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WHAT IS A MONOTONE LAGRANGIAN COBORDISM?

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Abstract. — We explain the notion of Lagrangian cobordism. A flexibility/rigidity dichotomy is illustrated by considering Lagrangian tori in $\mathbb{C}^2$. Towards the end, we present a recent construction by Cornea and the author [8], of monotone cobordisms that are not trivial in a suitable sense.

1. Introduction

In this note we explain the notion of Lagrangian cobordism that goes back to Arnol'd [1, 2]. We pay special attention to a so–called flexibility/rigidity dichotomy which is illustrated when studying Lagrangian tori in $\mathbb{C}^2$. The relevant definitions will be introduced in §2.

Lagrangian submanifolds play an essential role in the understanding of symplectic manifolds and it is therefore natural to try to classify them, up to Hamiltonian isotopy for example. Even for Lagrangians in $\mathbb{C}^n$ the complete classification is unknown. A more attainable goal is to study them up to Lagrangian cobordism.

If one considers only immersed Lagrangians and cobordisms, the problem has a flexible nature: it is completely solved in $\mathbb{C}^n$ and reduces to the computation of stable homotopy groups of certain Thom spaces, as shown by Eliashberg [11]. An account of this and generalization to any symplectic manifold can be found in the book by Audin [4]. This is briefly discussed in §3.1 via the Gromov–Lees theorem.

The embedded case behaves in the same way. Indeed, a Lagrangian surgery trick, see Polterovich [18], performed on self-intersection points of a Lagrangian immersion yields an embedded cobordism. An explicit example will be given in §3.2.

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To get purely symplectic phenomena and rigidity results, it is necessary to restrict the class of embedded cobordisms. As it turns out, the notion of monotonicity plays a key role and this is the content of §3.3 and 3.4, where techniques involving J-holomorphic curves enter the picture. This restriction gives some control on the space of holomorphic curves used to define algebraic invariants such as Lagrangian Floer and quantum homologies.

The last section presents, based on the monotone cobordism category recently introduced by Biran–Cornea [6, 7], a construction by Charette–Cornea [8] of monotone cobordisms which are non-trivial in a suitable sense.

2. Notions of symplectic topology

2.1. Classical mechanics and Hamiltonian isotopies

Here, we give the notions which will be useful for this note. For more context and motivations, the reader can consult the texts by Arnol'd [3], McDuff–Salamon [17], and Polterovich [19].

A symplectic manifold is a manifold M endowed with a two-form $\omega \in \Omega^2(M)$ such that:

- ω is closed, $d\omega = 0$
- ω is non-degenerate, i.e. it induces an isomorphism

$$\begin{aligned} \iota \colon TM &\to T^*M \\ X &\mapsto \omega(X, \cdot) \end{aligned}$$

The non degeneracy condition implies that M is even dimensional; we denote its dimension by $2n$. This notion comes from classical mechanics, where the role of M is played by phase-space, which locally looks like the set of variables $\{(q_i, p_i)\ i = 1, ..., n\}$, where q_i is the position of a particle and p_i its velocity. The associated symplectic form is $\omega_0 := \sum_i dq_i \wedge dp_i$.

Examples.

1. The typical example is any open subset of $\mathbb{R}^{2n}$ with the form ω_0 defined above. We will often identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, and write $\omega_0 = \sum_j dx_j \wedge dy_j$.
2. Any orientable surface Σ_g with an area form is symplectic.
3. The complex projective space $\mathbb{C}P^n$ admits a canonical symplectic structure ω_{FS} defined by restricting ω_0 to $S^{2n+1} \subset \mathbb{C}^{n+1}$ and quotienting by the action of S^1 given by complex multiplication.

4. The cotangent bundle of the interval $T^*[0,1]$ with the form $dt \wedge dy$ is symplectic; here t represents the base coordinate and y the fiber.

The last example will be used to define Lagrangian cobordisms.

Given a time dependent function $H\colon [0,1] \times M \to \mathbb{R}$, called a Hamiltonian, the Hamiltonian vector field X_t^H is uniquely defined by

$$\omega(X_t^H, \cdot) = -dH_t(\cdot)$$

and its associated flow starting at the identity is denoted ϕ_t^H. On $(\mathbb{R}^{2n}, \omega_0)$, the flow lines are the solutions of Hamilton's equations (as a quick check shows), hence symplectic geometry provides a mathematical framework to Hamiltonian mechanics.

Definition 2.1. — *The group of Hamiltonian diffeomorphisms, or isotopies, is*

$$Ham(M, \omega) = \{\phi \in Diff(M) \mid \phi = \phi_1^H \text{ for some } H\}$$

For more on this group, see Polterovich's book [19].

Example 2.2. — Rotation of the first coordinate in $\mathbb{C}P^n$ is Hamiltonian:

$$\phi_t\colon \mathbb{C}P^n \to \mathbb{C}P^n$$

$$[z_0 : z_1 : ... : z_n] \mapsto [e^{2\pi i t} z_0 : z_1 : ... : z_n]$$

Indeed, computations show that ϕ_t preserves the symplectic form, hence the vector field X_t generated by this isotopy is closed, meaning that $d(\omega(X_t, \cdot)) = 0$. Since $H^1(\mathbb{C}P^n; \mathbb{R}) = 0$, there exists a primitive for $\omega(X_t, \cdot)$.

2.2. Lagrangian submanifolds

Definition 2.3. — *Given a closed (i.e. compact without boundary) manifold L of dimension $n = \dim M/2$, an immersion (respectively embedding) $i\colon L \to M$ is called Lagrangian if $i^*\omega = 0$.*

Examples.

1. Any closed curve on a surface is Lagrangian, since a 2-form vanishes on a one-dimensional space.
2. The real projective space $\mathbb{R}P^n \subset (\mathbb{C}P^n, \omega_{FS})$ is Lagrangian, by definition of the symplectic form.
3. Whitney's immersion of the unit n-sphere is Lagrangian in $(\mathbb{C}^n, \omega_0)$: $w\colon S^n \to \mathbb{C}^n$, $(x_0, ..., x_n) \mapsto (x_1, ..., x_n, 2x_0x_1, ..., 2x_0x_n)$.

Immersed Lagrangians were studied by Lees and Gromov, who showed that they are flexible objects whose behaviour is governed by algebraic topology.

Theorem 2.4 (Gromov [14], Lees [16]). — *Fix L a manifold such that $\dim L = \dim M/2$. There exists a Lagrangian immersion $i\colon L \to (M, \omega)$ if and only if there exists a fiberwise Lagrangian monomorphism $F\colon TL \to TM$ such that the induced map $f\colon L \to M$, defined by precomposing F with the zero section and composing with projection to M, satisfies $[f^*\omega] = 0 \in H^2(L)$.*

An easy example is provided by any parallellizable manifold L of dimension n, e.g. a Lie group, where $F\colon L \times \mathbb{R}^n \to TM$ maps each fiber to a fixed Lagrangian linear subspace of $T_pM \cong (\mathbb{R}^{2n}, \omega_0)$ for some p. Notice that the induced map f is constant.

On the other hand, embedded Lagrangians exhibit rigidity properties, a symplectic phenomenon:

Theorem 2.5 ([13]).

- *Let L be a closed Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$, then $0 \neq [\omega_0] \in H^2(\mathbb{C}^n, L) \cong H^1(L)$. In particular, L cannot be a n-sphere, for $n \geqslant 2$.*
- *There exists a symplectic structure ω on $\mathbb{C}^n$ admitting a Lagrangian embedding of a closed simply connected manifold, for $n \geqslant 2$.*

2.3. J-holomorphic curves

The Hermitian product on $\mathbb{C}^n$ can be decomposed into a real part and an imaginary part, where the standard symplectic and complex structures are intertwined: $\langle z_1, z_2\rangle = z_1 \cdot \bar{z}_2 = \omega_0(z_1, iz_2) - i\omega_0(z_1, z_2)$.

Gromov's central idea in [13] was that on a symplectic manifold, integrability of the complex structure is not necessary, an almost complex structure is sufficient to obtain symplectic invariants.

Definition 2.6. — *An almost complex structure on M is an endomorphism $J\colon TM \to TM$ such that $J^2 = -Id$. It is called compatible with the symplectic structure if $\omega(\cdot, J\cdot)$ defines a Riemannian metric and if ω is J-invariant, i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$. The set of compatible almost complex structures is denoted by $\mathcal{J}_\omega$.*

Most rigidity results in symplectic topology use in some way the theory of J-holomorphic curves: maps from complex surfaces, possibly with boundary, $u : (\Sigma, \partial\Sigma) \to (M, L)$ that locally solve the equation $\partial_x u + J\partial_y u = 0$. The left-hand side is usually denoted by $\bar{\partial}_J u$. Note that the boundary condition involves a Lagrangian in M. The key to proving the first part of Theorem 2.5 is an existence result for holomorphic discs.

Theorem 2.7 ([13]). — *Let L be a closed embedded Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$. Then for generic $J \in \mathcal{J}_\omega$ there exists a non-constant J-holomorphic disc $u\colon (D^2, S^1) \to (\mathbb{C}^n, L)$ with boundary on L.*

An application of Stoke's theorem then yields the first part of Theorem 2.5.

3. Lagrangian cobordisms

Lagrangian cobordisms were introduced by Arnol'd [1, 2], although we will not define them in the same generality.

Given two Lagrangians $f_i\colon L_i \to M$, $i = 0, 1$, possibly immersed, a Lagrangian cobordism V between them is a smooth cobordism $(V; L_0, L_1)$ with a Lagrangian immersion

$$i\colon V \to (M \times T^*[0,1], \omega \oplus dt \wedge dy)$$

that is cylindrical:

$$\begin{aligned} V|_{M\times[0,\epsilon)\times\mathbb{R}} &= f_0(L_0) \times [0, \epsilon) \times \{0\} \\ V|_{M\times(1-\epsilon,1]\times\mathbb{R}} &= f_1(L_1) \times (1 - \epsilon, 1] \times \{0\}. \end{aligned}$$

The simplest example is the product of a fixed Lagrangian immersion with a line having endpoints at $(0, 0)$ and $(1, 0)$.

3.1. Flexibility

To illustrate the flexibility of this notion, let us restrict to $(M, \omega) = (\mathbb{C}^n, \omega_0)$. Note that ω_0 has a primitive $\lambda = \sum_i x_i \wedge dy_i$, such that $d\lambda = \omega$, called the Liouville form.

Following Audin [4], we define a L-regular homotopy of immersions $\phi\colon L\times [0, 1] \to \mathbb{C}^n$ to be a family of Lagrangian immersions $\phi_t\colon L \to \mathbb{C}^n$ such that $[\phi_t^*\lambda] \in H^1(L)$ is constant. This means that $\phi_t^*\lambda = \phi_0^*\lambda + dg_t$ for a smooth family of functions $g_t\colon L \to \mathbb{R}$. Denote by X_t the associated vector fields defined by $\frac{d}{dt}\phi_t$. Such a homotopy induces an immersed Lagrangian cobordism, called the suspension (see also §3.4):

$$\begin{aligned} \Phi\colon L \times [0,1] &\to \mathbb{C}^n \times T^*[0,1] \\ (x,t) &\mapsto (\phi_t(x), t, \frac{\partial g}{\partial t} + \lambda(X_t)). \end{aligned} \tag{3.1}$$

The Gromov–Lees theorem also applies to homotopies.

Theorem 3.1 (Gromov [14], Lees [16]). — *There is a bijection between L-regular homotopy classes of Lagrangian immersions $f\colon L \to \mathbb{C}^n$ with tuples consisting of the homotopy class of an isomorphism of complex bundles $f^*\mathbb{C}^n \cong TL \otimes \mathbb{C}$ and the class $[f^*\lambda]$.*

3.2. The Clifford and Chekanov tori

Now let us apply these observations to cobordisms of Lagrangian tori in $\mathbb{C}^2$. The description of the tori and the Lagrangian isotopy between them is taken from Auroux [5, §5], to which we refer for details.

Consider the map $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(x,y) \mapsto xy$. Away from $(0,0)$, π and ω_0 on $\mathbb{C}^2$ define a horizontal distribution

$$\mathrm{Hor} = \langle v \in T(\mathbb{C}^2) \mid \omega_0(v,\cdot)_{|\ker d\pi} = 0\rangle$$

whose associated parallel transport is symplectic. The fibers of π are identified with $\mathbb{C}\backslash\{0\}$ except for $\pi^{-1}(0)$, which is a union of the two complex lines $x = 0$ and $y = 0$. Let S denote the circle $|x| = |y|$ in $\pi^{-1}(1)$ and γ denote the unit circle in the base $\mathbb{C}$. Applying parallel transport to S along γ, one obtains an embedded Lagrangian 2-torus $f_0 : S^1 \times S^1 \to \mathbb{C}^2$ called the Clifford torus, whose image we denote by T^{Cliff}.

Doing the same construction with the circle $\gamma + 2$, one obtains another embedded Lagrangian 2-torus $f_1 : S^1 \times S^1 \to \mathbb{C}^2$ called the Chekanov torus, T^{Chek}. A computation shows that $[f_0^*\lambda] = [f_1^*\lambda]$.

Moreover, there is an isotopy through embedded Lagrangians between these two tori. Indeed, first dilate S in $\pi^{-1}(1)$ to a circle of radius $1+\epsilon$ to get a torus $T^{\mathrm{Cliff}}_{1+\epsilon}$ Lagrangian isotopic to T^{Cliff}_1. Then translate γ to $\gamma+2$ and use parallel transport above this translation to get an isotopy through Lagrangian tori between $T^{\mathrm{Cliff}}_{1+\epsilon}$ and a torus $T^{\mathrm{Chek}}_{1+\epsilon}$. Here it is essential not to use the circle S in the fiber, as parallel transport would then hit the singular point $(0,0)$ where the distribution Hor is not defined. Finally, rescale this last torus to T^{Chek} by shrinking the circle in the fiber.

The previous discussion shows that the trivializations $f_0^* T\mathbb{C}^n \cong T(S^1 \times S^1) \otimes \mathbb{C} \cong f_1^* T\mathbb{C}^n$ are homotopic. Since $[f_0^*\lambda] = [f_1^*\lambda]$, the Gromov–Lees theorem and the suspension construction yield an immersed Lagrangian cobordism $i\colon S^1 \times S^1 \times [0,1] \to \mathbb{C}^2$ between the Clifford and the Chekanov tori, which is an embedding when restricted to a collar neighbourhood of the boundary.

One can get an embedded Lagrangian cobordism from this by first noticing that there exists a Lagrangian immersion i_ϵ homotopic to i in the space of immersions, such that:

(1) the image of i_ϵ is embedded close to the boundary, since the Clifford and Chekanov tori are embedded;
(2) i_ϵ has only finitely many transversal double points of self-intersection, away from which it is an embedding.

Performing Lagrangian surgery (see Polterovich [18]) to get rid of double points, we get an embedded Lagrangian cobordism between these two tori, which might not be a product cobordism anymore. Summarizing the previous discussion, we have:

Proposition 3.2. — *There exists a connected embedded Lagrangian cobordism between the Clifford and the Chekanov tori.*

3.3. A rigid invariant of cobordisms

Last section showed that without any restrictions, immersed Lagrangian cobordisms between embedded Lagrangian submanifolds is a flexible notion governed by algebraic topology via the Gromov–Lees theorem. Moreover, embedded cobordisms are not really restrictive because of Lagrangian surgery. As it turns out, the class of monotone Lagrangians yields interesting rigidity results. First, we need to define the Maslov index of a disc.

Given a relative homotopy class $A \in \pi_2(M, L)$ and a disc representing it $u\colon (D^2, S^1) \to (M, L)$, fix a symplectic trivialization of the pullback bundle $u^*TM \cong D^2 \times (\mathbb{R}^{2n}, \omega_0)$ (this exists, since the 2-disc is contractible). Using this, one gets a loop in $\mathcal{L}(2n)$, the set of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$, given by $(u|_{S^1})^*TL$. As $\pi_1(\mathcal{L}(2n)) \cong \mathbb{Z}$ (see e.g. [17, §2.3]), the homotopy class of this loop is an integer, called the Maslov index of u. In fact, this integer is independent of the disc representing A and of the trivialization of the bundle, hence there is a well-defined morphism, also called the Maslov index

$$\mu\colon \pi_2(M, L) \to \mathbb{Z}.$$

Definition 3.3. — *A Lagrangian $L \subset (M, \omega)$ is monotone if:*

(1) *There is a constant $\rho > 0$ such that the two morphisms*

$$\omega \colon \pi_2(M, L) \to \mathbb{R}, \quad \mu \colon \pi_2(M, L) \to \mathbb{Z}$$

given respectively by integration of ω and the Maslov index, are proportional: $\omega = \rho\mu$;

(2) *The positive generator of the image of μ, called the minimal Maslov number N_L, is at least two.*

Direct computations show that the Clifford and Chekanov tori are monotone.

Now fix a compatible almost complex structure $J \in \mathcal{J}_\omega$ and let

$$\mathcal{M}(N_L; J) = \left\{u \colon (D^2, S^1) \to (M, L) \;\middle|\; \bar{\partial}_J u = 0 \text{ and } \mu[u] = N_L\right\}$$

denote the space of J-holomorphic discs with boundary on L and Maslov class N_L.

Proposition 3.4. — *Let L be a closed, monotone embedded Lagrangian. For a generic choice of $J \in \mathcal{J}_\omega$, $\mathcal{M}(N_L; J)$ is a compact manifold without boundary, of dimension $\dim L + N_L = n + N_L$. Moreover, given two generic J_0 and J_1, there exists a compact cobordism between $\mathcal{M}(N_L; J_0)$ and $\mathcal{M}(N_L; J_1)$.*

The proof relies on Gromov compactness for holomorphic discs, see Frauenfelder [12] for example.

Remark. — This result is false for non-monotone Lagrangians, although the dimension formula still holds. An explicit example was hiding in §3.2; along the isotopy between $T^{\mathrm{Cliff}}_{1+\epsilon}$ and $T^{\mathrm{Chek}}_{1+\epsilon}$, consider the torus T obtained by lifting the circle running through the origin. Then it is possible to show that the cobordism class of $\mathcal{M}(2; J)$ depends on J.

Assume that $N_L = 2$ and L is monotone. This will be the case for the Chekanov or Clifford tori for instance. There is an evaluation map

$$\begin{aligned} ev \colon \mathcal{M}(2; J) \times S^1 &\to L \\ (u, \theta) &\mapsto u(\theta) \end{aligned}$$

The group of biholomorphisms of the 2-disc, denoted by $\mathcal{G}$, has real dimension 3 and acts freely on the domain of ev via $g \cdot (u, \theta) = (u \circ g, g^{-1}(\theta))$. Moreover, ev is obviously invariant under this action, hence there is an induced map:

$$ev \colon \mathcal{M}(2; J) \times S^1 / \mathcal{G} \to L$$

between two compact n-dimensional manifolds without boundary. Denote the mod 2 degree of this map by $d(J;L)$; it represents the algebraic number, modulo 2, of J-holomorphic discs of Maslov index two going through a generic point of L. By Proposition 3.4, this degree is independent of J, since cobordant maps have the same degree, denoted by $m_0(L)$.

Example. — The invariant $m_0(L)$ is not easy to compute in general, but one can check that $m_0(T^{\mathrm{Chek}}) = 1$ and $m_0(T^{\mathrm{Cliff}}) = 0$. See Chekanov–Schlenk [10] for details.

Theorem 3.5 (Chekanov [9]). — *Let V be a connected, monotone embedded Lagrangian cobordism between two monotone Lagrangians L_0 and L_1. Then $m_0(L_0) = m_0(L_1)$.*

Corollary 3.6. — *The Chekanov and Clifford tori are not monotone cobordant.*

3.4. Lagrangian suspension and non trivial cobordisms

Chekanov's result was the first instance of rigidity for cobordisms. Recently, Biran and Cornea [6, 7] gave strong rigidity properties involving not only holomorphic discs, but also the triangulated structure of the derived Fukaya category, which is an invariant related to Lagrangian Floer homology.

To apply their results, it is therefore important to have non trivial examples of monotone cobordisms. At the moment, there are essentially two approaches to this problem. The first one is Lagrangian surgery applied to pairs of Lagrangians intersecting transversally, as noticed by Biran and Cornea.

In this section we explain the second approach, which is related to Lagrangian suspension and is due to Cornea and the author [8].

From here on, it is convenient to use the language of category theory, taken from [6]. Let $\mathcal{C}ob(M)$ denote the category whose objects are ordered families of closed embedded monotone Lagrangians of M. Given two such objects L_0 and L_1, the space of morphisms between them is the set $\mathrm{Mor}_M(L_0, L_1)$ of all monotone Lagrangian cobordisms with boundary $L_0 \coprod L_1$, modulo Hamiltonian isotopies of $M \times T^*[0,1]$ that are constant close to the boundary of $T^*[0,1]$.

Given a Hamiltonian isotopy, $\{\phi_t\}_{t\in[0,1]}$, $\phi_t \in \mathrm{Ham}(M)$, its Hamiltonian functions $\{H_t\}$ and a monotone Lagrangian submanifold L, the suspension of L by ϕ_t is the morphism $\Sigma(\phi_t)(L)\colon \phi_0(L) \to \phi_1(L)$ given by:

$$\begin{aligned}\Sigma(\phi_t)(L)\colon L \times [0,1] &\to M \times T^*[0,1]\\ (x,t) &\mapsto (\phi_t(x), t, H_t(\phi_t(x)))\end{aligned}$$

Compare with Equation (3.1). Although these cobordisms are diffeomorphic to products, they might not be Hamiltonian isotopic to $\Sigma(Id)(L)$.

Example. — Recall from Example 2.2 that rotating the first coordinate in $\mathbb{C}P^n$ is Hamiltonian. Consider the following family of rotations ϕ_t^k, each satisfying $\phi_1^k(\mathbb{R}P^n) = \mathbb{R}P^n$:

$$\begin{aligned}\phi_t^k\colon \mathbb{C}P^n &\to \mathbb{C}P^n\\ [z_0 : z_1 : ... : z_n] &\mapsto [e^{k\pi it} z_0 : z_1 : ... : z_n]\end{aligned}$$

They yield a family of suspension morphisms $V_k = \Sigma(\phi_t^k)(\mathbb{R}P^n)\colon \mathbb{R}P^n \to \mathbb{R}P^n$.

Theorem 3.7 (Charette–Cornea [8], Corollary C). — *The monoid $Mor_{\mathbb{C}P^n}(\mathbb{R}P^n, \mathbb{R}P^n)$ contains an element u such that $\{1, u, u^2, ..., u^n\}$ are pairwise distinct, where u^k is represented by the cobordism V_k.*

The proof of this result relies on the fact that the paths ϕ_t^k, $k = 0, 1, ..., n$, represent n different homotopy classes (rel. endpoints) of paths in $\mathrm{Ham}(\mathbb{C}P^n)$; this in turn uses computations of so-called Lagrangian Seidel elements in Floer homology due to Hyvrier [15]. Finally, the tricky part of the proof relates suspension cobordisms to these Seidel elements.

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