb{Z}/2)$. We give a brief sketch here; see [11, Section 3] for details. First, we invoke a theorem of Putman [48] that enables us to write $\mathrm{Sp}[2]$ as a quotient of free products of vertex stabilizer subgroups in $\mathrm{Sp}[2]$. We can then restrict the quotient map π defined above to a corresponding subgroups of PB_n modulo θ that stabilize a curve in the disk D_n. Cutting along this curve allows us to reduce to disks with fewer marked points. The proof then proceeds by induction and requires a Birman Exact Sequence for the hyperelliptic Torelli group [13, Theorem 4.2].

6. Level 4

Before we begin our discussion of the level-4 braid congruence group $\mathrm{B}_n[4]$, we briefly recap our key results on braid congruence groups so far. Our first result on level-2 is due to Arnol'd and Artin:

$$\mathrm{B}_n[2] = \mathrm{PB}_n = \mathcal{N}_{\mathrm{B}_n}\langle T_c \mid \ c \text{ is any simple closed curve in } D_n \ \rangle$$

where $\mathcal{N}\langle - \rangle$ denotes the normal closure of the group generated by the types of elements indicated. (Equivalently, we could take c to be any 2-curve in the above.) The result above was then used by Brendle-Margalit-Putman to characterize the level-0 / braid Torelli case:

$$\mathrm{B}_n[0] = \mathcal{BI}_n = \mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \mid \ c \text{ is an odd curve in } D_n \ \rangle$$

As noted in Section 5, one does not in fact need all odd-curves; one can normally generate $\mathcal{BI}_n$ using just 3-curves and 5-curves.

Exercise 7. Recall the following elementary exercise from group theory: any group in which every nontrivial element has order 2 is necessarily abelian.

Now, if G is a group, we denote by G^2 the subgroup of G generated by the squares of all the elements. It follows from Exercise 7 that G/G^2 is universal among 'mod two abelian quotients' of G, that is, abelian quotients of G in which every nontrivial element has order two. With this notation in hand, we can state the following characterization of the level-4 braid congruence group.

Theorem 6.1 (Brendle-Margalit [14])**.**

$$\mathrm{B}_n[4] = PB^2 = \mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \mid c \text{ is any simple closed curve in } D_n\rangle$$

In other words, the level-4 braid congruence group arises both as the kernel of the universal mod two abelianization of the pure braid group, and as the subgroup generated by squares of Dehn twists. Note the latter equivalence in the statement is not immediate; *a priori* the subgroup generated by squares of Dehn twists could be a proper subgroup of the subgroup of the pure braid group generated by all squares.

Abelianization of $\mathbf{PB_n}$. We begin our sketch of the proof of Theorem 6.1 with a description of the abelianization of the pure braid group PB_n.

Recall from Section 4 that Artin gave a finite generating set $\{A_{ij} \mid 1 \le i < j \le n\}$ for PB_n consisting of $\binom{n}{2}$ elements. One can then use, say, the presentation given by Birman [9, Lemma 1.8.2] or Artin's original presentation [4] to deduce the abelianization of PB_n. The result is just $\binom{n}{2}$ copies of $\mathbb{Z}$, with each pure braid generator A_{ij} generating one of the $\binom{n}{2}$ summands in the abelianization. We can further reduce modulo any integer ℓ in order to obtain a family of finite quotients of PB_n. We will denote the abelianization map by α, and its further mod ℓ reduction by α_ℓ. We will be particularly concerned with the case $\ell = 2$, as summarized in the following diagram.

$$\begin{tikzcd} \mathrm{PB}_n \arrow[r, "\alpha"] \arrow[rr, bend right, "\alpha_2"'] & \mathbb{Z}^{\binom{n}{2}} \arrow[r, "\mathrm{mod}", "2"'] & (\mathbb{Z}/2)^{\binom{n}{2}}. \end{tikzcd} \tag{6.1}$$

We will prove both of the equalities in the statement of Theorem 6.1 using the characterization of PB_n^2 as the kernel of α_2; see Exercise 7 above. For brevity, in what follows we will use the following simplified notation:

$$\mathcal{N}\langle T_c^2\rangle := \mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \mid c \text{ is any simple closed curve in } D_n\rangle.$$

We prove the second equality first. Note that the image of $\mathcal{N}\langle T_c^2\rangle$ under α is $2\mathbb{Z}^{\binom{n}{2}}$. Further, the full pre-image of $2\mathbb{Z}^{\binom{n}{2}}$ under α is PB_n^2. We already know that $\mathcal{N}\langle T_c^2\rangle < \mathrm{PB}_n^2$. In order to obtain the reverse inclusion, it suffices to show that $\mathcal{N}\langle T_c^2\rangle$ contains the kernel of α, that is, the commutator subgroup of the pure braid group PB_n. We can prove this via a case-by-case examination of the different commutators of Artin's generators, according to the respective topological types of the curves involved in the Dehn twists A_{ij}. There is only one case that is not straightforward to check, namely $[A_{12}, A_{23}]$. Note that any commutator $[a, b]$ can be realized as a product of squares, for example as follows:

$$[a, b] = aba^{-1}b^{-1} = (a^2)(a^{-1}b)^2b^{-2}.$$

In this case, however, we are able to use the Squared Lantern Relation given in Figure 6.1 to write the commutator $[A_{12}, A_{23}]$ explicitly as the product of squares of Dehn twists rather than squares of products of twists.

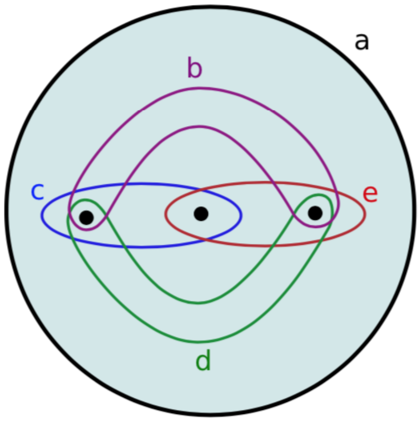

Figure 6.1. The squared lantern relation: $[T_c, T_d] = T_c^2 T_b^2 T_e^2 T_a^2$.

We now establish the first equality by showing that $\mathrm{B}_n[4]$ is in fact equal to PB_n^2, the kernel of α_2. Consider the following sequence of maps.

$$\begin{array}{ccccc} \mathrm{PB}_n & \xrightarrow{\beta} & \mathrm{Sp}(2g,\mathbb{Z})[2] & \longrightarrow & \mathfrak{sp}_{2g}(\mathbb{Z}/2) \\ & & \mathrm{Id}+2A & \longmapsto & A \end{array} \tag{6.2}$$

The group $\mathfrak{sp}_{2g}(\mathbb{Z}/2)$ is the additive group of persymmetric matrices, that is, matrices that are symmetric along the anti-diagonal. The first map is just the restriction of the symplectic representation β to PB_n; recall from Section 5 that A'Campo has shown that the map shown here is surjective. The second map is defined as in the diagram and is well known to be surjective; see for example [42, Chapter VII, 33-34].

Consider the kernel of the composition of these two maps. On the one hand, this kernel is clearly the level-4 braid congruence group $\mathrm{B}_n[4]$, since the kernel of the second map is just $\mathrm{Sp}(2g,\mathbb{Z})[4]$. On the other hand, it is easy to check that the kernel of the composition is just the kernel of the mod two abelianization map α_2, since as an abelian group we have that $\mathfrak{sp}_{2g}(\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{\binom{n}{2}}$; to see this, consider the constraints on matrix entries imposed by the persymmetric condition.

7. Connections and applications

We will end these notes by recording some relationships between various braid congruence subgroups we have studied, as well as congruence subgroups of the symplectic group, and well known subgroups of braid groups such as point pushing groups and Brunnian braid groups.

Forgetful maps. Let $p_1, \cdots, p_n$ denote the n punctures in the disk D_n. If we "fill in" one of the punctures, say, p_i, we get a 'forgetful map' F_i between corresponding mapping class groups, with kernel corresponding to all possible ways to "push" the point p_i around the disk, that is, all possible loops based at p_i. This gives rise to a version of the Birman Exact Sequence [9, Section 4.1] (or see discussion in Farb-Margalit [18, Section 4.2]) as follows.

$$1 \longrightarrow \pi_1(D_n, p_i) \longrightarrow \mathrm{PB}_n \xrightarrow{F_i} \mathrm{PB}_{n-1} \longrightarrow 1$$

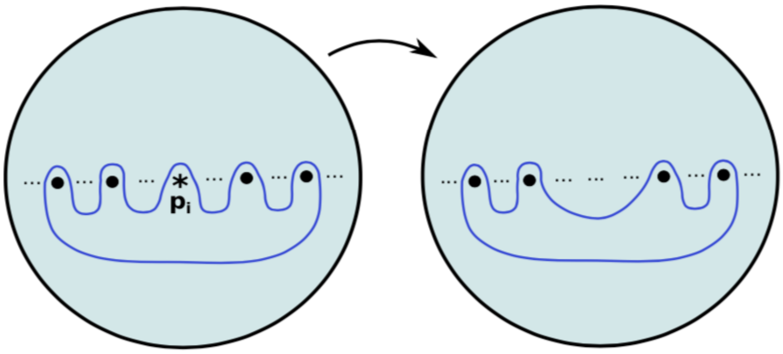

Figure 7.1. The given $(2h)$-curve on the right is the image of the odd curve on the left under the "forget p_i" map.

We can extend our notion of a forgetful map to any subset $\mathcal{S} \subseteq \{1, \ldots, n\}$, and define a map $F_{\mathcal{S}} : \mathrm{PB}_n \to \mathrm{PB}_{n-|\mathcal{S}|}$ by filling in all punctures p_i for $i \in \mathcal{S}$. The following proposition relates level zero braid congruence groups to level four braid congruence groups via forgetful maps.

Proposition 7.1. *For any $\mathcal{S} \subseteq \{1, \ldots, n\}$, we have $F_{\mathcal{S}}(\mathcal{BI}_n) = \mathrm{B}_k[4]$, where $k = n - |\mathcal{S}|$.*

The proof is elementary now that we have our topological characterizations of $\mathcal{BI}_n$ and $\mathrm{B}_n[4]$.

Proof. We have the following sequence of inclusions and equalities:

$$F_{\mathcal{S}}(\mathcal{BI}_n) \subseteq F_{\mathcal{S}}(\mathrm{B}_n[4]) = F_{\mathcal{S}}(\mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \rangle) \subseteq \mathcal{N}_{\mathrm{B}_k}\langle T_c^2 \rangle = \mathrm{B}_k[4].$$

The equalities all follow from Theorem 6.1. The first inclusion follows from the fact that $\mathcal{BI}_n \subseteq \mathrm{B}_n[m]$ for all integers m. The second inclusion is due to the fact that Dehn twists always map to (possibly trivial) Dehn twists under any forgetful map.

To see the reverse inclusion, suppose $T_c^2 \in \mathrm{B}_k[4]$. If c is an odd curve in D_k, then under the forgetful map $F_{\mathcal{S}}$, the twist T_c is the image of the twist about an odd curve $\tilde{c}$ in D_n that surrounds precisely the same "unforgotten" punctures, and hence $T_c^2 = F_{\mathcal{S}}(T_{\tilde{c}}^2)$. If instead c is a $(2h)$-curve in D_k for some integer h, then T_c is the image under $F_{\mathcal{S}}$ of the twist about a $(2h+1)$-curve $\tilde{c}$ surrounding the same $2h$ unforgotten punctures as well as one additional forgotten puncture p_i for $i \in \mathcal{S}$; see Figure 7.1. Hence by Theorems 6.1 and 5.1 we have:

$$\mathrm{B}_k[4] \subseteq F_{\mathcal{S}}(\mathcal{N}_{\mathrm{B}_n}\langle T_c^2 \mid c \text{ an odd curve } \rangle) = F_{\mathcal{S}}(\mathcal{BI}_n).$$

□

More on point pushing subgroups. As a further application, we can prove the somewhat curious fact that, under the symplectic representation ρ, the image of point pushing subgroups in always contains the level four congruence subgroup of the symplectic group. Indeed, we can say a bit more, and give a sample result here. The first statement in this theorem is due to Yu [58, Theorem 7.3(iii)], and the second is due to Brendle-Margalit [12]. Recall from Section 3 that the notation $(\mathrm{Sp}(2g+2, \mathbb{Z})[m])_\partial$ denotes the subgroup stabilizing the vector corresponding to one of the two boundary components of S_g^2.

Theorem 7.2 (Yu, Brendle-Margalit)**.** *Let $g \geq 2$.*

1. *If $n = 2g+1$, then $\rho(\pi_1(D_n, p_i))$ contains $\mathrm{Sp}(2g, \mathbb{Z})[4]$ and we have*

$$\rho(\pi_1(D_n, p_i)) / \mathrm{Sp}(2g, \mathbb{Z})[4] \cong (\mathbb{Z}/2)^{2g}$$

2. *If $n = 2g + 2$, then $\rho(\pi_1(D_n, p_i))$ contains* $(\mathrm{Sp}(2g+2,\mathbb{Z})[4])_\partial$ *and*

$$\rho(\pi_1(D_n, p_i))/(\mathrm{Sp}(2g+2,\mathbb{Z})[4])_\partial \cong (\mathbb{Z}/2)^{2g+1}$$

In particular, this theorem answers a question of user "JSE" posed on MathOverflow [32]. A more general version of Theorem 7.2 can be found in [12], correcting a misstated version of this result given in the original paper [14]. The proof uses A'Campo's result described in Section 5 that PB_n surjects onto $\mathrm{Sp}(2g,\mathbb{Z})[2]$, a result due to Mennicke [39] that the level m congruence subgroup $\mathrm{Sp}(2g,\mathbb{Z})[m]$ is generated by m^{th} powers of transvections, and a symmetric version of homology realization (which was described in Section 3 above).

One can use Theorem 7.2 to compute the indices of $\rho(\pi_1(D_n, p_i))$ in $\mathrm{Sp}(2g,\mathbb{Z})[2]$ and $(\mathrm{Sp}(2g+2,\mathbb{Z})[2])_\partial$, for the different parity cases respectively, using the facts that

$$[\mathrm{Sp}(2g,\mathbb{Z})[2] : \mathrm{Sp}(2g,\mathbb{Z})[4]] = 2^{\binom{2g+1}{2}}$$

and

$$[(\mathrm{Sp}(2g+2,\mathbb{Z})[2])_\partial : (\mathrm{Sp}(2g+2,\mathbb{Z})[4])_\partial] = 2^{\binom{2g+2}{2}}.$$

For example, when $n = 2g + 1$, we have:

$$[\mathrm{Sp}(2g,\mathbb{Z})[2] : \rho(\pi_1(D_n, p_i))] = 2^{g(2g+1)} - 2^{2g}.$$

Remarks on Brunnian braids. A *Brunnian braid* is a (pure) braid that becomes trivial when any one of its n strands is deleted. The set of all Brunnian braids forms a subgroup of PB_n, denoted Brun_n, and can be defined in terms of point pushing subgroups as follows:

$$\mathrm{Brun}_n := \pi_1(D_n, p_1) \cap \cdots \cap \pi_1(D_n, p_n).$$

Finally, it follows that $\rho(\mathrm{Brun}_3)$ has infinite index in $\mathrm{SL}(2,\mathbb{Z})$; again, see [12] for a correction of a misstatement in [14]. Let Z denote the center of B_3; this is an infinite cyclic group generated by a Dehn twist about the boundary of the disk D_n. The group Z is also the kernel of the symplectic representation $\rho : \mathrm{B}_3 \to \mathrm{SL}(2,\mathbb{Z})$ (recalling that $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$). Let $M = \rho(\sigma_1)$. No element of the coset $\sigma_1^k Z$ is Brunnian, and hence no power of M lies in $\rho(\mathrm{Brun}_3)$.

We remark that Cohen-Wu [15] previously showed that, in contrast to the previous result, the group of Brunnian 3-braids on the sphere rather than the disk, specifically the Brunnian subgroup of $\mathrm{Mod}(S^0_{0,3})$, maps isomorphically onto $\mathrm{SL}(2,\mathbb{Z})[4]$ under the symplectic representation.

8. Further directions

The beauty of braid groups is that they lie at the intersection of many different fields, and as we have seen, braid congruence groups are similarly pervasive in mathematics. In these notes we have described them from mainly the viewpoint of geometric group theory, via mapping class groups, but along the way we have encountered work coming from areas such as algebra and representation theory (e.g. Squier [53], Newman [42], Cohen-Wu [15]), 4-manifold topology (Gambaudo-Ghys [21]), algebraic geometry (Hain [23]), and number theory (Yu [58]). Each of these viewpoints represents a promising direction for further study of braid congruence groups; see the introduction of [14] for additional references.

In these notes, we have focused exclusively on the braid congruence groups $\mathrm{B}_n[m]$ for just three values of m: $0, 2$, and 4, and we have focused on basic questions such as generation. There is no doubt much more to say about each of these groups. Indeed, Kordek-Margalit have already vastly expanded our understanding of $\mathrm{B}_n[4]$, recently proving a number of results concerning the cohomology of $\mathrm{B}_n[4]$ and its representation theory [33].

We are also starting to better understand $\mathrm{B}_n[m]$ for other values of m. Stylianakis has given a topological interpretation of generators for $\mathrm{B}_n[3]$, along with several other related results for $\mathrm{B}_n[p]$ where p is a prime number [54]. The work of Stylianakis is largely based on Wajnryb's presentation [57] for $\mathrm{Sp}[\mathbb{Z}/p]$ for primes $p \geq 3$, given as a quotient of the pure braid group, which in turn incorporates work of Sunday [55] and Assion [6]. Building on work of Stylianakis, Damiani-McLeay-Stylianakis have announced some preliminary results on crystallographic quotients of braid congruence groups, while Appel-Bloomquist-Gravel-Holden [2] recently announced generalisations of some of Stylianakis's results to $\mathrm{B}_n[m]$ for arbitrary m.

In a different direction, our level zero braid congruence group $\mathcal{BI}_n$ arose from studying the Burau representation at the particular value $t = -1$. There is a certain amount of literature relating to different choices of t; we have already mentioned Squier's work, for example. More recently, Scherich has looked at other real specializations of the Burau representation of the three-strand braid group [52]. It would be interesting to study further the kernels arising from different various natural choices of the parameter t.

Finally, in the broader context of geometric group theory, there are natural analogies that arise between mapping class groups and automorphism groups of free groups, providing yet another avenue of exploration. It is well known that braid groups embed in $\mathrm{Aut}(F_n)$ via their action on $\pi_1(D_n)$, and we have somewhat exploited this viewpoint in obtaining some of the results described herein, but no doubt additional insights remain to be gained along these lines. As one example, Fullarton has proven an analogue of the Hain-Morifuji conjecture in the $\mathrm{Aut}(F_n)$ setting [20], where palindromic automorphisms play the role of hyperelliptic mapping classes; it would be interesting to determine what further analogies might hold.

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