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four-dimensional hyperbolic manifolds**

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NEGATIVELY CURVED EINSTEIN METRICS ON RAMIFIED COVERS OF CLOSED FOUR-DIMENSIONAL HYPERBOLIC MANIFOLDS

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ABSTRACT. — This paper is a shortened version of the recent article *Examples of compact Einstein four-manifolds with negative curvature* [11] written in collaboration with J. Fine (ULB). Its content was presented by the author at the *Séminaire de Théorie Spectrale et Géométrie* in Grenoble in December 2017. In [11], new examples of compact, negatively curved Einstein manifolds of dimension 4 have been obtained. These are seemingly the first such examples which are not locally homogeneous. The Einstein metrics we construct are carried by a sequence of 4-manifolds (X_k), previously considered by Gromov and Thurston [13], and obtained as ramified coverings of closed hyperbolic 4-manifolds. Our proof relies on a deformation procedure. We first find an approximate Einstein metric on X_k by interpolating between a model Einstein metric near the branch locus and the pull-back of the hyperbolic metric from the base hyperbolic manifolds. We then perturb to a genuine solution to Einstein's equations, by a parameter dependent version of the inverse function theorem.

1. Introduction and Statement of the results

1.1. State of the art of Compact Einstein manifolds with negative scalar curvature

A Riemannian manifold (M, g) is called *Einstein* if $\mathrm{Ric}(g) = \lambda g$, for some $\lambda \in \mathbb{R}$. This article gives a new construction of compact Einstein 4-manifolds with negative scalar curvature, that is with $\lambda < 0$. Currently known methods for constructing compact Einstein manifolds with $\lambda < 0$ are:

(1) *Locally homogeneous Einstein manifolds.* These are Einstein manifolds whose universal cover is homogeneous, i.e, acted on transitively by isometries. Negatively curved examples include hyperbolic and complex-hyperbolic manifolds.

(2) *Kähler–Einstein metrics.* A compact Kähler manifold with $c_1 < 0$ carries a Kähler–Einstein metric with $\lambda < 0$. This is due to Aubin [2] and Yau [24].

(3) *Dehn fillings of hyperbolic cusps.* Given a finite volume hyperbolic n-manifold with cusps, one can produce a compact manifold by Dehn filling: each cusp is cut off at finite distance to produce a boundary component diffeomorphic to a torus T^{n-1}; this is then filled in by gluing $D^2 \times T^{n-2}$ along their common T^{n-1} boundary. For appropriate choices, the Dehn filling carries an Einstein metric with $\lambda < 0$. When $n = 3$ this is due to Thurston [22] and the Einstein metric is in fact hyperbolic. For $n \geqslant 4$, the Einstein metrics are no longer locally homogeneous. In these dimensions the original idea is due to Anderson [1] and was later refined by Bamler [3] (see also the excellent exposition of Biquard [6]).

Of these three constructions, only the first is known to produce Einstein metrics which are *negatively curved*, i.e., with all sectional curvatures negative. We refer to Fine–Premoselli [11] for the details. It is therefore of great interest to find new constructions of Einstein metrics and, in particular, examples with negative curvature which are not locally homogeneous. We address this question in this article.

1.2. Statement of the results

In [13] Gromov and Thurston investigated pinching for negatively curved manifolds of dimension $n \geqslant 4$. They showed that for any $\epsilon > 0$, there exists a compact Riemannian n-manifold (X, g) with sectional curvatures pinched by $-1 - \epsilon < \sec(h) \leqslant -1$ and which does not admit a hyperbolic metric. The positively curved analogue of this statement is false, see [4, 16, 7].

A natural question, which motivated this work is: *do Gromov and Thurston's manifolds carry Einstein metrics?* We answer this question positively, at least in dimension 4. In order to state our main result we first construct a particular family of hyperbolic manifolds which belong to the class investigated in [13]:

PROPOSITION 1.1. — *For each $n \in \mathbb{N}$, there exists a sequence (M_k) of compact hyperbolic n-manifolds with the following properties.*

(1) *The injectivity radius $i(M_k)$ satisfies $i(M_k) \to \infty$ as $k \to \infty$.*

(2) *For each k, there is a nullhomologous totally-geodesic codimension-2 submanifold $\Sigma_k \subset M_k$. Moreover, the normal injectivity radius of Σ_k satisfies $i(\Sigma_k, M_k) \geqslant \frac{1}{2} i(M_k)$.*

(3) *There is a constant C, independent of k such that for all sufficiently large k, the volume of Σ_k with respect to the hyperbolic metric satisfies*

$$(1.1) \qquad \mathrm{vol}(\Sigma_k) \leqslant C \exp\left(\frac{n^2 - 3n + 6}{4} i(M_k)\right)$$

There actually are infinitely many such sequences (M_k). As will be proven in Section 2, these are obtained as a Spin analogue of the original Gromov–Thurston construction. Since $[\Sigma_k] = 0$, given any fixed integer $l \geqslant 2$, there is an l-fold cover $X_k \to M_k$ branched along Σ_k. One way to see this is to take a hypersurface H_k bounding Σ_k, cut M_k along H_k to produce a manifold-with-boundary M_k'; now take l copies of M_k' and glue their boundaries appropriately. It is important to note that Σ_k may have many separate connected components.

The pull-back of the hyperbolic metric from M_k to X_k is singular along the branch locus, with cone angle $2\pi l$. It is in particular a *singular* Einstein metric since the covering $X_k \to M_k$ is a local isometry outside of the ramification locus. The main result of this article is that, in dimension 4 at least, the manifolds X_k carry *smooth* Einstein metrics:

THEOREM 1.2. — *Fix $l \geqslant 2$ and let (M_k) denote a sequence of compact hyperbolic 4-manifolds satisfying the conclusions of Proposition 1.1. Let X_k be the l-fold cover of M_k branched along Σ_k. For all sufficiently large k (depending on l), X_k carries an Einstein metric of negative sectional curvature which is not locally homogeneous.*

It is not difficult to see that manifolds X_k carry no locally homogenous Einstein metrics whatsoever (we refer to Fine–Premoselli [11] for the details). Theorem 1.2, together with deep 4-dimensional rigidity results for Einstein metrics on compact manifolds, gives another way to see this. Indeed, if a compact 4-manifold is either hyperbolic or complex-hyperbolic then the locally homogeneous metric is the only possible Einstein metric (up to overall scale). This was proved in the hyperbolic case by Besson, Courtois and Gallot [5], whilst the complex-hyperbolic case is due to LeBrun [18].

It is worth mentioning that with Theorem 1.2 we find *infinitely many compact 4-manifolds that carry negatively curved Einstein metrics, but that admit no locally homogeneous Einstein metrics.* This is the first occurrence, in the compact case, of a negatively curved Einstein metric which is not locally homogeneous. *Non-compact* examples are relatively easy to find: an infinite dimensional family of Einstein deformations of the hyperbolic metric – negatively curved when the deformation is small – was found

by Graham and Lee [12], and in 4 dimensions, a 1-parameter family of such deformations with an explicit formula was given by Pedersen [21]. It was recently observed by Cortés and Saha [10] that Pedersen's metrics are also negatively curved, even when far from the hyperbolic metric. Other explicit examples can also be found in [10].

1.3. Outline of the proof

The proof of Theorem 1.2 has two steps. The first, carried out in Section 3, is similar in spirit to that of the tightly pinched Gromov–Thurston metrics. We smooth out the pull-back of the hyperbolic metric from M_k to X_k to find a metric which is approximately Einstein. The larger the injectivity radius, the better we can make the approximation. It is important to note, however, that our approximate Einstein metrics are *not* the same as the tightly pinched metrics that Gromov and Thurston consider in [13]. Inside a tubular neighborhood of Σ_k of width at most the normal injectivity radius we use *a Riemannian Kottler metric* whose sectional curvatures satisfy sec $\leqslant -c$ for some constant $0 < c < 1$ which depends only on l (and not on k). The least negative sectional curvatures, sec $= -c$, are attained at points on the branch locus. At large distances from the branch locus, this metric is asymptotic to the pull-back of the hyperbolic metric. We interpolate between these two metrics at a distance which tends to infinity with k. This gives a metric g_k on X_k which is Einstein everywhere except for an annular region of large radius and fixed width in the tubular neighbourhood of the branch locus. In these annular regions g_k is close to Einstein, with error that can be explicitly controlled in terms of the glueing parameter (and that tends to zero as k tends to infinity).

The second step of the proof is to use the inverse function theorem to prove that for all large k, the approximately Einstein metric g_k can be perturbed to a genuine Einstein metric. This new Einstein metric has sectional curvatures which are very close to those of g_k and so, in particular, are also negative. The analysis involved turns out to be quite delicate. The fact that g_k has negative sectional curvatures leads to the fact that the linearised Einstein equations (in Bianchi gauge) are invertible, with L^2-control. However, the volume and diameter of X_k are rapidly increasing with k and so weighted Hölder spaces, rather than Sobolev spaces, are seemingly the appropriate choice of Banach spaces in which to apply the inverse function theorem. Even with these spaces, however, we are unable to obtain control over the derivative in every direction. We circumvent this

as follows. Since the metric g_k is made by interpolating two Einstein metrics, the error is supported in a subset $S_k \subset X_k$. We reduce the problem in Section 4 to controlling the inverse of the linearised Einstein equations on sections supported in S_k and we state the key estimate on which the proof is based. We prove this estimate in Section 5. Starting from the uniform L^2 control given by the linearised Einstein equations we perform an involved bootstrap procedure and transform it into a weighted Hölder control. This relies on Carleman-type estimates for the Green's operator of the linearised equations and on a precise control of the volume of the branch locus Σ_k provided by (1.1).

We close this brief outline with a comment on dimension. The model Einstein metric exists in all dimensions $n \geqslant 4$ and gives approximately Einstein metrics on Gromov–Thurston manifolds for all $n \geqslant 4$. It is likely that these approximate metrics can be deformed into genuine Einstein metrics in every dimension $n \geqslant 4$. Unfortunately, the control of the volume of the branch locus provided by Proposition 1.1 is only sufficient for our analytic arguments to work in dimension four.

Acknowledgements

2. A spin version of Gromov–Thurston manifolds

In this section we sketch the proof of Proposition 1.1. The original construction of Gromov and Thurston uses arithmetic hyperbolic geometry to produce (M_k) satisfying the first two properties of Proposition 1.1. We review this in the next Subsection 2.1. To bound the volume of Σ_k we will make use of recent work of Murillo [20], which does not apply to all the sequences arising from Gromov–Thurston's original construction, but instead to a special subclass of them. Put loosely, we need the manifolds M_k to be spin.

Following Gromov–Thurston [13], consider the following quadratic form on $\mathbb{R}^{n+1}$:

$$f(x_0, \ldots, x_n) = -\sqrt{2}x_0^2 + x_1^2 + \cdots + x_n^2$$

The corresponding pseudo-Riemannian metric on $\mathbb{R}^{n+1}$ restricts to a genuine Riemannian metric on the hyperboloid $H = \{x : f(x) = -1,\ x_0 > 0\}$. This makes H isometric to hyperbolic space $\mathbb{H}^n$ and gives an identification between the group $\mathrm{SO}(\mathbb{R}^{n+1}, f)$ of orientation-preserving isometries of $(\mathbb{R}^{n+1}, f)$ which are isotopic to the identity, and the group of orientation preserving isometries of $\mathbb{H}^n$.

We write Γ for those automorphisms of f which are defined over the ring of integers $\mathbb{Z}[\sqrt{2}]$. Explicitly,

$$\Gamma = \mathrm{SO}\left(\mathbb{R}^{n+1}, f\right) \cap \mathrm{GL}\left(\mathbb{Z}[\sqrt{2}], n+1\right)$$

It is important that the action of Γ on $\mathbb{H}^n$ is discrete and cocompact. (This is a classical result in the study of arithmetic hyperbolic manifolds; the use of $\sqrt{2}$ is precisely to ensure the action is cocompact.) The quotient $\mathbb{H}^n/\Gamma$ is a compact hyperbolic orbifold with singularities corresponding to the fixed points of Γ. Given $k \in \mathbb{N}$, write $\Gamma_k \leqslant \Gamma$ for the kernel of the homomorphism given by reducing the entries mod k, i.e., the homomorphism

$$\Gamma \hookrightarrow \mathrm{GL}\left(\mathbb{Z}[\sqrt{2}], n+1\right) \to \mathrm{GL}\left((\mathbb{Z}/k\mathbb{Z})[\sqrt{2}], n+1\right)$$

induced by $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. The kernel Γ_k is a finite-index normal subgroup and for any k sufficiently large, $M_k = \mathbb{H}^n/\Gamma_k$ is a compact hyperbolic manifold with injectivity radius $i(M_k) \to \infty$.

The next step is to find the totally geodesic submanifolds $\Sigma_k \subset M_k$. Reflection in the coordinate x_1 gives an isometry of $(\mathbb{R}^{n+1}, f)$. Since the subgroups $\Gamma_k \leqslant \Gamma$ normalise it, this descends to give an isometric involution $r_1 \colon M_k \to M_k$. For each k, the fixed locus is a totally geodesic hypersurface $H_k^1 \subset M_k$. Similarly reflection in x_2 gives an isometric involution $r_2 \colon M_k \to M_k$ with totally geodesic fixed set H_k^2. The fixed sets H_k^1, H_k^2 meet transversely, since their preimages in $\mathbb{H}^n$ are transverse. We let $\Sigma_k = H_k^1 \cap H_k^2$, a totally-geodesic codimension 2 submanifold. By passing twice to double covers if necessary, we can assume that both H_k^1 and H_k^2 separate M_k; then each H_k^i is orientable, and so Σ_k is too; moreover, Σ_k is the boundary of the part of H_k^2 which lies on one side of H_k^1. This shows that $[\Sigma_k] = 0 \in H_{n-2}(M_k, \mathbb{Z})$. It is also easily seen that the normal injectivity radius satisfies $i(\Sigma_k, M_k) \geqslant \frac{1}{2} i(M_k)$ (see [11]).

The important new ingredient is a bound of the form

$$\mathrm{vol}(M_k) \leqslant A e^{C i(M_k)}$$

for constants A and C which are independent of k. It is not difficult to show that for the sequence M_k constructed by Gromov and Thurston, such constants $A(n), C(n)$ exist, depending only on the dimension n. There is a

short self-contained proof of this fact in [14, Section 4.1]. However, for our purposes it is important to know the precise value of $C(n)$.

For the congruence coverings discussed here, Katz–Schaps–Vishne [15] proved that $C(2) = 3/2$ and $C(3) = 3$. (See also the related work [8] of Buser–Sarnack.) For us, of course, the interest is in $n \geqslant 4$. This was treated in a recent article by Murillo [20], which shows that, with a couple of caveats, the optimal inequality has

$$C(n) = \frac{n(n+1)}{4}$$

The first caveat is that Murillo's argument works for sequences of congruence subgroups starting with $\mathrm{Spin}(\mathbb{R}^{n+1}, f)$ rather than $\mathrm{SO}(\mathbb{R}^{n+1}, f)$. The second is that the congruences must be determined by *prime* ideals $I \leqslant \mathbb{Z}[\sqrt{2}]$ and not just reduction modulo an arbitrary integer.

We give a brief description of Murillo's theorem, following the original article closely (where the reader can also find detailed justifications for everything in this section). In order to keep the arithmetic to a minimum, we continue to work with the quadratic form f which is defined over $\mathbb{Q}(\sqrt{2})$ but in fact everything holds much more generally, for admissible quadratic forms defined over totally real number fields. The interested reader can find the details neatly summarised in Murillo's article.

We first give the analogue of $\Gamma \subset \mathrm{SO}(\mathbb{R}^{n+1}, f)$, namely the subgroup $\overline{\Gamma} \leqslant \mathrm{Spin}(\mathbb{R}^{n+1}, f)$ of elements defined over the ring of integers $\mathbb{Z}[\sqrt{2}]$. To do so, we use the Clifford representation of $\mathrm{Spin}(\mathbb{R}^{n+1}, f)$. The group $\mathrm{Spin}(\mathbb{R}^{n+1}, f)$ is a subgroup of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$ and so acts on it by left multiplication. To use this to get a matrix representation of $\mathrm{Spin}(\mathbb{R}^{n+1}, f)$ we fix a basis of $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$. Choose first a basis $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$ with respect to which the innerproduct g_f defined by f is standard:

$$g_f(e_i, e_j) = \begin{cases} -1 & \text{if } i = j = 0 \\ 1 & \text{if } i = j > 0 \\ 0 & \text{if } i \neq j \end{cases}$$

Then $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$ has as basis the 2^{n+1} elements of the form $e_{i_1} \cdots e_{i_r}$ where $i_1 < \cdots < i_r$ and $r = 0, \ldots, n+1$. With respect to this basis, left multiplication by Spin on Cliff gives a faithful representation

$$\rho_L \colon \mathrm{Spin}\left(\mathbb{R}^{n+1}, f\right) \hookrightarrow \mathrm{GL}\left(\mathbb{R}, 2^{n+1}\right)$$

We now set $\overline{\Gamma} \subset \mathrm{Spin}(\mathbb{R}^{n+1}, f)$ to be

$$\overline{\Gamma} = \mathrm{Im}\, \rho_L \cap \mathrm{GL}\left(\mathbb{Z}[\sqrt{2}], 2^{n+1}\right)$$

$\overline{\Gamma}$ has an explicit description in terms of the above basis for $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$. Given $J = (i_1, \ldots, i_r)$ we write $e_J = e_{i_1} \cdots e_{i_r}$ for the corresponding basis element of the Clifford algebra. Then

$$\overline{\Gamma} = \left\{ \gamma = \sum_{|J|\ \mathrm{even}} c_J e_J \in \mathrm{Spin}(\mathbb{R}^{n+1}, f) : c_J \in \mathbb{Z}[\sqrt{2}] \right.$$
$$\left. \mathrm{for}\ J \neq \emptyset,\ c_\emptyset - 1 \in \mathbb{Z}[\sqrt{2}] \right\}$$

$\overline{\Gamma}$ acts on $\mathbb{H}^n$ via

$$\overline{\Gamma} \hookrightarrow \mathrm{Spin}\left(\mathbb{R}^{n+1}, f\right) \to \mathrm{SO}\left(\mathbb{R}^{n+1}, f\right)$$

(where the second arrow is the standard double cover). The crucial fact is that *the resulting action of $\overline{\Gamma}$ is discrete and cocompact* and so $\mathbb{H}^n/\overline{\Gamma}$ is a compact hyperbolic orbifold. (Again this is a foundational fact in the study of arithmetic hyperbolic manifolds.)

We now pass to finite covers. Let $I \subset \mathbb{Z}[\sqrt{2}]$ denote an ideal. We obtain a normal subgroup $\Gamma_I \leqslant \overline{\Gamma}$ as the kernel of the homomorphism

$$\overline{\Gamma} \to \mathrm{GL}\left(\mathbb{Z}[\sqrt{2}]/I, 2^{n+1}\right)$$

Explicitly,

$$(2.1) \qquad \Gamma_I = \left\{ \gamma = \sum_{|J|\ \mathrm{even}} c_J e_J \in \overline{\Gamma} : c_J \in I\ \mathrm{for}\ J \neq \emptyset,\ c_\emptyset - 1 \in I \right\}$$

We are now in a position to state Murillo's Theorem.

THEOREM 2.1 (Murillo's volume bound [20]). — *Let $I_k \subset \mathbb{Z}[\sqrt{2}]$ be a sequence of prime ideals with $|\mathbb{Z}[\sqrt{2}]/I_k| \to \infty$ and write $\Gamma_{I_k} \leqslant \overline{\Gamma}$ for the corresponding normal subgroups of $\overline{\Gamma}$. Then for sufficiently large k, the quotient $M_k = \mathbb{H}^n/\Gamma_{I_k}$ is smooth and there is a constant A such that for all k,*

$$\mathrm{vol}(M_k) \leqslant A \exp\left(\frac{n(n+1)}{4} i(M_k)\right)$$

The key observation in Murillo's proof is to control the hyperbolic displacement of an element $s \in \mathrm{Spin}(\mathbb{R}^{n+1}, f)$ by the size of the coefficient of $e_\emptyset$ in the expression $s = \sum c_J e_J$ of s in terms of the chosen basis of $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$. From here he is able to control the minimal displacement $i(\Gamma_I)$ from below in terms of the cardinality of the quotient $\mathbb{Z}[\sqrt{2}]/I$. At the same time, the index $[\overline{\Gamma} : \Gamma_I]$ can be controlled from above in terms of

this same quantity and, since volume is proportional to index, this leads to Theorem 2.1.

2.1. Proof of Proposition 1.1

We now give the proof of Proposition 1.1. Let $I_k \subset \mathbb{Z}[\sqrt{2}]$ be a sequence of prime ideals as in Murillo's Theorem, let $\Gamma_{I_k} \subset \overline{\Gamma}$ and write $M_k = \mathbb{H}^n/\Gamma_{I_k}$ for the corresponding hyperbolic manifolds. We can for instance take $I_k = p_k \mathbb{Z}[\sqrt{2}]$ for a suitable increasing sequence of prime numbers (p_k). Just as before, $i(M_k) \to \infty$ as $k \to \infty$.

To find the nullhomologous totally geodesic codimension 2 submanifold $\Sigma_k \subset M_k$ we copy the same argument. Recall that the natural action $\rho \colon \mathrm{Spin}(\mathbb{R}^{n+1}, f) \to \mathrm{SO}(\mathbb{R}^{n+1}, f)$ is given by $\rho(s)(v) = svs^{-1}$ where we treat $v \in \mathbb{R}^{n+1}$ as an element of the Clifford algebra, and the product on the righthand side of this formula is the Clifford product. We can also represent reflections in a similar way. Let $e_0, \ldots, e_n$ be a basis of $\mathbb{R}^{n+1}$ for which f is standard and consider the linear transformation r_1 of $\mathrm{Cliff}(\mathbb{R}^{n+1}, f)$ given by $r_1(c) = e_1 c e_1$. Note that for any multi-index J,

$$r_1(e_J) = \begin{cases} -1^{|J|+1} e_J & \text{if } 1 \notin J \\ -1^{|J|} e_J & \text{if } 1 \in J \end{cases}$$

In particular, r_1 preserves $\mathbb{R}^{n+1} \subset \mathrm{Cliff}(\mathbb{R}^{n+1}, f)$ where it acts as reflection in the hyperplane orthogonal to e_1. Moreover, from the description (2.1) of Γ_{I_k}, it follows that $r_1(\Gamma_{I_k}) = \Gamma_{I_k}$. Now for any $s \in \mathrm{Spin}(\mathbb{R}^{n+1}, f)$,

$$r_1\left(\rho(s)(v)\right) = \rho\left(r_1(s)\right)(r_1(v))$$

So r_1 descends to an isometry of M_k, with fixed locus a totally geodesic hypersurface $H_k^1 \subset M_k$. Similarly, there is a second totally geodesic hypersurface $H_k^2 \subset M_k$ coming from reflection orthogonal to e_2. Just as in the Gromov–Thurston construction, we can assume that the H_k^i each separate M_k by passing twice to a double cover if necessary. Then $\Sigma_k = H_k^1 \cap H_k^2$ is totally geodesic and bounds the part of H_k^2 which lies on one side of H_k^1 and so the homology class $[\Sigma_k]$ vanishes. Again, as in the Gromov–Thurston situation, we have that the normal injectivity radius satisfies $i(\Sigma_k, M_k) \geqslant \frac{1}{2} i(M_k)$.

It remains to prove the volume bound (1.1) on Σ_k. We control the volume of Σ_k in two steps. Write $i(H_k^1, M_k)$ for the normal injectivity radius of $H_k^1 \subset M_k$. By considering the volume of an embedded tubular

neighbourhood of H_k^1 of maximal radius we have

$$(2.2) \qquad \mathrm{vol}\left(H_k^1\right) e^{(n-1)i\left(H_k^1, M_k\right)} \leqslant A_1 \, \mathrm{vol}(M_k)$$

where A_1 is independent of k. Similarly, by considering an embedded tubular neighbourhood of Σ_k in H_k^1 of maximal radius, we find a constant A_2 such that

$$(2.3) \qquad \mathrm{vol}\left(\Sigma_k\right) e^{(n-2)i\left(\Sigma_k, H_k^1\right)} \leqslant A_2 \, \mathrm{vol}\left(H_k^1\right)$$

Now $i(\Sigma_k, H_k^1) \geqslant i(\Sigma_k, M_k) \geqslant \frac{1}{2} i(M_k)$ and, similarly, $i(H_k^1, M_k) \geqslant \frac{1}{2} i(M_k)$. Using this and putting (2.2) and (2.3) together we see that

$$(2.4) \qquad \mathrm{vol}(\Sigma_k) \leqslant A_1 A_2 \, \mathrm{vol}(M_k) e^{\frac{3-2n}{2} i(M_k)}$$

The bound 1.1 now follows from (2.4) and Theorem 2.1. (Note that if we passed to double covers to ensure the H_k^i separated and $[\Sigma_k] = 0$, then the volume would at worst quadruple and at worst the injectivity radius is unchanged.)

3. The approximate solution

In this section we give the construction of the approximate solutions to Einstein's equations. Recall that in the previous section we constructed a sequence of hyperbolic n-manifolds (M_k, h_k), each containing a totally geodesic hypersurface $H_k \subset M_k$ whose boundary Σ_k is also totally geodesic. The injectivity radius $i(M_k)$ of M_k tends to infinity with k and $\frac{1}{2} i(M_k)$ is a lower bound for the normal injectivity radius of $\Sigma_k \subset M_k$. We denote by $p \colon X_k \to M_k$ the l-fold cover branched along Σ_k. We abuse notation by using Σ_k to also denote the branch locus in X_k.

Define a function $r \colon X_k \to \mathbb{R}$ by setting $r(x)$ to be the distance of $p(x) \in M_k$ from the branch locus Σ_k. As a notational convenience, we set $u = \cosh(r)$. Write $U_{k, \max} = \cosh(\frac{1}{2} i(M_k))$ and pick a sequence (U_k) which tends to infinity, with $U_k < \frac{1}{2} U_{k, \max}$. The main result of this section is the following.

PROPOSITION 3.1. — *For each k, there is a smooth Riemannian metric g_k on X_k with the following properties:*

(1) *For any $m \in \mathbb{N}$ and $0 \leqslant \eta < 1$, there is a constant A such that for all k,*
$$\|\mathrm{Ric}(g_k) + (n-1)g_k\|_{C^{m,\eta}} \leqslant A U_k^{1-n}$$

(2) *There is a constant $c > 0$ such that for all k, $\sec(g_k) \leqslant -c$.*

(3) $\operatorname{Ric}(g_k) + (n-1)g_k$ is supported in the region $\frac{1}{2}U_k < u < U_k$.
(4) For any $m \in \mathbb{N}$, there exists a constant C such that for all k,
   $\| \operatorname{Rm}(g_k) \|_{C^m} \leqslant C$.

The metric g_k will be given by interpolating in the region $\frac{1}{2}U_k < u < U_k$ between a model Einstein metric defined on a tubular neighbourhood of $\Sigma_k \subset X_k$ and the hyperbolic metric p^*h_k pulled back via the branched cover $p \colon X_k \to (M_k, h_k)$ on the complement of this tubular neighbourhood. In Proposition 3.1 the Hölder norms are defined with respect to the metric g_k (see Definition 4.4 for the explicit definition used in this paper for the Hölder norms). We begin by describing the model.

3.1. The model Einstein metric

Write $(\mathbb{H}^n, h)$ for hyperbolic space of dimension n. Denote by $S \subset \mathbb{H}^n$ a totally geodesic copy of $\mathbb{H}^{n-2}$. We can write h as

$$h = \mathrm{d}r^2 + \sinh^2(r)\mathrm{d}\theta^2 + \cosh^2(r)h_S$$

where h_S is the hyperbolic metric on S. Here, $(r, \theta) \in (0, \infty) \times S^1$ are polar coordinates on the totally geodesic copies of $\mathbb{H}^2$ which are orthogonal to S. The hypersurfaces given by setting θ constant are the totally geodesic copies of $\mathbb{H}^{n-1}$ containing S. In fact, it will be more convenient to use the coordinate $u = \cosh(r)$; the hyperbolic metric then becomes

$$(3.1) \qquad h = \frac{\mathrm{d}u^2}{u^2 - 1} + (u^2 - 1)\mathrm{d}\theta^2 + u^2 h_S$$

This expression is valid for $(u, \theta) \in (1, \infty) \times S^1$.

We will consider a family g_a of Einstein metrics depending on a parameter $a \in \mathbb{R}$. When $a = 0$, we recover h, whilst for $a \neq 0$ the metric has a cone singularity along S, with cone angle varying with a. By an appropriate choice of a, the metric will have the correct cone angle to become smooth when pulled back by an l-fold cover ramified along S.

The metrics we will consider all have the form

$$(3.2) \qquad g = \frac{\mathrm{d}u^2}{V(u)} + V(u)\mathrm{d}\theta^2 + u^2 h_S$$

where V is a smooth positive function.

PROPOSITION 3.2. — *The metric (3.2) solves* $\mathrm{Ric}(g) = -(n-1)g$ *precisely when*

(3.3)
$$V(u) = u^2 - 1 + \frac{a}{u^{n-3}}$$

for some $a \in \mathbb{R}$.

Proof. — This follows from the expression for the curvatures of g. We use the convention that our indices i, j run between $1, \ldots, n-2$. Let f^i be an orthonormal coframe for (S, h_S), and write ω^i_j for the connection matrix of the Levi–Civita connection of h_S, i.e, $\nabla^{h_S} f^i = \omega^i_j \otimes f^j$. Let $W^2 = V$, then

$$e^i = u f^i, \quad e^{n-1} = W^{-1}\mathrm{d}u, \quad e^n = W\mathrm{d}\theta$$

is an orthonormal coframe for g. We use the convention that a Roman index takes the values $1, \ldots, n-2$, whilst a Greek index takes the values $n-1, n$. A standard calculation that we omit here gives the following expression for the curvatures of g:

$$R_{ijkl} = -\frac{1+W^2}{u^2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$
$$R_{i\mu i\mu} = -\frac{W'W}{u}$$
$$R_{\mu\nu\mu\nu} = -\left(W''W + (W')^2\right)$$

whilst the remaining components are zero:

$$R_{i\mu\nu\rho} = 0$$
$$R_{ij\mu\nu} = 0$$
$$R_{i\mu j\nu} = 0 \ \text{ unless } \ i = j \ \text{ and } \ \mu = \nu$$
$$R_{ijk\mu} = 0$$
$$; R_{\mu\nu\rho\sigma} = 0 \ \text{ unless } \ \mu = \rho \ \text{ and } \ \nu = \sigma$$

The proposition easily follows from these computations.   □

When V is given by (3.3), we denote the metric (3.2) by g_a. We next consider the singularity of the metric g_a. The metric is smooth for those values of u for which $0 < V(u) < \infty$. Write u_a for the largest root of V. At least when $u_a > 0$, the metric g_a is defined for $u \in (u_a, \infty)$. The metric g_a has a cone singularity at $u = u_a$. The next Lemma 3.3, whose proof is straightforward, describes how the cone angle depends on a.

LEMMA 3.3. — *Let*

(3.4)
$$v = \sqrt{\frac{n-3}{n-1}}, \qquad a_{\max} = \frac{2}{n-1}v^{n-3}$$

(1) We have $u_a > 0$ if and only if $a \in (-\infty, a_{\max}]$. The map $a \mapsto u_a$ is a decreasing homeomorphism $(-\infty, a_{\max}] \to [v, \infty)$.

(2) When $a \in (-\infty, a_{\max})$, the metric g_a has a cone singularity along S at $u = u_a$, with cone angle $2\pi c_a \in (0, \infty)$.

(3) The map $a \mapsto c_a$ is a decreasing homeomorphism $(-\infty, a_{\max}] \to [0, \infty)$. In particular, as a runs from 0 to $a_{\max}$, the cone angle takes every value from 2π to 0 precisely once.

As a consequence of the proof of Proposition 3.2 we also obtain the following Lemma 3.4:

LEMMA 3.4. — *For $a \in (0, a_{\max})$, the metric g_a is negatively curved, with all sectional curvatures satisfying*

$$\sec \leqslant -1 + \frac{n-3}{2} a u_a^{1-n} < 0$$

Remark 3.5. — Notice that as $u \to \infty$, the metric g_a approaches the hyperbolic metric. The metrics g_a are Riemannian analogues of static generalized Kottler metrics, which are solutions of the Lorentzian Einstein's equations with a negative cosmological constant. We refer for instance to Chruściel–Simon [9] for a study of Kottler spacetimes.

3.2. The sequence of interpolated metrics

We now transfer the model metric to our sequence (M_k) of compact hyperbolic n-manifolds. We keep the notation introduced at the beginning of Section 3. We let (U_k) be a sequence tending to infinity with k with $U_k < \frac{1}{2} U_{k,\max}$.

Let Σ_k^0 denote a connected component of Σ_k. Using geodesics orthogonal to Σ_k^0, we can set up a tubular neighbourhood of Σ_k^0 in which the hyperbolic metric on M_k is given by

$$\frac{du^2}{u^2 - 1} + (u^2 - 1)d\theta^2 + u^2 h_\Sigma$$

Here h_Σ is the hyperbolic metric on Σ_k^0 and $u = \cosh(r)$ where r is the distance to Σ_k^0. The hypersurface $\theta = 0$ corresponds to the totally geodesic hypersurface H_k; in general $\theta(p)$ is the angle that the shortest geodesic from p to Σ_k^0 makes with H_k. This expression is valid for $(u, \theta) \in [1, U_{k,\max}] \times S^1$.

Let $a \in (0, a_{\max})$. We define a new metric near Σ_k^0 interpolating between g_a and the hyperbolic metric as follows. Let $\chi \colon \mathbb{R} \to [0, \infty)$ be a smooth

function with $\chi(u) = 1$ for $u \leqslant 1/2$ and $\chi(u) = 0$ for $u \geqslant 1$. Write

$$(3.5) \qquad\qquad V(u) = u^2 - 1 + \frac{a}{u^{n-3}}\chi\left(\frac{u}{U_k}\right)$$

and consider the corresponding metric

$$\frac{\mathrm{d}u^2}{V} + V\mathrm{d}\theta^2 + u^2 h_\Sigma$$

The factor $\chi(u/U)$ has the effect of interpolating between the Einstein model of the previous section for $u \leqslant \frac{1}{2}U_k$ and the hyperbolic metric for $u \geqslant U_k$. Since the model is close to hyperbolic at large distances from Σ_k^0, when U_k is large this interpolation does not change the metric very much. As we will see, this means that the result is close to Einstein.

We also note that in terms of the intrinsic distance r from Σ_k^0, we are using the Einstein model for $r < \log U_k$ and the hyperbolic metric for $r > \log U_k + \log 2$, so the band on which the interpolation takes place has fixed geodesic width, independent of k.

The expression for the interpolated metric is valid for $(u, \theta) \in (u_a, U_{k,\,max}) \times S^1$, where $u_a < 1$ (since $a > 0$). We remove the tubular neighbourhood of Σ_k^0 at a distance U_k and glue this new metric in. The result is a metric on the same manifold, which is smooth across $u = U_k$ and which has a cone singularity along Σ_k^0 of angle $2\pi c_a$. Carrying out this procedure at every connected component of Σ_k we obtain a metric $\widetilde{g}_k$ on M_k which is Einstein near Σ_k, hyperbolic at long distances from Σ_k, and has cone singularities along each component of Σ_k, of angle $2\pi c_a$.

We now pass to the l-fold branched cover $p\colon X_k \to M_k$. By Proposition 3.3, there is a unique value of $a \in (0, a_{\max})$ for which the cone angles of $\widetilde{g}_k$ are $2\pi/l$. It follows that the pull-back metric $g_k = p^*\widetilde{g}_k$ is smooth on the whole of X_k, even across the branch locus. Proposition 3.1 now follows from the above construction and the arguments developed in the proof of Proposition 3.2.

4. The inverse function theorem

Our aim in this and the next section is to show that for all sufficiently large k there is an Einstein metric on X_k near to g_k. In this section we will set this up as a question about the inverse function theorem and reduce it to a key analytic estimate. We will then prove this estimate in the case $n = 4$ in the following Section 5.

4.1. Bianchi gauge condition and invertibility of the linearization

We will apply the implicit function theorem to a non-linear elliptic map between appropriate Banach spaces. Einstein's equations are diffeomorphism invariant and so not directly elliptic. We deal with this in the standard way (appropriate for Einstein metrics with negative scalar curvature) by adding an additional term, a technique called *Bianchi gauge fixing*. We describe this briefly here and refer to [1, 6] for proofs of the results we use.

The following applies to arbitrary closed Riemannian manifolds (X, g), and so we momentarily drop the k subscript to ease the notation. We write $\mathrm{div}_g \colon C^\infty(S^2 T^* X) \to C^\infty(T^* X)$ for the divergence of a symmetric 2-tensor. In abstract index notation, we have $(\mathrm{div}_g h)_p = -\nabla^q h_{pq}$, where ∇ is the Levi–Civita connection. We write div_g^* for the L^2-adjoint. Again, in index notation, $(\mathrm{div}_g^* \alpha)_{ab} = \nabla_{(a}\alpha_{b)} := \frac{1}{2}(\nabla_a \alpha_b + \nabla_b \alpha_a)$.

A computation gives that the linearisation of the Ricci curvature is

$$(4.1) \qquad (\mathrm{d}_g \mathrm{Ric})(s) = \frac{1}{2}\Delta_L(s) - \mathrm{div}_g^* \mathrm{div}_g(s) - \frac{1}{2}\nabla \mathrm{d} \mathrm{Tr}_g(s)$$

where Δ_L is the Lichnerowicz Laplacian:

$$\Delta_L(s) = \nabla^* \nabla + \mathrm{Ric}_g \circ s + s \circ \mathrm{Ric}_g - 2 \mathrm{Rm}_g(s)$$

In index notation this is

$$(\Delta_L s)_{ab} = \nabla^p \nabla_p s_{ab} + R_a{}^p s_{pb} + R_b{}^p s_{pa} - 2 s^{pq} R_{apbq}$$

Define the Bianchi operator $B_g \colon C^\infty(S^2 T^* X) \to C^\infty(T^* X)$ by

$$(4.2) \qquad B_g(h) = \mathrm{div}_g h + \frac{1}{2}\mathrm{d}(\mathrm{Tr}_g h)$$

Note that the contracted Bianchi identity gives $B_g(\mathrm{Ric}(g)) = 0$. Now given a pair of metrics g, h, we write

$$(4.3) \qquad \Phi_g(h) = \mathrm{Ric}(h) + (n-1)h + \mathrm{div}_h^*(B_g(h))$$

Here div_h^* is the formal adjoint of div_h, taken with respect to the L^2 innerproduct defined by h. We call Φ_g the *Einstein operator in Bianchi gauge relative to g*.

One can check that the addition of this second term produces an elliptic map. We write L_h for the derivative of Φ_g at h. The case $h = g$ is the simplest:

$$(4.4) \qquad L_g(s) = \frac{1}{2}\Delta_L + (n-1)s$$

The derivative at a general point is slightly more awkward. To describe it, we introduce the following notation. Given a metric h, a section $s \in C^\infty(S^2 T^* X)$ and a 1-form $\alpha \in C^\infty(T^* X)$ we consider the quantity

$$\lim_{t \to 0} \frac{(\mathrm{div}^*_{h+ts} - \mathrm{div}^*_h) \alpha}{t}$$

i.e., the infinitesimal change of $\mathrm{div}^*_h(\alpha)$ when h moves in the direction s. This has an expression of the form $\alpha * \nabla s$ where ∇ is the Levi–Civita connection of h and $*$ denotes some universal algebraic contraction. (See for example, the discussion in [23, Section 2.3.1].) We can now give the formula for L_h:

$$(4.5) \quad L_h(s) = \frac{1}{2} \Delta_{L,h}(s) + (n-1)s + \mathrm{div}^*_h \left(\mathrm{div}_g - \mathrm{div}_h \right)(s)$$
$$+ \frac{1}{2} \nabla_h \mathrm{d} \left(\mathrm{Tr}_g(s) - \mathrm{Tr}_h(s) \right) + B_g(h) * \nabla_h(s)$$

(where ∇_h denotes the Levi–Civita connection of h). This follows by direct differentiation of (4.3) together with the linearisation of Ricci curvature (4.1). Since $B_g(g) = 0$, (4.4) follows from (4.5).

The precise expression for L_h is not important in what follows. What is essential is that it is locally Lipschitz continuous in h. More precisely:

LEMMA 4.1. — *Fix an integer $m \geqslant 2$ and $0 \leqslant \eta < 1$. Given $K > 0$ there exist constants $\delta, C > 0$ such that if $g, h, \widetilde{h}$ are Riemannian metrics on the same n-dimensional manifold with*

$$\|g - h\|_{C^{m,\eta}}, \left\| g - \widetilde{h} \right\|_{C^{m,\eta}} < \delta$$
$$\|\mathrm{Rm}_g\|_{C^{m-2,\eta}} < K$$

where the norms are defined by g, then

$$\|(L_h - L_{\widetilde{h}})(s)\|_{C^{m-2,\eta}} \leqslant C \left\| h - \widetilde{h} \right\|_{C^{m,\eta}} \|s\|_{C^{m,\eta}}$$

for all symmetric 2-tensors $s \in C^{m,\eta}$. (Here L_h and $L_{\widetilde{h}}$ are both defined using g as the reference metric for the Bianchi gauge.)

This is again a standard result and we omit the proof. When there is no ambiguity we write $C^{m,\eta}$ for the space of sections of $S^2 T^* X$ of regularity $C^{m,\eta}$, and the Hölder norms in Lemma 4.1 are measured with respect to g (see Definition 4.4 for the explicit definition of the Hölder norms used in this article). Lemma 4.1 implies that $\Phi_g \colon C^{m,\eta} \to C^{m-2,\eta}$ is a continuously differentiable map of Banach spaces. (Strictly speaking, the domain of Φ_g is the open subset of $C^{m,\eta}$ consisting of positive definite sections.)

We next recall another important fact about Bianchi gauge: at least in the case of negative Ricci curvature, zeros of Φ_g are precisely Einstein metrics. To see this, one computes that

$$(4.6) \qquad\qquad 2B_h \circ \operatorname{div}_h^* = \nabla_h^* \nabla_h - \operatorname{Ric}(h)$$

In particular, when $\operatorname{Ric}(h)$ is negative, $B_h \circ \operatorname{div}_h^*$ is an isomorphism. From this, the next result follows easily.

LEMMA 4.2. — *Let (X, g) be a closed Riemannian manifold and h a second metric on X with $\operatorname{Ric}(h) < 0$. If $\Phi_g(h) = 0$ then in fact $\operatorname{Ric}(h) = -(n-1)h$ and $B_g(h) = 0$.*

Proof. — Since $B_h(\operatorname{Ric}_h) = 0 = B_h(h)$ the fact that $B_h(\Phi_g(h)) = 0$ implies $B_h(\operatorname{div}_h^* B_g(h)) = 0$. Equation (4.6) and integration by parts then implies that $B_g(h) = 0$ and so $\operatorname{Ric}_h = -(n-1)h$. □

We return to our sequence (X_k, g_k) of approximately Einstein metrics, constructed in Section 3. Write $\Phi_k = \Phi_{g_k}$ for the Einstein operator in Bianchi gauge relative to g_k. By Proposition 3.1, we have that $\Phi_k(g_k) = O(U_k^{1-n})$. We would like to apply the inverse function theorem to Φ_k to show that for sufficiently large k there is a metric h near to g_k with $\Phi_k(h) = 0$. Since g_k has negative curvature, the same will be true of h and so, by Lemma 4.2, h will be the Einstein metric we seek.

The first step in applying the inverse function theorem is to show that the linearised operator is an isomorphism. The following result proves this with a certain amount of uniformity. We show that the linearisation L_g of Φ_k at g is invertible for g on a definite neighbourhood of g_k whose diameter is bounded below independently of k. We also obtain uniform L^2 estimates on the inverse operator.

PROPOSITION 4.3. — *There exist constants $\delta > 0$ and $C > 0$ such that for all sufficiently large k, if g is a Riemannian metric on X_k with*

$$\|g - g_k\|_{C^2} \leqslant \delta$$

then, for any C^2 symmetric bilinear form s,

$$(4.7) \qquad\qquad \int_{X_k} \langle L_g(s), s \rangle_g \, \mathrm{dvol}_g \geqslant C \int_{X_k} |s|_g^2 \, \mathrm{dvol}_g$$

It follows that for any $m \geqslant 2$ and $0 < \eta < 1$, the linearisation $L_g \colon C^{m,\eta} \to C^{m-2,\eta}$ is an isomorphism.

In the statement of Proposition 4.3 it is implicit that the Hölder norm is taken with respect to the metric g_k. Throughout the proof we use the fact that, provided δ is small enough, the C^0 norms defined by g and g_k are

equivalent uniformly in k. We will switch between them without further comment. The proof of Proposition 4.3 is an adaptation of an argument used by Koiso [17] to investigate the rigidity of negatively curved Einstein metrics. In our situation, there are additional complications. Firstly, g_k is not an Einstein metric, merely close to Einstein, and secondly we do not linearise at g_k (the metric used to define Bianchi gauge) but instead at nearby metrics g. We push through Koiso's argument via a series of technical Lemmas that we omit here; for the detailed proof of Proposition 4.3 we refer to [11].

4.2. Weighted Hölder spaces

The crux to applying the inverse function theorem to find a zero of Φ_k is to obtain uniform control over the inverse L_g^{-1}. Proposition 4.3 shows that the lowest eigenvalue of L_g is uniformly bounded away from zero and this immediately gives good control of the inverse in L^2. This is not sufficient for our purposes, however. The volume of (X_k, g_k) grows rapidly with k, so much so that even though we have strong pointwise control of $\Phi_{g_k}(g_k)$ it does not even imply that $\Phi_{g_k}(g_k)$ tends to zero in L^2. Instead we work in Hölder spaces. Moreover, since the diameter of (X_k, g_k) tends to infinity, weighted Hölder spaces are required.

We begin with a word on the definition of *unweighted* Hölder spaces. When one deals with a sequence (X_k, g_k) of Riemannian manifolds, a little care must be taken with Hölder norms. In order to make it clear that no problems arise in our situation we will be very explicit about the way in which we define the Hölder norm.

DEFINITION 4.4. — *Let (X, g) be a compact Riemannian manifold. Write $\rho(g)$ for the conjugacy radius of g and fix $\rho_0 < \rho(g)$. Given $x \in X$ write $\exp_x \colon T_x X \to X$ for the exponential map, which is a local diffeomorphism on the ball $B(0, \rho_0) \subset T_x X$. Let s be a tensor field on X. Then $\exp_x^*(s)$ is a tensor field on the Euclidean vector space $T_x X$ and we can use the Euclidean metric to define the Hölder coefficient of s near x:*

$$[s]_{\eta, x} \coloneqq \sup_{p \neq q \in B(0, \rho_0)} \frac{|\exp_x^*(s)(p) - \exp_x^*(s)(q)|}{|p - q|^\eta}$$

We then take the supremum over all points x and combine with derivatives to take the full Hölder norm:

$$\|s\|_{C^{m, \eta}} \coloneqq \sum_{j \leqslant m} \sup_{x \in X} |\nabla^j s(x)| + \sup_{x \in X} [\nabla^m s]_{\eta, x}$$

This definition of the Hölder norm is well adapted to studying sequences (X_k, g_k) for which there is a uniform bound for the curvature and its derivatives: for all $m \in \mathbb{N}$ there exists $C > 0$ such that $\| \operatorname{Rm}(g_k) \|_{C^m} \leqslant C$. Our sequence of approximately Einstein metrics have uniform C^m bounds on $\operatorname{Rm}(g_k)$, thanks to part 4 of Proposition 3.1.

We now move to the weighted norms. We begin by defining the weight function w. Near each component of the branch locus we have a distinguished coordinate u, used in the construction of the model metric, given by (3.2) and (3.3). It is defined for $u_a \leqslant u < U_{k, \max}$ (where a is chosen so that the metric on M_k has cone angles $2\pi/l$). We extend this to a function $w \colon X_k \to \mathbb{R}$ by first setting it to be constant, equal to $U_{k, \max}$ outside the region $\{ u < U_{k, \max} \}$. We then modify it in the region $u \geqslant \frac{1}{2} U_{k, \max}$ to make this extension smooth. The smoothing is done so as to ensure the following:

LEMMA 4.5. — *For all large k, there exists a smooth function $w \colon X_k \to \mathbb{R}$ such that*

(1) *In the region $\{ u < \frac{1}{2} U_{k, \max} \}$, $w = u$.*
(2) *Outside the region $\{ u < U_{k, \max} \}$, $w = U_{k, \max}$.*
(3) *For each m, there is a constant C (not depending on k), such that $|\nabla^m w| \leqslant C|w|$ (where the norm is taken with g_k).*

This lemma follows again easily from the arguments developed in Section 3.

DEFINITION 4.6. — *Let $m \in \mathbb{N}$, $0 \leqslant \eta < 1$ and $\alpha > 0$. Given a section s of $S^2 T^* X_k$, we define the weighted Hölder norm of s to be*

$$\|s\|_{C_\alpha^{m, \eta}} := \|w^\alpha s\|_{C^{m, \eta}}$$

where w is the weight function of Lemma 4.5 and the norm is taken for g_k.

We have been careful in our choice of weight function to ensure that we have the following local control between weighted and unweighted norms. This sort of argument is typical in the use of weight functions and we refer for instance to [6, Section 3.8], . Recall that the uniform control on sectional curvatures of g_k gives a uniform lower bound on the conjugacy radius $\rho(g_k) \geqslant \rho_0$ of the manifolds (X_k, g_k).

LEMMA 4.7. — *Let $m \in \mathbb{N}$, $0 \leqslant \eta < 1$ and $0 < \rho \leqslant \rho_0$. Then there exists a constant $C = C(m, \eta, \rho) > 0$ such that for any $x \in X_k$ and any symmetric bilinear form s of regularity $C^{m, \eta}$, we have*

$$(4.8) \quad \frac{1}{C} w(x)^\alpha \|s\|_{C^{m, \eta}(B_x(\rho))} \leqslant \|s\|_{C_\alpha^{m, \eta}(B_x(\rho))} \leqslant C w(x)^\alpha \|s\|_{C^{m, \eta}(B_x(\rho))}$$

where $B_x(\rho) \subset (X_k, g_k)$ denotes the geodesic ball centred at x with radius ρ. In particular, C is independent of both x and k.

An easy consequence is the following:

COROLLARY 4.8. — *For any m, $0 \leqslant \eta < 1$ and $\alpha > 0$ there is a constant C, independent of k, such that for all $s \in C^{m,\eta}$,*

$$\|s\|_{C^{m,\eta}} \leqslant C \|s\|_{C_\alpha^{m,\eta}}$$

Proof. — This follows from Lemma 4.7, taking the supremum over x, together with the fact that $w \geqslant u_a > 0$, a lower bound which is independent of k. □

We now give the effect of the weight in measuring the failure of g_k to be an Einstein metric.

LEMMA 4.9. — *For all integers $m \geqslant 0$ and real numbers $0 \leqslant \eta < 1$, there is a constant A such that*

$$\|\mathrm{Ric}(g_k) + (n-1)g_k\|_{C_\alpha^{m,\eta}} \leqslant A U_k^{1-n+\alpha}$$

Proof. — Recall that $\mathrm{Ric}(g_k) + (n-1)g_k$ is supported in the region $S_k = \{U_k/2 \leqslant u \leqslant U_k\}$. We cover this by a finite family of geodesic balls (B_i) of fixed radius $\rho < \rho_0$, and with centres $x_i \in S_k$. By Proposition 3.1, for each i we have

$$\|\mathrm{Ric}(g_k) + (n-1)g_k\|_{C^{m,\eta}(B_i)} \leqslant A U_k^{1-n}$$

Since the x_i lie in S_k we have $w^\alpha(x_i) \leqslant C U_k^\alpha$ for some constant C. Now we multiply the previous inequality by $w^\alpha(x_i)$, use Lemma 4.7 and take the supremum over all i. □

We conclude this section with the weighted analogues of standard elliptic estimates, whose proof is standard and can again be found in [11]:

LEMMA 4.10 (Uniform Lipschitz continuity of the linearisation). — *Fix an integer $m \geqslant 2$, and real numbers $0 \leqslant \eta < 1$ and $\alpha > 0$. There are constants $\delta, C > 0$, independent of k, such that if g and h are Riemannian metrics on X_k with*

$$\|g - g_k\|_{C^{m+2,\eta}}, \|h - g_k\|_{C^{m+2,\eta}} \leqslant \delta$$

then for all symmetric bilinear forms s of regularity $C_\alpha^{m+2,\eta}$ we have

$$\|(L_g - L_h)(s)\|_{C_\alpha^{m,\eta}} \leqslant C \|g - h\|_{C^{m+2,\eta}} \|s\|_{C_\alpha^{m+2,\eta}}$$

(where all norms are taken with respect to g_k).

Remark 4.11. — Note that by Corollary 4.8, we are free to replace the unweighted norms on the metrics in this result by weighted ones (at the expense of shrinking δ). We will frequently do this in applications of this result.

Lemma 4.12 (Uniform elliptic estimate). — *For any integer m and real numbers $0 < \eta < 1$ and $\alpha \geqslant 0$ there are constants $\delta, C > 0$, independent of k, such that if g is a Riemannian metric on X_k with*

$$\|g - g_k\|_{C_\alpha^{m+2, \eta}} \leqslant \delta$$

then for all sections s of $S^2 T^* X_k$ of regularity $C^{m+2, \eta}$, we have

$$\|s\|_{C_\alpha^{m+2, \eta}} \leqslant C \left(\|L_g(s)\|_{C_\alpha^{m, \eta}} + \|s\|_{C_\alpha^0} \right)$$

(where all norms are taken with respect to g_k).

Fix an integer $m \geqslant 2$, and real numbers $0 < \eta < 1$ and $\alpha > 0$. We claim that there are constants $\delta > 0$ (independent of k) and $C_k > 0$ (depending on k) such that if g is a Riemannian metric on X_k with

$$\|g - g_k\|_{C_\alpha^{m+2, \eta}} \leqslant \delta$$

then for all symmetric bilinear forms s of regularity $C^{m, \eta}$ we have

(4.9) $$\|s\|_{C_\alpha^{m+2, \eta}} \leqslant C_k \|L_g(s)\|_{C_\alpha^{m, \eta}}$$

This follows from the previous results of this section and is essentially the standard contradiction argument used to remove the C^0 term in the elliptic estimate, based on the fact that L_g is invertible by Proposition 4.3. Note that the constant in (4.9) depends on k, because the contradiction argument must be carried out on each X_k separately.

4.3. The inverse function theorem assuming a key estimate

We now explain how to perturb g_k to an Einstein metric, assuming temporarily one critical estimate, Theorem 4.14 below. We will prove this estimate in the case $\dim X_k = 4$ in the following section.

The first step in the proof is to apply a version of the inverse function theorem to Φ_k with uniformity in g, if not in k. We state the result here:

Proposition 4.13. — *Fix an integer $m \geqslant 0$ and real numbers $0 < \eta < 1$ and $\alpha > 0$. There exist constants $\delta > 0$ (independent of k) and $r_k > 0$ (depending on k) such that if g is a Riemannian metric on X_k with*

$$\|g - g_k\|_{C_\alpha^{m+2, \eta}} \leqslant \delta$$

then $B(\Phi_k(g), r_k) \subset C_\alpha^{m,\eta}$ is contained in the image of Φ_k and there is a differentiable map

$$\Psi_k \colon B\left(\Phi_k(g), r_k\right) \to C_\alpha^{m+2,\eta}$$

inverting Φ_k on a neighbourhood of $g \in C_\alpha^{m+2,\eta}$.

Proof. — This is just an application of the inverse function theorem to Φ_k at g, where the invertibility is given by Proposition 4.3. The inverse function theorem provides a quantitative estimate on the radius of a ball centred at $\Phi_k(g)$ which is contained in the image of Φ_k. By the uniform Lipschitz continuity of Φ_k (Lemma 4.10) this radius depends only on the square inverse of the operator norm of $L_g = \mathrm{d}\Phi_k(g)$. By (4.9) we can choose such a bound to only depend on k. □

Of course, this is far from enough to prove the existence of an Einstein metric. Whilst $\Phi_k(g_k)$ tends to zero as k tends to infinity, the radius r_k may tend to zero even quicker. To remedy this problem we will use a much sharper estimate on L_g^{-1}. *It is at this point our argument requires* $\dim X_k$ = 4. As a matter of notation, write

$$S_k = \left\{ \frac{1}{2} U_k \leqslant u \leqslant U_k \right\}$$

Recall that $\mathrm{Ric}(g_k) + 3g_k$ is supported in S_k. Recall also that until now, our choice of gluing parameter U_k has only had to satisfy the requirements that $U_k \to \infty$ and $U_k \leqslant \frac{1}{2} U_{k,\max}$. We will need to be more careful in our choice of U_k in order to prove the estimate we want.

THEOREM 4.14. — *Let* $\dim X_k = 4$. *There is a choice* (U_k) *of gluing parameters such that for the corresponding approximately Einstein manifolds* (X_k, g_k) *the following holds. For any integer* $m \geqslant 1$ *and real number* $0 < \eta < 1$ *there exists real numbers* $0 < \alpha < 3$ *and* $\delta > 0$ *and a sequence* (ϵ_k) *of positive real numbers, with* $\epsilon_k \to 0$ *as* $k \to \infty$ *which have the following property. For all large* k, *if* g *is a Riemannian metric on* X_k *with*

$$\|g - g_k\|_{C_\alpha^{m+2,\eta}} \leqslant \delta$$

then for any symmetric bilinear tensor $s \in C_\alpha^{m+2,\eta}$, *with* $L_g(s)$ *supported in* S_k *we have*

$$\|s\|_{C_\alpha^0} \leqslant \epsilon_k U_k^{3-\alpha} \|L_g(s)\|_{C_\alpha^{m,\eta}}$$

It is crucial in Theorem 4.14 that we restrict attention to those s with $L_g(s)$ supported in S_k. It seems that the sought-after estimate will not hold otherwise. Whilst our proof does not extend to arbitrary dimensions, it seems plausible that the analogous estimate could hold in dimension n (with the power $U_k^{n-1+\alpha}$ on the right-hand side). This would then imply

the existence of Einstein metrics for these higher dimensional Gromov–Thurston manifolds.

Theorem 4.14 is the core of the analysis of this paper and we prove it in the following section. For the remainder of this section we show how this refined estimate proves the existence of an Einstein metric.

Proof of Theorem 1.2, assuming Theorem 4.14. — Let

$$\gamma(t) = (1 - t)\Phi_k(g_k).$$

Proposition 4.13 gives a smooth path of Riemannian metrics $g(t)$ solving $\Phi_k(g(t)) = \gamma(t)$ for

$$0 \leqslant t < \frac{r_k}{\|\Phi_k(g_k)\|_{C_\alpha^{m,\eta}}}$$

We will show this path can be extended up to $t = 1$, and then $g(1)$ is the Einstein metric we seek.

Let $\delta > 0$ be small enough so that Lemma 4.12, Proposition 4.13 and Theorem 4.14 all apply simultaneously. Write $B_\delta \subset C_\alpha^{m+2,\eta}$ for the ball of radius δ centred at g_k. Consider the set

$$T = \big\{\tau > 0 : \text{ there is a differentiable map } g \colon [0, \tau] \to B_\delta$$
$$\text{with } \Phi_k(g(t)) = \gamma(t), g(0) = g_k\big\}$$

Let $\sigma = \sup T$. By Proposition 4.13 we know that

$$\sigma \geqslant \frac{r_k}{\|\Phi_k(g_k)\|_{C_\alpha^{m,\eta}}}$$

We will show that $\sigma < 1$ gives a contradiction. Consider

$$\tau = \sigma - \frac{r_k}{2\|\Phi_k(g_k)\|_{C_\alpha^{m,\eta}}}$$

We have $0 < \tau < \sigma$ and so the path $g(t)$ exists on $[0, \tau]$ and stays inside B_δ. By Proposition 4.13, Φ_k is a local diffeomorphism at $g(\tau)$ and its image contains the ball of radius r_k centred at $\gamma(\tau)$. In particular it contains $\gamma(t)$ for

$$t \in \left[\tau, \sigma + \frac{r_k}{2\|\Phi_k(g_k)\|_{C_\alpha^{m,\eta}}}\right]$$

So we can actually extend $g(t)$ smoothly to solve $\Phi_k(g(t)) = \gamma(t)$ for values of t slightly larger than σ. The crux is to show that in doing so we do not leave B_δ.

To prove this, differentiate $\Phi_k(g(t)) = (1 - t)\Phi_k(g_k)$ with respect to t to get

$$L_{g(t)}(g'(t)) = -\Phi_k(g_k)$$

For $t \in [0, \sigma)$, $g(t) \in B_\delta$ and so, for these times, we can apply the elliptic estimate Lemma 4.12 and Theorem 4.14. This, together with the error estimate Lemma 4.9, gives

$$\|g'(t)\|_{C_\alpha^{m+2,\eta}} \leqslant C \left(\|\Phi_k(g_k)\|_{C_\alpha^{m,\eta}} + \|g'(t)\|_{C_\alpha^0} \right)$$
$$\leqslant AC \left(1 + \epsilon_k U_k^{n-1-\alpha} \right) U_k^{1-n+\alpha}$$

This bound tends to zero as k tends to infinity and so for all large k we have $\|g'(t)\|_{C_\alpha^{m+2,\eta}} < \delta$. Integrating this from $t = 0$ to $t = \sigma$, we see that

$$\|g(\sigma) - g_k\|_{C_\alpha^{m+2,\eta}} \leqslant \sigma\delta$$

So the assumption that $\sigma < 1$ means $g(\sigma) \in B_\delta$ and hence $g(t) \in B_\delta$ for t slightly larger that σ. This is a contradiction with the fact that $\sigma = \sup T$.

We write g for the Einstein metric on X_k found in this way. To check that the sectional curvatures of g are negative, recall that there is a constant $c > 0$ such that the sectional curvatures of the approximate solution g_k all satisfy $\sec(g_k) \leqslant -c$. By construction, our Einstein metric g is of the form $g = g_k + s_k$ where $\|s_k\|_{C^{2,\eta}} \to 0$. From this it follows that the sectional curvatures of g satisfy $\sec(g) \leqslant -c/2$.

We now check that g is not simply locally homogeneous. There is a constant $b > 0$ such that the model metric of Proposition 3.2 has at least one sectional curvature at finite distance from the branch locus, which satisfies $\sec \geqslant -1 + b$. (This follows from the explicit form of the sectional curvatures given in the proof of Lemma 3.4.) Since g is a C^2-small perturbation of this metric near the branch locus, it must have a a sectional curvature which satisfies $\sec \geqslant -1 + b/2$. However, the approximate solution g_k is genuinely hyperbolic at large distances and so at these distances all sectional curvatures of the Einstein metric g satisfy $\sec < -1 + b/2$. It follows that g near the branch locus is not locally isometric to g at large distances and hence g is not locally homogeneous. $\qquad\square$

5. Proving the key estimate

In this section we prove Theorem 4.14, which completes the proof of Theorem 1.2. The proof in Fine–Premoselli [11] goes through a contradiction argument and requires to study three different cases corresponding the different possible locations of the maximum point of an hypothetical tensor field s_k failing Theorem 4.14. For the sake of clarity, in the present paper, we will discuss the ideas involved in the proof but will only sketch the proof in one of these three cases.

5.1. Setting the problem

We quickly recall some of our notation. $U_{k,\max} = \cosh(\frac{1}{2}i(M_k))$, where $i(M_k)$ is the injectivity radius of M_k with the hyperbolic metric; the gluing is carried out in the region $\frac{1}{2}U_k \leqslant u \leqslant U_k$, where $U_k < \frac{1}{2}U_{k,\max}$. In the course of the proof, it will be important how we choose the gluing parameter U_k. For now, we stipulate only that $U_k \to \infty$ whilst $U_k/U_{k,\max} \to 0$. The precise choice will be made later.

The proof is by contradiction and so we assume Theorem 4.14 is false. I.e.:

HYPOTHESIS WHICH WILL LEAD TO A CONTRADICTION. — *Let $m \geqslant 1$. Let $\alpha > 0$, $\delta_0 > 0$, (ϵ_k) be a sequence of positive real numbers with $\epsilon_k \to 0$ and let $N_0 \in \mathbb{N}$. Then there exist $k_0 \geqslant N_0$, $\widetilde{g}_{k_0}$ a metric on X_k with*

$$\|\widetilde{g}_{k_0} - g_{k_0}\|_{C_\alpha^{m+2,\eta}} \leqslant \delta_0$$

and $s_{k_0} \in C_\alpha^{m+2,\eta}$ with $L_{\tilde{g}_{k_0}}(s_{k_0})$ supported in S_{k_0}, such that

$$(5.1) \qquad \|s_{k_0}\|_{C_\alpha^0} > \epsilon_{k_0} U_{k_0}^{3-\alpha} \|L_{\tilde{g}_{k_0}}(s_{k_0})\|_{C_\alpha^{m,\eta}}$$

We fix α and take $\epsilon_k = U_k^{-p}$ for some small positive number p. Both α and p will be determined in the course of the proof. We now apply our hypothesis with δ_0 replaced by a sequence $\delta_m > 0$ with $\delta_m \to 0$ and N_0 replaced by a sequence $N_m \in \mathbb{N}$ with $N_m \to \infty$. This gives a sequence $\widetilde{g}_{k_m}$ of metrics and symmetric bilinear forms s_{k_m} on X_{k_m} such that the conclusions of the hypothesis are satisfied. To ease the notation, we pass to this subsequence and drop the m subscript. This leads to a sequence $(\widetilde{g}_k)$ of metrics with

$$(5.2) \qquad \|\widetilde{g}_k - g_k\|_{C_\alpha^{m+2,\eta}} \to 0$$

as $k \to \infty$, and a sequence (s_k) of symmetric bilinear forms for which $L_{\tilde{g}_k}(s_k)$ is supported in S_k and

$$(5.3) \qquad \|s_k\|_{C_\alpha^0} > U_k^{3-\alpha-p} \|L_{\tilde{g}_k}(s_k)\|_{C_\alpha^{m,\eta}}$$

We will prove that (5.3) actually never holds, giving our contradiction. To do this, for each k we pick $x_k \in X_k$ at which

$$w^\alpha(x_k)\,|s_k(x_k)|_{g_k} = \|s_k\|_{C_\alpha^0}$$

where w is the weight constructed in Lemma 4.5.

First, a word on notation. Given sequences (p_k) and (q_k) of real numbers, We write $p_k \lesssim q_k$ to mean that there is a constant $C > 0$ such that for all

k, $p_k \leqslant Cq_k$. In a chain of such inequalities, $p_k \lesssim q_k \lesssim r_k$, the constant C may change, but will always be independent of k.

The first step in the proof is a preliminary lemma, showing that the C_α^0-norm of $L_{\tilde{g}_k}(s_k)$ gives control of s_k in $W^{1,2}$. It is at this point that the crucial bound on the volume of the branch locus, derived in Section 2, enters the analysis. Recall part (3) of Proposition 1.1, which says that, in arbitrary dimension,

$$\mathrm{vol}(\Sigma_k) \leqslant A \exp\left(\frac{n^2 - 3n + 6}{4} i(M_k) \right)$$

In our case, $n = 4$. By definition of $U_{k,max}$ we deduce that

$$(5.4) \qquad \mathrm{vol}(\Sigma_k) \lesssim U_{k,\max}^5$$

LEMMA 5.1. — *We have*

$$(5.5) \qquad \|L_{\tilde{g}_k}(s_k)\|_{L^2} \lesssim U_{k,\max}^{\frac{5}{2}} U_k^{\frac{3}{2} - \alpha} \|L_{\tilde{g}_k}(s_k)\|_{C_\alpha^0}$$

$$(5.6) \qquad \|s_k\|_{L^2} + \|\nabla s_k\|_{L^2} \lesssim U_{k,\max}^{\frac{5}{2}} U_k^{\frac{3}{2} - \alpha} \|L_{\tilde{g}_k}(s_k)\|_{C_\alpha^0}$$

Both the L^2 and Hölder norms here are taken with respect to the metric g_k.

Proof. — $L_{\tilde{g}_k}(s_k)$ is supported in S_k, so

$$\|L_{\tilde{g}_k}(s_k)\|_{L^2} \lesssim \mathrm{vol}(S_k)^{\frac{1}{2}} \|L_{\tilde{g}_k}(s_k)\|_{C^0}$$

But, by definition of S_k, at all points of S_k the weight function w satisfies $w \gtrsim U_k$ from which we have $\|L_{\tilde{g}_k}(s_k)\|_{C^0} \lesssim U_k^{-\alpha} \|L_{\tilde{g}_k}(s_k)\|_{C_\alpha^0}$. We have $\mathrm{vol}(S_k) \lesssim U_k^3 \mathrm{vol}(\Sigma_k)$. Now (5.4) implies (5.5).

From here, Proposition 4.3 gives $\|s_k\|_{L^2} \lesssim \|L_{\tilde{g}_k}(s_k)\|_{L^2}$. Proposition 4.3 gives this with the L^2-norms defined by $\tilde{g}_k$ but by (5.2) these norms are equivalent to those defined by g_k. By (5.5) this proves (5.6) for $\|s_k\|_{L^2}$.

We now use (4.7) together with a Böchner formula, which gives

$$\int_{X_k} |\nabla s_k|_{\tilde{g}_k}^2 \, \mathrm{dvol}_{\tilde{g}_k} \lesssim \int_{X_k} \langle L_{\tilde{g}_k}(s_k), s_k \rangle_{\tilde{g}_k} + \int_{X_k} |s_k|_{\tilde{g}_k}^2 \, \mathrm{dvol}_{\tilde{g}_k},$$

from which (5.6) follows by the previous arguments (we have used here the fact that $\mathrm{Rm}_{\tilde{g}_k}$ is bounded uniformly in k.) $\qquad\square$

At this point we divide the argument into three separate cases.

(1) There exists a constant $C > 0$ such that, after passing to a subsequence, for all large k we have

$$w(x_k) \geqslant \frac{1}{C} U_{k,\max}$$

The points x_k are further and further from the branch locus. More-over, since we choose the gluing distance with $U_k/U_{k,\,\mathrm{max}} \to 0$, for large k the points x_k lie in the region of X_k where g_k is genuinely hyperbolic.

(2) There exists a constant C such that, after passing to a subsequence,

$$w(x_k) \leqslant C$$

The points x_k remain at bounded distance from the branch locus $\Sigma_k \subset X_k$ and so lie in the region where g_k is given by the model Einstein metric of Section 3.

(3) The remaining possibility is that $w(x_k) \to \infty$ and $w(x_k)/U_{k,\,\mathrm{max}} \to 0$. In this case the points x_k live in a region where the model coordinate system near the branch locus makes sense, but they are moving further and further from the branch locus.

We will treat each of these cases separately, but each time the argument follows similar lines. We translate the problem onto a non-compact space (either $\mathbb{H}^4$ or the model metric of Section 3). We use a Green's representation formula in this non-compact space to give an expression for $s_k(x_k)$. We then prove estimates for the Green's operator. In cases 1 and 2 these are weighted integral estimates which enable us to turn L^2 estimates on s_k into pointwise ones. In case 3 we can even use pointwise estimates on the Green's operator.

In order to fix our notation and conventions, we quickly recall the general form of the representation formula for systems which are not necessarily self adjoint. More details are given in [11, the Appendix]. Suppose D is an elliptic operator on sections of a vector bundle E with a fibrewise metric, over a Riemannian manifold. Let $G(y, x) \in \mathrm{Hom}(E_x, E_y)$ be defined for all $x, y \in X$ with $x \neq y$, depending smoothly on x and y. We say that G is a fundamental solution for D if it satisfies the following distributional equation: let $\sigma \in E_x$ and write $G(\cdot, x)(\sigma)$ for the section $y \mapsto G(y, x)(\sigma)$ of E; then

$$D\left(G(\cdot, x)(\sigma)\right) = \delta_x \sigma$$

Explicitly, for any compactly supported section s of E,

$$\int_X \langle G(y, x)(\sigma), D^* s(y) \rangle \, \mathrm{dvol}_y = \langle s(x), \sigma \rangle$$

This is equivalent to the following representation formula: for any compactly supported section s of E,

$$(5.7) \qquad s(x) = \int_X G(y, x)^t \left(D^* s(y)\right) \mathrm{dvol}_y$$

Notice in particular that a fundamental solution for D gives a representation formula for s in terms of D^*s.

In the following, for simplicity, we only sketch the proof of the contradiction when x_k satisfies Case 2 listed above. A detailed account of the other cases can again be found in [11].

5.2. Case 2

We assume that, after passing to a subsequence, $w(x_k) \leqslant C$ for some constant C independent of k. By definition of g_k, this means that the geodesic distance from x_k to Σ_k is uniformly bounded and so less than the normal injectivity radius (for all large k). We denote by Σ_k' the nearest component of Σ_k to x_k. Just as in Section 3, we identify a tubular neighbourhood of Σ_k' with

$$\frac{[u_a, U_{k,\max}) \times S^1 \times \Sigma_k'}{\sim}$$

The quotient by the relation $\sim$ denotes that we have collapsed the S^1 factor over $\{u_a\} \times \Sigma_k$ to produce a smooth manifold without boundary. We use $u \in [u_a, U_{k,\max})$ for the corresponding coordinate function in the radial direction, as in Section 3, which behaves at large distances as the exponential of the distance to Σ_k. Here the minimal value u_a is the constant defined in the course of Lemma 3.3. We recall that it depends only on the degree l of the cover, and not on k. We then transfer everything to the non-compact manifold

$$Y_k = \frac{[u_a, \infty) \times S^1 \times \Sigma_k'}{\sim}$$

The approximate Einstein metric g_k restricts from X_k to the region $u \leqslant U_{k,\max}$ of Y_k; it is hyperbolic for $u \geqslant U_{k,\max}/2$ and so extends directly, remaining hyperbolic, to the rest of Y_k. We continue to denote this extension by g_k. The metric $\widetilde{g}_k$, satisfying (5.2) restricts to the region $u \leqslant U_{k,\max}/2$ of Y_k; we then extend it to the whole of Y_k by interpolating with g_k over the region $U_{k,\max}/2 \leqslant u \leqslant U_{k,\max}$. This gives a metric on the whole of Y_k which we continue to denote by $\widetilde{g}_k$. We remark that we still have the analogue of (5.2), namely

$$(5.8) \qquad \|\widetilde{g}_k - g_k\|_{C_\alpha^{m+2,\eta}(Y_k)} \to 0$$

The strategy is the following: we prove weighted L^2 estimates on the Green's operator which, together with the global $W^{1,2}$-estimates of Lemma 5.1 lead to a contradiction with (5.3). Here we use the function u as

a weight. This choice is motivated by the fact that for the asymptotically hyperbolic model of Section 3, u^{-1} is a boundary defining function, and such functions are the appropriate weight to use in that context.

For any pair x, y of disjoint points of Y_k, denote by $\widetilde{G}_k(y, x)$ the fundamental solution of $L^*_{\tilde{g}_k}$ in Y_k, centred at x. (See [11, the Appendix] for the construction of $\widetilde{G}_k$ and for a description of its properties.) We have the following weighted L^2 estimate on $\widetilde{G}_k$.

PROPOSITION 5.2. — *Given $0 < \epsilon < 3$, there exists a constant C (depending only on ϵ, but not on k) such that for any $x \in Y_k$ and for all large k,*

$$\int_{Y_k \setminus B_{g_k}(x,1)} u(y)^{3-\epsilon} \left| \widetilde{G}_k(y, x) \right|^2_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k} \leqslant C u(x)^{3-\epsilon}$$

Proof. — The proof follows the ideas in [19]. More precisely, [19, Lemma 7.14] shows that for any $\delta > 0$ there exists U_0 such that for any smooth s compactly supported in $\{u \geqslant U_0\}$ we have

(5.9)
$$\int_{Y_k} \left\langle L^*_{\tilde{g}_k}(s), s \right\rangle \mathrm{dvol}_{\tilde{g}_k} = \int_{Y_k} \left\langle L_{\tilde{g}_k}(s), s \right\rangle \mathrm{dvol}_{\tilde{g}_k} \geqslant \left(\frac{9}{8} - \frac{\delta}{2} \right) \int_{Y_k} |s|^2_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k}$$

This follows because $\widetilde{g}_k$ is exactly hyperbolic at large distances, with boundary defining function u^{-1}.

Fix $\delta > 0$ small and an associated U_0 as in (5.9). Let $\eta \colon [0, \infty) \to [0, \infty)$ be a smooth function with $\eta(u) = 0$ when $u \leqslant U_0$ and $\eta(u) = 1$ for $u \geqslant 2U_0$. We will use η to support the Green's function in the region $\{u \geqslant U_0\}$. Meanwhile we will use a second cut-off function χ to cut-off at large values of u. For this, pick $M \gg 1$ and let $\chi \colon [0, \infty) \to [0, \infty)$ be smooth with $\chi(u) = U_0 + 1$ when $u \leqslant U_0$, $\chi(u) = u$ for $U_0 + 1 \leqslant u \leqslant M - 1$ and $\chi(u) = M$ when $u \geqslant M + 1$. We choose χ so that $|\chi'| \leqslant 1$.

Let $x \in Y_k$ and $\sigma \in S^2 T^*_x Y_k$ with $|\sigma|_{\tilde{g}_k} = 1$ and as before, put

$$F(y) = \widetilde{G}_k(y, x)(\sigma)$$

Let ψ be a smooth cut-off function centred at x, with $\psi \equiv 0$ in $B_{\tilde{g}_k}(x, 1/2)$ and $\psi \equiv 1$ in $Y_k \setminus B_{\tilde{g}_k}(x, 1)$. We put

$$\widetilde{F}(y) = \psi(y)\eta(u(y))F(y)$$

$L^*_{\tilde{g}_k}(\widetilde{F})$ is supported in the union of the annulus $B_{\tilde{g}_k}(x, 1) \setminus B_{\tilde{g}_k}(x, 1/2)$ and the region $\{U_0 \leqslant u \leqslant 2U_0\}$. Straightforward computations using the distributional equation satisfied by $\widetilde{G}_k(\cdot, x)$ show that

(5.10)
$$\int_{Y_k} \chi(u(y))^{3-\epsilon} \left| L^*_{\tilde{g}_k} \widetilde{F}(y) \right|^2_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k} \leqslant C u(x)^{3-\epsilon}$$

where C does not depend on ϵ, M or k but does depend on U_0. Two points in the proof of (5.10) need special care. On the one hand, to control the contribution of the integrand supported in $B_{\tilde{g}_k}(x,1) \setminus B_{\tilde{g}_k}(x,1/2)$ we must first bound uniformly the volume of the unit ball:

$$\mathrm{vol}(B_{\tilde{g}_k}(x,1)) \leqslant C$$

This follows from the Bishop–Gromov inequality and the fact that the Ricci curvature of $\tilde{g}_k$ is uniformly bounded below. Then, we use that for large k we have $|du|_{\tilde{g}_k} \leqslant 2V(u)^{1/2} \leqslant 2u$, where V is defined in (3.5). It follows that there is a constant C such that for any $y \in B_{\tilde{g}_k}(x,1)$, $u(y) \leqslant Cu(x)$.

On the other hand, the other possible support of the integrand in (5.10) is the region $\{U_0 \leqslant u \leqslant 2U_0\}$. For such y, we also have $u(y) \leqslant Cu(x)$ where C now depends on U_0 (this is because $u(x) \geqslant u_a > 0$), and therefore $\chi(u(y))^{3-\epsilon} \leqslant Cu(x)^{3-\epsilon}$. Together with the global uniform L^2 control on $\tilde{G}_k(y,x)$ and its covariant derivative (see again [11, the Appendix]), this proves (5.10).

A variation on [19, the proof of Lemma 7.14] now gives that, for any $0 < \beta < 3/2$,

$$\int_{Y_k} \chi(u(y))^{2\beta} \left| \widetilde{F}(y) \right|_{\tilde{g}_k}^2 \, \mathrm{dvol}_{\tilde{g}_k} \leqslant Cu(x)^{2\beta}$$

Here C depends only on β and U_0, but not on M, k or x. Now letting $M \to \infty$ gives, by definition of η and ψ:

$$\int_{\{u \geqslant 2U_0\} \setminus B_{\tilde{g}_k}(x,1)} u(y)^{2\beta} |F(y)|_{\tilde{g}_k}^2 \, \mathrm{dvol}_{\tilde{g}_k} \leqslant Cu(x)^{2\beta}$$

Finally, the integral over the region $\{u \leqslant 2U_0\} \setminus B_{\tilde{g}_k}(x,1)$ is independently estimated by a global L^2 bound on $\tilde{G}_k(y,x)$ which is uniform in k. This completes the proof of Proposition 5.2. $\qquad\qquad\square$

With this weighted L^2-estimate in hand, we prove the following bound on $s_k(x_k)$ which gives a contradiction with (5.3) (again with $p = 1/8$). We let $0 < \alpha < 1/4$ and choose the gluing parameter U_k so that

(5.11) $$U_k^{-\frac{3}{2}} U_{k,\,\mathrm{max}}^{\frac{5}{4}+\alpha} \to 0$$

If we only cared about Case 2, we could have used a weaker constraint on U_k, but this choice turns out to be also suitable for Cases 1 and 3.

PROPOSITION 5.3. — *Let $1/8 < \alpha < 1/4$ and choose the gluing parameter U_k so that*

$$U_k^{-\frac{3}{2}} U_{k,\,\mathrm{max}}^{\frac{5}{4}+\alpha} \to 0$$

Then for all sufficiently large k,

$$\|s_k\|_{C^0_\alpha} \leqslant U_k^{3-\alpha-\frac{1}{8}} \|L_{\tilde{g}_k}(s_k)\|_{C^0_\alpha}$$

Proof. — We first transport s_k from X_k to the model space Y_k. Let $\eta_k \colon [0,\infty) \to [0,\infty)$ be a smooth cut-off function with $\eta_k(u) = 1$ for $u \leqslant U_{k,\max}/4$, $\eta_k(u) = 0$ when $u \geqslant U_{k,\max}/2$ and

$$|\eta_k'| + U_{k,\max} |\eta_k''| \lesssim \frac{1}{U_{k,max}}$$

For such a choice of η_k we have, in particular, that

(5.12) $$|\nabla \eta_k(u(\cdot))|_{\tilde{g}_k} + |\Delta \eta_k(u(\cdot))|_{\tilde{g}_k} \lesssim 1$$

Let $\widetilde{s}_k = \eta_k s_k$. This defines a symmetric tensor field supported in the region $u \leqslant U_{k,\max}$ and we extend it by zero to a tensor on the whole of Y_k. We now take the Green's representation formula for $\widetilde{s}_k$:

$$s_k(x_k) = \widetilde{s}_k(x_k) = \int_{Y_k} \widetilde{G}_k(y, x_k)^t \, (L_{\tilde{g}_k}(\widetilde{s}_k)(y)) \, \mathrm{dvol}_{\tilde{g}_k}(y)$$

Now $L_{\tilde{g}_k}(s_k)$ is by assumption supported in the region $S_k = \{U_k/2 \leqslant u \leqslant U_k\}$, where $\eta_k = 1$. It follows that

(5.13) $$|s_k(x_k)|_{g_k} \lesssim \left(\int_{S_k} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k} \right) U_k^{-\alpha} \|L_{\tilde{g}_k}(s_k)\|_{C^0_\alpha}$$

$$+ \left(\int_{\frac{U_{k,\max}}{4} \leqslant u \leqslant \frac{U_{k,\max}}{2}} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k}^2 \right)^{1/2} (\|s_k\|_{L^2} + \|\nabla s_k\|_{L^2})$$

(we used here the uniform bounds (5.12) on the derivatives of η_k.)

We make use of Proposition 5.2. Since we are assuming here that $u(x_k) \leqslant C$ is uniformly bounded, we can replace the bound of this result by a uniform constant, and we in particular have that $x_k \notin S_k$.

By Cauchy–Schwarz,

$$\int_{S_k} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k}$$

$$\leqslant \left(\int_{S_k} u(y)^{\epsilon-3} \, \mathrm{dvol}_{\tilde{g}_k}(y) \right)^{1/2} \left(\int_{S_k} u(y)^{3-\epsilon} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k}^2 \mathrm{dvol}_{\tilde{g}_k}(y) \right)^{1/2}$$

$$\lesssim U_k^{\frac{1}{2}(\epsilon-3)} \mathrm{vol}(S_k)^{1/2}$$

Now $\mathrm{vol}(S_k) \leqslant U_k^3 \mathrm{vol}(\Sigma_k)$ where $\mathrm{vol}(\Sigma_k)$ is the hyperbolic volume of the branch locus Σ_k. By part (3) of Proposition 1.1, we have $\mathrm{vol}(\Sigma_k) \lesssim U_{k,\max}^5$

(as discussed before (5.4)) and from here we have that

$$(5.14) \qquad \int_{S_k} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k} \mathrm{dvol}_{\tilde{g}_k} \lesssim U_k^{\frac{\epsilon}{2}} U_{k,\,\mathrm{max}}^{\frac{5}{2}} = o\left(U_k^{3 - \frac{1}{8}} \right)$$

(where we have used the condition on U_k in the hypotheses, with $\epsilon > 0$ chosen sufficiently small).

This deals with the first term in (5.13). For the second term we use Proposition 5.2 to write

$$\int_{\frac{U_{k,\,\mathrm{max}}}{4} \leqslant u \leqslant \frac{U_{k,\,\mathrm{max}}}{2}} \left| \widetilde{G}_k(y, x_k) \right|_{\tilde{g}_k}^2 \lesssim U_{k,\,\mathrm{max}}^{\epsilon - 3}$$

From this and Lemma 5.1 we get that the second term in (5.13) is bounded by

$$U_{k,\mathrm{max}}^{\frac{\epsilon}{2}+1} U_k^{\frac{3}{2}-\alpha} \| L_{\tilde{g}_k}(s_k) \|_{C^0_\alpha} = \left(U_{k,\,\mathrm{max}}^{\frac{\epsilon}{2}+1} U_k^{-\frac{3}{2}} \right) U_k^{3-\alpha} \| L_{\tilde{g}_k}(s_k) \|_{C^0_\alpha}$$

When ϵ is sufficiently small, using the hypothesis on the choice of U_k we have

$$U_{k,\,\mathrm{max}}^{\frac{\epsilon}{2}+1} U_k^{-\frac{3}{2}} = o\left(U_k^{-\frac{1}{8}} \right)$$

Together with (5.13) and (5.14), this completes the proof of Proposition 5.3. □

Proposition 5.3 contradicts assumption (5.1) and shows in particular that Case 2 can actually never occur. Hence, x_k satisfies either Case 1 or Case 3. By repeating the same kind of arguments (with, however, substantial modifications depending on the underlying geometry) we prove that neither Case 1 nor Case 3 can occur, and obtain a contradiction. This concludes the proof of the key estimate in Theorem 4.14.

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