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THE TEICHMÜLLER GEODESIC FLOW AND THE GEOMETRY OF THE HODGE BUNDLE

Carlos Matheus

ABSTRACT. — The Teichmüller geodesic flow is the flow obtained by quasiconformal deformation of Riemann surface structures. The goal of this lecture is to show the strong connection between the geometry of the Hodge bundle (a vector bundle over the moduli space of Riemann surfaces) and the dynamics of the Teichmüller geodesic flow. In particular, we shall provide geometric criterions (based on the variational formulas derived by G. Forni) to detect some special orbits ("totally degenerate") of the Teichmüller geodesic flow. These results have been obtained jointly with J.-C. Yoccoz [MY] and G. Forni, A. Zorich [FMZ1], [FMZ2].

RÉSUMÉ. — Le flot géodésique de Teichmüller est le flot obtenu par déformation quasiconforme des structures de surface de Riemann. Le but de cet exposé est montrer la fort connexion entre la géométrie du fibré de Hodge (un fibré vectoriel audessus de l'espace de modules de surfaces de Riemann) et la dynamique du flot géodésique de Teichmüller. En particulier, on fournira des critères géométriques (basé sur les formules variationnelles derivés par G. Forni) pour detecté certaines orbites speciales ("totalement dégénérées") du flot géodésique de Teichmüller. Ces resultats sont en colaboration avec J.-C. Yoccoz [MY] et G. Forni, A. Zorich [FMZ1], [FMZ2].

CONTENTS

Keywords: Teichmüller dynamics, Kontsevich-Zorich cocycle, Geometry of Hodge bundle, Gauss-Manin connection, variations along geodesics, second fundamental form, totally degenerate origamis.
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The present text corresponds to extended lecture notes of a talk (on November 25, 2010) by the author at the "Séminaire de Théorie Spectrale et Géométrie" of the Institut Fourier - Grenoble. The main goal of the talk (and hence of these notes) was to discuss the relationship between the Teichmüller geodesic flow on the (cotangent bundle of the) moduli space of curves and the geometry of Gauss-Manin connection. To do so, we divide this text into two sections: in the first section we'll spend our time with the introduction of the main actors (e.g., Teichmüller and moduli spaces of curves and Abelian differentials, Teichmüller geodesic flow, Kontsevich-Zorich cocycle, Gauss-Manin connection and its second fundamental form/Kodaira-Spencer map, etc.), and in the second section we show how the tools developed in the first section can be used to detect "totally degenerate" orbits of both Teichmüller flow and the natural $SL(2,\mathbb{R})$-action on the moduli space of Abelian differentials.

At this point, I had two options: either to pursue the "dynamical" consequences of this discuss (e.g., its consequences to Lyapunov exponents of the Kontsevich-Zorich cocycle) or to stop the discussion. I've chosen the second option for two reasons: firstly, I wanted these notes to be as close as possible to the content of the talk, and secondly, the audience of the talk was mainly interested in geometrical aspects of this subject rather than dynamical ones. So, I apologize in advance the "dynamical" readers, but this time I'll make no mention neither to Lyapunov exponents of Kontsevich-Zorich cocycle nor to the Ergodic Theory of Teichmüller flow (and its applications to the deviations of ergodic averages and dynamics of interval exchanges, translation flows and billiards). Instead, I refer them to the excellent survey of A. Zorich [Z] for a nice account of the topics I'm omitting here.

Finally, let me say that the (introductory) material of these notes have a large intersection with some texts I wrote in my mathematical blog (see [DM]), even though there the texts are mostly focused in dynamical aspects of the subject, and the material in Subsection 1.6 and Section 2 are largely inspired by the joint work [FMZ2] (still in preparation) of G. Forni, A. Zorich and the author.

1. Teichmüller flow, Hodge bundle and Kontsevich-Zorich cocycle

1.1. Quasiconformal maps between Riemann surfaces

Given two Riemann surface structures S_0 and S_1 on a given compact topological surface S of genus $g \geqslant 1$, in general there is no *conformal* map $f : S_0 \to S_1$ (i.e., a holomorphic map with non-vanishing derivative). However, we can try to produce maps $f : S_0 \to S_1$ as "nearly conformal" as possible. To do so, one needs a way to "measure" the amount of "non-conformality" of f. A fairly standard procedure is the following one. Given $x \in S_0$ and some (holomorphic) coordinates around $x \in S_0$ and $f(x) \in S_1$, we can write the derivative $Df(x)$ of f at x as $Df(x)u = \frac{\partial f}{\partial z}(x)u + \frac{\partial f}{\partial \overline{z}}(x)\overline{u}$, so that $Df(x)$ maps circles into ellipses of eccentricity

$$\frac{\left|\frac{\partial f}{\partial z}(x)\right| + \left|\frac{\partial f}{\partial \overline{z}}(x)\right|}{\left|\frac{\partial f}{\partial z}(x)\right| - \left|\frac{\partial f}{\partial \overline{z}}(x)\right|} = \frac{1+k(f,x)}{1-k(f,x)} := K(f,x)$$

where $k(f,x) := \left|\frac{\frac{\partial f}{\partial \overline{z}}(x)}{\frac{\partial f}{\partial z}(x)}\right|$. See Figure 1 below.

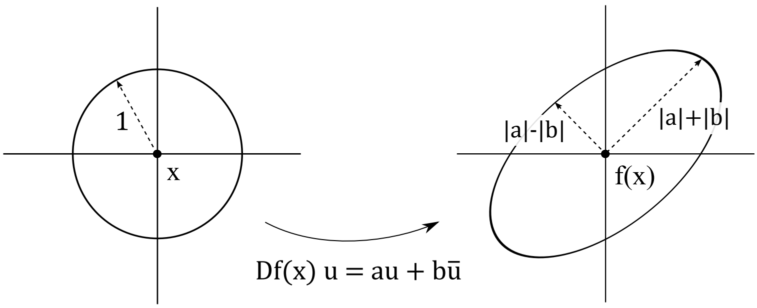

Figure 1.1. Derivative of a quasiconformal C^1 map f.

In the literature, $K(f,x)$ is called the *eccentricity coefficient* of f at x, while

$$K(f) = \sup_{x \in S_0} K(f,x)$$

is the *eccentricity coefficient* of f. Note that, by definition, $K(f) \geqslant 1$ and f is a conformal map if and only if $K(f) = 1$ (or, equivalently, $k(f,x) = 0$ for all $x \in S_0$). Hence, $K(f)$ accomplishes the task of measuring the amount of "non-conformality" of f. Any *reasonably smooth*[(1)] map f is called *quasiconformal* whenever $K(f) < \infty$.

Once we dispose of a good measurement of non-conformality, namely $K(f)$, it is natural to try to measure the distance between two Riemann

[(1)] For instance, any C^1 diffeomorphism f is quasiconformal. In general, a K-quasiconformal map f is a homeomorphism whose distributional derivatives are locally in L^2 and satisfy $\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 \geqslant \frac{1}{K}\left(\left|\frac{\partial f}{\partial z}\right| + \left|\frac{\partial f}{\partial \overline{z}}\right|\right)^2$ locally in L^1.

surfaces structures S_0 and S_1 by minimizing the eccentricity coefficient $K(f)$ among "all" maps $f : S_0 \to S_1$. That is, it is tempting to say that S_0 and S_1 are "close" if we can produce quasiconformal maps $f : S_0 \to S_1$ between them with eccentricity coefficient $K(f)$ "close" to 1. To formalize this, we need first to investigate the "nature" of the quantities $k(f,x) := \left|\frac{\frac{\partial f}{\partial \overline{z}}(x)}{\frac{\partial f}{\partial z}(x)}\right|$.

We start by recalling that $k(f,x)$ *doesn't* provide a *globally defined* function on S_0: indeed, since the definition of $k(f,x)$ *depended* on the choice of local coordinates around $x \in S_0$ and $f(x) \in S_1$, the quantity $k(f,x)$ can only (globally) define a function if it doesn't change under change of coordinates (which is not the case in general). By checking how $k(f,x)$ transforms under changes of coordinates, one can see that the quantities $k(f,x)$ can be collected to globally defined a tensor $\mu(x)$ (of type $(-1,1)$) via the formula:

$$\mu(x) = \frac{\frac{\partial f}{\partial \overline{z}}(x) d\overline{z}}{\frac{\partial f}{\partial z}(x) dz}$$

In the literature, $\mu(x)$ is called a *Beltrami differential*. Note that $\|\mu\|_{L^\infty} < 1$ whenever f is an orientation-preserving quasiconformal map. The intimate relationship between Beltrami differentials and quasiconformal maps is revealed by the following profound theorem of Ahlfors and Bers:

Theorem 1 (Measurable Riemann mapping theorem). — *Let $U \subset \mathbb{C}$ be an open set and consider $\mu \in L^\infty(U)$ verifying $\|\mu\|_{L^\infty} < 1$. Then, there exists a quasiconformal map $f : U \to \mathbb{C}$ such that the Beltrami equation*

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

is satisfied (in the sense of distributions). Furthermore, f is unique modulo composition with conformal maps: if g is another solution of the Beltrami equation above, there exists $\varphi : f(U) \to \mathbb{C}$ such that $g = \varphi \circ f$.

A direct consequence of this striking theorem is the following theorem (whose proof we left as an exercise to the reader):

Proposition 1. — *Let X be a Riemann surface and μ be a Beltrami differential on X. Given an atlas $\varphi_i : U_i \to \mathbb{C}$ (compatible with the complex structure on X), let us denote by μ_i the function defined by $\mu|_{U_i} := \varphi_i^*\left(\frac{\mu_i d\overline{z}}{dz}\right)$. Then, there exists a family of maps $\psi_i(\mu) : V_i \to \mathbb{C}$ solving the Beltrami equations*

$$\frac{\partial \psi_i(\mu)}{\partial \overline{z}} = \mu_i \frac{\partial \psi_i(\mu)}{\partial z}$$

such that $\psi_i : V_i \to \psi_i(V) \subset \mathbb{C}$ *are homeomorphisms. Moreover,* $\psi_i \circ \varphi_i : U_i \to \mathbb{C}$ *form an atlas associated to a well-defined Riemann surface structure* X_μ *in the sense that it doesn't depend on the initial choice of atlas* $\varphi_i : U_i \to \mathbb{C}$ *and the choice of* ψ_i *verifying the corresponding Beltrami equations.*

In simpler terms, this proposition (a by-product of Ahlfors-Bers theorem) permits to deform Riemann surface structures X using Beltrami differentials μ. Actually, this is part of a more general phenomenon: given two Riemann surface structures S_0 and S_1, we can always relate them by quasiconformal maps with "optimal" eccentricity coefficient. More precisely, we have the following remarkable theorem of Teichmüller:

THEOREM 2. — *Given two Riemann surface structures* S_0 *and* S_1 *on a compact topological surface* S *of genus* $g \geqslant 1$ *and a homeomorphism* $h : S \to S$, *there exists a quasiconformal map* $f : S_0 \to S_1$ *minimizing the eccentricity coefficient* $K(g)$ *among all quasiconformal maps* $g : S_0 \to S_1$ *isotopic to* h. *Furthermore, whenever* $f : S_0 \to S_1$ *minimizes the eccentricity coefficient in a given isotopy class, the eccentricity coefficient of* f *at "typical" points* $x \in S_0$ *is constant, i.e.,* $K(f,x) = K(f)$ *for all but finitely many* $x_1, \dots, x_n \in S_0$. *Also, quasiconformal maps minimizing eccentricity in an isotopy class are unique modulo (pre and/or post) composition with conformal maps isotopic to identity.*

In the literature, quasiconformal maps minimizing eccentricity in a given isotopy class are called *extremal maps*. Using extremal quasiconformal maps, we can define a *distance* between two Riemann surface structures S_0 and S_1 by the formula:

$$d(S_0, S_1) = \frac{1}{2}\inf\{\log K(g) : g \text{is isotopic to identity}\}$$

The metric d is the so-called *Teichmüller metric*.

In this way, we have a natural metric on the *Teichmüller space* of curves, that is, the space $\mathcal{T}(S)$ of Riemann surface structures on S *modulo* conformal maps *isotopic to identity*. Also, since the Teichmüller metric is equivariant with respect to the action of the so-called *mapping class group* $\Gamma_g = \Gamma(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}^+_0(S)$ of isotopy classes of (orientation-preserving) diffeomorphisms[(2)], the Teichmüller metric induces a natural metric on the

[(2)] Here $\mathrm{Diff}^+(S)$ denotes the space of orientation-preserving diffeomorphisms of S and $\mathrm{Diff}^+_0(S)$ denotes the connected component of the identity inside $\mathrm{Diff}^+(S)$, i.e., the set of orientation-preserving diffeomorphisms isotopic to identity.

moduli space of curves, that is, the space $\mathcal{M}(S)$ of Riemann surface structures on S *modulo* conformal maps (i.e., $\mathcal{M}(S) = \mathcal{T}(S)/\Gamma(S)$).

Remark 1.1. — It is known that the Teichmüller metric is *not* a Riemannian metric but only a Finsler metric. We will come back to this point in the next subsection.

Remark 1.2. — It is also known that the Teichmüller space $\mathcal{T}(S)$ is a complex manifold of complex dimension 1 when $g = 1$, and $3g - 3$ when $g \geqslant 2$, which is *homeomorphic* (but not diffeomorphic) to the unit open ball of $\mathbb{C}^{\dim_{\mathbb{C}}(\mathcal{T}(S))}$. However, the moduli space $\mathcal{M}(S)$ is only a complex orbifold (due to the fact that there are Riemann surfaces which are "more symmetric" than others). Indeed, this lack of smoothness is already present in the genus 1 case: since the Teichmüller space of tori (equipped with Teichmüller metric) can be identified with the upper-half plane $\mathbb{H} \subset \mathbb{C}$ (equipped with the hyperbolic metric) and the mapping class group Γ_1 can be identified with $SL(2,\mathbb{Z})$, it follows that the moduli space of tori is $\mathbb{H}/SL(2,\mathbb{Z})$ (where $SL(2,\mathbb{Z})$ acts by Möbius transformations), an *orbifold* with conical points at $i, e^{\pi i/3} \in \mathbb{H}$ (because the $SL(2,\mathbb{Z})$ stabilizer of these points have orders 4 and 6 resp. while it is trivial at other points). See the author's mathematical blog [DM] for an illustrated discussion.

In the sequel, we will study the Teichmüller geodesic flow (i.e., the geodesic flow associated to the Teichmüller metric). In particular, it is important to understand the cotangent bundle of Teichmüller and moduli spaces of curves.

1.2. Cotangent bundle of $\mathcal{T}(S)$ and $\mathcal{M}(S)$

Recall from the discussion of the previous subsection that the Teichmüller space of curves can be modeled by the space of Beltrami differentials. By definition, Beltrami differentials μ are tensor of type $(-1,1)$ with $\|\mu\|_{L^\infty} < 1$. Therefore, the tangent bundle of $\mathcal{T}(S)$ can be naturally identified with the space of essentially bounded (L^∞) tensors of type $(-1,1)$ (because Beltrami differentials form the unit open ball of this Banach space). Hence, the cotangent bundle $\mathcal{Q}(S)$ of the Teichmüller space of curves $\mathcal{T}(S)$ can be naturally identified with the space of *integrable quadratic differentials* on S, i.e., the space of (integrable) tensors q of type $(2,0)$ (that is, locally q has the form $q(z)dz^2$). Intuitively, the cotangent bundle consists of objets q (tensors of some type) such that the pairing

$$\langle q, \mu \rangle = \int_S q\mu$$

is well-defined. When μ is a tensor of type $(-1,1)$ and q is a tensor of type $(2,0)$, we can write $q\mu = q(z)\mu(z)dz^2\frac{\overline{dz}}{dz} = q(z)\mu(z)dz\overline{dz} = q(z)\mu(z)|dz|^2$, i.e., $q\mu$ is a tensor of type $(1,1)$ (that is, an area form). Thus, since μ is essentially bounded, the pairing is well-defined whenever q is integrable.

Remark 1.3. — It is possible to prove that the Teichmüller metric on Teichmüller space is compatible with the *Finsler metric* associated to the family of L^1 norms on the space of integrable quadratic differentials (cotangent bundle of Teichmüller space). However, this Finsler metric is *not* a Riemannian metric: indeed, by developing appropriate asymptotic formulas, one can show this Finsler metric (family of norms on the cotangent space) depends C^1 on the base point but not C^2! See [H] for more details.

Remark 1.4. — An alternative popular metric on Teichmüller space is the *Weil-Petersson metric* coming from the Hermitian inner-product $\langle q_1, q_2\rangle_{WP} = \int_S \frac{\overline{q_1}q_2}{\rho_S^2}$ where ρ_S is the hyperbolic metric (of curvature -1) of the Riemann surface S and ρ_S^2 is the associated area form. This is a Riemannian metric such that 2-form associated to its imaginary part $\mathrm{Im}\langle .,.\rangle_{WP}$ is closed, i.e., the Weil-Petersson metric is a Kähler metric. Some important facts about the Weil-Petersson geodesic flow are:

- it is a negatively curved incomplete metric with unbounded curvature (i.e., the sectional curvatures can approach either 0 or $-\infty$);
- S. Wolpert showed that the geodesic flow is defined for all times in a full measure subset of the cotangent bundle of the Teichmüller space;
- J. Brock, H. Masur and Y. Minsky showed that this geodesic flow is transitive, its set of periodic orbits is dense and it has infinite topological entropy;
- based on important previous works of S. Wolpert and C. McMullen, K. Burns, H. Masur and A. Wilkinson [BMW] proved that this geodesic flow is ergodic with respect to Weil-Petersson volume form.

We recommend the article of [BMW] and references therein for the reader interested in the Weil-Petersson flow.

Next, let's see how the Teichmüller flow looks like after this identification of $\mathcal{Q}(S)$ with the space of integrable quadratic differentials. To do so, we need to better understand the geometry of extremal quasiconformal maps. For this task, we invoke another remarkable theorem of Teichmüller:

THEOREM 3 (Teichmüller). — *Given an extremal map $f : S_0 \to S_1$, there is an atlas φ_i on S_0 such that*

- *outside the neighborhoods of finitely many points $x_1, \dots, x_n \in S_0$, the changes of coordinates of the atlas φ_i have the form $z \mapsto \pm z + c$, $c \in \mathbb{C}$;*
- *the horizontal (resp. vertical) foliation $\{Im\varphi_i \equiv constant\}$ (resp. $\{Re\varphi_i \equiv constant\}$) is tangent to the major (resp. minor) axis of the infinitesimal ellipses which are mapped by Df into infinitesimal circles, and*
- *in the coordinates provided by the atlas φ_i, f expands the horizontal direction by $\sqrt{K}$ and contracts the vertical direction by $1/\sqrt{K}$ (where $K = K(f)$).*

The figure 1.2 below illustrates the action of an extremal map f in appropriate coordinates φ_i.

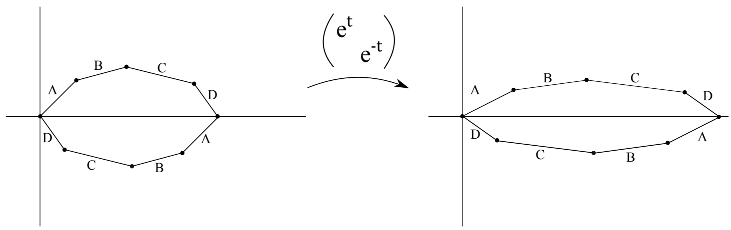

Figure 1.2. Action of extremal map with eccentricity $K = e^{2t}$

An atlas φ_i satisfying the property in the first item of Teichmüller's theorem is called a *half-translation structure*. In this language, Teichmüller's theorem says that extremal maps $f : S_0 \to S_1$ are very easy to describe in terms of half-translation structures: it suffices to expand (resp. contract) the horizontal (resp. vertical) direction by a factor of $e^{d(S_0,S_1)} = \sqrt{K(f)}$. This provides an elegant way of describing Teichmüller geodesic flow in terms of half-translation structures.

Thus, it remains to relate half-translation structures to quadratic differentials to complete the description of Teichmüller geodesic flow in the cotangent bundle of $\mathcal{T}(S)$. Given a half-translation structure $\varphi_i : U_i \to \mathbb{C}$, we can construct a quadratic differential q by pulling back the canonical quadratic differential dz^2 on $\mathbb{C}$ through φ_i: indeed, this procedure leads to a well-defined quadratic differential because the changes of coordinates between the several φ_i always are of the form $z \mapsto \pm z + c$ (outside the neighborhoods of finitely many points). Conversely, given a quadratic differential q, we take an atlas $\varphi_i : U_i \to \mathbb{C}$ such that $q|_{U_i} = \varphi_i^*(dz^2)$ outside the neighborhoods of the finitely many singularities (zeros and/or poles) of q. Because q is obtained by pulling back dz^2, we have that the changes of coordinates $z \mapsto z'$ send $(dz)^2$ to $(dz')^2$ (outside the neighborhoods of finitely many points), and hence they have the form $z \mapsto \pm z + c$, $c \in \mathbb{C}$, that is, φ_i is a half-translation structure.

Remark 1.5. — Generally speaking, a quadratic differential on a Riemann surface can be either *orientable* or *non-orientable*. More precisely, given a quadratic differential q and denoting by φ_i the corresponding half-translation structure, we say that q is orientable if the horizontal and vertical foliations $\{\mathrm{Im}\varphi_i = constant\}$ and $\{\mathrm{Re}\varphi_i = constant\}$ are orientable (and q is non-orientable otherwise). Alternatively, q is *orientable* if the changes of coordinates of the atlas φ_i outside the singularities of q have the form $z \mapsto z + c$, $c \in \mathbb{C}$, that is, φ_i is a *translation structure*. Equivalently, q is orientable if it is the global square of a Abelian differential (i.e., holomorphic 1-form) ω, that is, $q = \omega^2$.

For the sake of simplicity, these notes will be mostly focused on the case of *orientable* quadratic differentials. Actually, each time our quadratic differential q is orientable, we will immediately forget about q and we will concentrate on a choice ω of global square root of q.

Remark 1.6. — In general, there is not a great loss of generality by restricting to the case of orientable quadratic differentials: in fact, given a non-orientable quadratic differential, there is a canonical double-cover procedure such that the lift of q is the global square of a holomorphic 1-form.

Before passing to the next subsection, let's introduce some notation. We denote by $\widehat{\mathcal{H}_g}$ the Teichmüller space of Abelian differentials of genus $g \geqslant 1$, that is, the space of pairs (S_0, ω) of Riemann surface structure on a genus $g \geqslant 1$ compact topological surface S and a choice of (non-zero) Abelian differential (holomorphic 1-form) ω on S_0 *modulo* conformal maps isotopic to identity. Similarly, we denote by $\mathcal{H}_g$ the moduli space of Abelian differentials of genus $g \geqslant 1$, that is, the space of pairs (S_0, ω) as above *modulo* conformal maps. Again, we have that $\mathcal{H}_g = \widehat{\mathcal{H}_g}/\Gamma_g$ (where the mapping class group Γ_g acts on (S_0, ω) by pullback on S_0 *and* ω).

1.3. Teichmüller flow and $SL(2,\mathbb{R})$ action on $\widehat{\mathcal{H}_g}$ and $\mathcal{H}_g$

In the case of Abelian differentials ω, the discussion of the previous subsection says that the orbit $g_t(S_0, \omega)$ of the Teichmüller geodesic flow at a point (S_0, ω) of the cotangent bundle of $\mathcal{T}(S)$ is given by $g_t(S_0, \omega) = (S_t, \omega_t)$ where S_t is the unique Riemann surface structure such that the Abelian differential

$$\omega_t = e^t \cdot \operatorname{Re}\omega + i \cdot e^{-t} \cdot \operatorname{Im}\omega$$

is holomorphic.

In other words, by writing

$$\omega_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \operatorname{Re}\omega \\ \operatorname{Im}\omega \end{pmatrix},$$

we see that the Teichmüller geodesic flow corresponds to the action of the diagonal subgroup $\operatorname{diag}(e^t, e^{-t})$ of $SL(2,\mathbb{R})$.

Of course, the previous equation hints that Teichmüller flow is part of a $SL(2,\mathbb{R})$-action: indeed, given a matrix $M \in SL(2,\mathbb{R})$ and a Abelian differential ω, we can define

$$M(\omega) := M \cdot \begin{pmatrix} \operatorname{Re}\omega \\ \operatorname{Im}\omega \end{pmatrix}$$

Actually, the attentive reader may complain that one can even define a $GL^+(2,\mathbb{R})$-action on the space of Abelian differentials (by the same formula). In fact, even though one disposes of this larger action, we refrain from using it because geodesic flows live naturally in the *unit* cotangent bundle. In the case of the Teichmüller space, the discussion of the previous subsection implies that the Abelian differentials living on the unit cotangent bundle are precisely the *unit area* Abelian differentials, i.e., Abelian differentials ω such that the *total area function* A evaluated at ω

$$A(\omega) := \frac{i}{2} \int_S \omega \wedge \overline{\omega}$$

equals to 1. In particular, since the total area function is invariant precisely under the $SL(2,\mathbb{R})$-action, we prefer to stick to it (as we're going to move to the unit cotangent bundle sooner or later). In any event, we denote by $\widehat{\mathcal{H}_g}^{(1)}$, resp. $\mathcal{H}_g^{(1)}$, the Teichmüller, resp. moduli, space of unit area Abelian differentials.

Besides the total area function, the Teichmüller flow and the $SL(2,\mathbb{R})$ action on the Teichmüller and moduli space of Abelian differentials preserves the so-called *singularity pattern* of Abelian differentials. In the next subsection, we recall this notion and we review some important structures on $\widehat{\mathcal{H}_g}$ and $\mathcal{H}_g$ related to it.

1.4. Stratification of $\widehat{\mathcal{H}_g}$ and $\mathcal{H}_g$, and period coordinates

Given a non-zero Abelian differential ω on a genus $g \geqslant 1$ Riemann surface S_0, we can list the orders of its zeros, say $\varkappa = (k_1, \dots, k_\sigma)$. Recall that, by Poincaré-Hopf formula, this list $\varkappa$ is subjected to the constraint $\sum\limits_{m=1}^{\sigma} k_m = 2g-2$. We denote by $\widehat{\mathcal{H}(\varkappa)}$, resp. $\mathcal{H}(\varkappa)$, the subset of $\widehat{\mathcal{H}_g}$, resp. $\mathcal{H}_g$ consisting of Abelian differentials whose list of orders of zeros coincides with $\varkappa$. By definition, we have

$$\widehat{\mathcal{H}_g} = \bigcup_{\substack{\varkappa=(k_1,\dots,k_\sigma)\\ \sum\limits_{m=1}^{\sigma} k_m=2g-2}} \widehat{\mathcal{H}(\varkappa)} \quad \text{and} \quad \mathcal{H}_g = \bigcup_{\substack{\varkappa=(k_1,\dots,k_\sigma)\\ \sum\limits_{m=1}^{\sigma} k_m=2g-2}} \mathcal{H}(\varkappa)$$

In the literature, the sets $\widehat{\mathcal{H}(\varkappa)}$ and $\mathcal{H}(\varkappa)$ are called the *strata* of $\widehat{\mathcal{H}_g}$ and $\mathcal{H}_g$. The terminology is justified by the fact that each stratum $\widehat{\mathcal{H}(\varkappa)}$ is a complex *manifold* of complex dimension $2g+\sigma-1$, and each stratum $\mathcal{H}(\varkappa) = \widehat{\mathcal{H}(\varkappa)}/\Gamma_g$ is a complex *orbifold* with the same complex dimension. Indeed, this can be proved with the aid of the so-called *period coordinates* on $\widehat{\mathcal{H}(\varkappa)}$. More precisely, given $\omega_0 \in \widehat{\mathcal{H}(\varkappa)}$, we denote by Σ_0 the set of its zeros. The relative homology $H_1(S, \Sigma_0, \mathbb{C})$ is generated by $2g$ absolute homology classes $a_1, \dots, a_g, b_1, \dots, b_g$ and $\sigma - 1$ relative cycles $c_1, \dots, c_{\sigma-1}$ connecting an arbitrarily fixed point in Σ_0 to the other points in Σ_0. In particular, for every $\omega \in \widehat{\mathcal{H}(\varkappa)}$ nearby ω_0, we have a map

$$\omega \mapsto \left(\int_{a_1} \omega, \dots, \int_{a_g} \omega, \int_{b_1} \omega, \dots, \int_{b_g} \omega, \int_{c_1} \omega, \dots, \int_{c_{\sigma-1}} \omega\right) \in \mathbb{C}^{2g+\sigma-1}$$

Alternatively, by integration, we have a local map from some neighborhood of ω_0 to $\operatorname{Hom}(H_1(S, \Sigma_0, \mathbb{Z}), \mathbb{C}) \simeq H^1(S, \Sigma_0, \mathbb{C})$. Such a local map is called a *period coordinate* because it is obtained from the periods of ω and it is a local homeomorphism (so that it can be used as local coordinates in $\widehat{\mathcal{H}(\varkappa)}$). The reader can check that the change of coordinates between two period coordinates always corresponds to an affine map of $\mathbb{C}^{2g+\sigma-1}$. Therefore, $\widehat{\mathcal{H}(\varkappa)}$ equipped with period coordinates is a complex affine manifold of dimension $2g+\sigma-1$ (as claimed). Also, since these period coordinates are compatible with the action of the mapping class group Γ_g, the period coordinates endow $\mathcal{H}(\varkappa)$ with a structure of complex affine orbifold of dimension $2g+\sigma-1$.

Remark 1.7. — It is known that the strata are *not* connected in general (for instance, W. Veech showed that $\mathcal{H}(4)$ is not connected and P. Arnoux showed that $\mathcal{H}(6)$ is not connected). After the work of M. Kontsevich and A. Zorich [KZ], we dispose nowadays of a *complete* classification of connected components of strata: for instance, every stratum has 3 connected components at most and they can be distinguished by certain invariants (parity of spin and hyperellipticity).

Closing this subsection, the reader is invited to check that the $SL(2,\mathbb{R})$-action (and, *a fortiori*, the Teichmüller geodesic flow) on $\mathcal{H}_g$ preserves each stratum $\mathcal{H}(\varkappa)$ (and hence its connected components).

For the next subsection, we will use the period coordinates to reduce the study of the *derivative* of the Teichmüller flow g_t to the so-called *Kontsevich-Zorich cocycle* on the *Hodge bundle* over $\mathcal{H}_g^{(1)}$.

1.5. Derivative of Teichmüller flow and Kontsevich-Zorich cocycle

Using the period coordinates, we see that the derivative Dg_t of the Teichmüller geodesic flow on the Teichmüller space of Abelian differentials $\widehat{\mathcal{H}_g}$ can be identified with the trivial product map

$$g_t \times id : \widehat{\mathcal{H}_g} \times H^1(S,\Sigma,\mathbb{C}) \to \widehat{\mathcal{H}_g} \times H^1(S,\Sigma,\mathbb{C})$$

of the Teichmüller flow on the first entry and the identity map on the second entry. Now, when passing to the moduli space of Abelian differentials $\mathcal{H}_g$, one should do the quotient of this trivial product map by the action of the mapping class group Γ_g on both factors. In particular, the bundle $(\widehat{\mathcal{H}_g} \times H^1(S,\Sigma,\mathbb{C}))/\Gamma_g$ and the derivative of Dg_t on $\mathcal{H}_g$ are no longer trivial. However, we claim that the possibly interesting part of the action of Dg_t occurs only on the absolute part of the cohomology. In other words, we affirm that the action of Dg_t on $(\widehat{\mathcal{H}_g} \times (H^1(S,\Sigma,\mathbb{C})/H^1(S,\mathbb{C})))/\Gamma_g$ is "boring".(3) Indeed, this is more clearly seen by using duality and considering the action of Dg_t on the quotient $H_1(S,\Sigma_0,\mathbb{C})/H_1(S,\mathbb{C})$ of the relative homology by the absolute homology: by writing $H_1(S,\Sigma_0,\mathbb{C})/H_1(S,\mathbb{C}) = \mathbb{C}\otimes H_1(S,\Sigma_0,\mathbb{R})/H_1(S,\mathbb{R})$, one can write

$$Dg_t|(\widehat{\mathcal{H}_g} \times (H_1(S,\Sigma_0,\mathbb{C})/H_1(S,\mathbb{C})))/\Gamma_g = \operatorname{diag}(e^t,e^{-t})\otimes G_t^{rel}.$$

(3) Here we consider the quotient $H^1(S,\Sigma,\mathbb{C})/H^1(S,\mathbb{C})$ because, generally speaking, the absolute cohomology $H^1(S,\mathbb{C})$ doesn't admit an equivariant supplement inside the relative cohomology. Indeed, if this were the case our arguments concerning the relative part would be easier, but the example in Appendix B of the article [MY] shows that this is not always the case.

In this way, our claim that the action of Dg_t on the "purely relative homology" part is "boring" corresponds to the fact that G_t^{rel} is the identity. To show this, we observe that the image $c_t := G_t^{rel}(c_0)$ under G_t^{rel} of a relative cycle c_0 joining two points p and q in Σ_0 is again a relative cycle joining the *same* points p and q. Therefore, $c_t - c_0$ is a cycle in absolute homology, that is, c_0 and c_t represent the same element of $H_1(S,\Sigma_0,\mathbb{R})/H_1(S,\mathbb{R})$. Hence, G_t^{rel} acts by the identity on $H_1(S,\Sigma_0,\mathbb{R})/H_1(S,\mathbb{R})$ and the claim is proved.

Therefore, the "interesting" part of the action of Dg_t occurs on the "absolute part" $(\widehat{\mathcal{H}_g} \times H^1(S,\mathbb{C}))/\Gamma_g$. In the literature, $H_g^1(\mathbb{C}) := (\widehat{\mathcal{H}_g} \times H^1(S,\mathbb{C}))/\Gamma_g$ is called the *(complex) Hodge bundle*. Similarly, $H_g^1 := (\widehat{\mathcal{H}_g} \times H^1(S,\mathbb{R}))/\Gamma_g$ is called the *(real) Hodge bundle*.

As before, using the fact that $H^1(S,\mathbb{C}) = \mathbb{C} \otimes H^1(S,\mathbb{R})$, we can write

$$Dg_t = \operatorname{diag}(e^t, e^{-t}) \otimes G_t^{KZ}$$

where $G_t^{KZ} : H_g^1 \to H_g^1$ is the quotient of the trivial product $g_t \times id : \widehat{\mathcal{H}_g} \times H^1(S,\mathbb{R}) \to \widehat{\mathcal{H}_g} \times H^1(S,\mathbb{R})$ by the action of the mapping class group Γ_g. In the literature, G_t^{KZ} is called the *Kontsevich-Zorich cocycle*.(4) To see that this cocycle is far from trivial in general, we observe that the cohomology classes of the real and imaginary parts of any Abelian differential ω defines a plane $H_{st}^1 := \mathbb{R}\cdot[\operatorname{Re}\omega] \oplus \mathbb{R}\cdot[\operatorname{Im}\omega]$ in the absolute cohomology $H^1(S,\mathbb{R})$ which is equivariant under the action of G_t^{KZ}. Furthermore, by identifying $[\operatorname{Re}\omega]$ with the vector $e_1 = (1,0) \in \mathbb{R}^2$ and $[\operatorname{Im}\omega]$ with the vector $e_2 = (0,1) \in \mathbb{R}^2$ (so that $H_{st}^1 \simeq \mathbb{R}^2$), one sees that $G_t^{KZ}|_{H_{st}^1} \simeq \operatorname{diag}(e^t, e^{-t})$. In particular, the action of G_t^{KZ} is *never* completely trivial!

In resume, the Kontsevich-Zorich cocycle captures the "essence" of the derivative of the Teichmüller flow, so that we can safely restrict our attention exclusively to the study of G_t^{KZ} when trying to understand the Teichmüller flow.

For the task of understanding the Kontsevich-Zorich cocycle, it will be useful to introduce the *Gauss-Manin connection* and the *Hodge norm* on the (real and complex) Hodge bundle. This is the main concern of the next section.

(4) These definitions of the Hodge bundle and Kontsevich-Zorich cocycle work *only* for Abelian differentials with no non-trivial automorphisms. For the general definitions see [MYZ].

1.6. Gauss-Manin connection and geometry of Hodge bundle

The fibers $H^1(S,\mathbb{C})$ of the complex Hodge bundle $H^1_g(\mathbb{C})$ over any Abelian differential ω come equipped with a natural *Hermitian* intersection form

$$(\omega_1,\omega_2)=\frac{i}{2}\int_S \omega_1\wedge\overline{\omega_2}$$

Given (S,ω), we denote by $H^{1,0}(S,\omega)$ the subspace (of complex dimension g) of $H^1(S,\mathbb{C})$ consisting of holomorphic 1-forms and by $H^{0,1}(S,\omega)$ the subspace (of complex dimension g) of $H^1(S,\mathbb{C})$ consisting of anti-holomorphic 1-forms. In this way, we can decompose the complex Hodge bundle as a direct sum

$$H^1_g(\mathbb{C})=H^{1,0}\oplus H^{0,1}$$

of two orthogonal (with respect to $(.,.)$) subbundles. Furthermore, the restriction of $(.,.)$ to $H^{1,0}(S,\omega)$ is positive-definite, while its restriction to $H^{0,1}(S,\omega)$ is negative-definite. In particular, $(.,.)$ is a Hermitian form of signature (g,g) on $H^1_g(\mathbb{C})$.

The fundamental *Hodge representation theorem* asserts that any *real* cohomology class $c\in H^1(S,\mathbb{R})$ can be written as $c=[\operatorname{Re} h(c)]$ for an *unique* $h(c)=h_\omega(c)\in H^{1,0}(S,\omega)$ (holomorphic 1-form). In the literature, the real cohomology class $[\operatorname{Im} h(c)]$ is denote by $*c$ and the operator $c\mapsto *c$ is called the *Hodge * operator*.

By combining Hodge's representation theorem with the fact that the Hermitian form $(.,.)$ is positive-definite on $H^{1,0}(S,\omega)$, we can introduce an inner-product (and hence a norm) on the *real* Hodge bundle H^1_g. This norm is called the *Hodge norm*. By defintion, we have

$$\|c\|^2=\frac{i}{2}\int_S h(c)\wedge\overline{h(c)}=\int_S \operatorname{Re} h(c)\wedge \operatorname{Im} h(c)$$

In the sequel, the Hodge *inner-product* on the real Hodge bundle will be denoted by $(.,.)_\omega=(.,.)$ (slightly abusing of the notation) while the *symplectic* intersection form on the real cohomology $H^1(S,\mathbb{R})$ (and the real Hodge bundle) will be denoted by $\langle .,.\rangle$. In this language, one has

$$(h(c_1),h(c_2))=(c_1,c_2)+i\langle c_1,c_2\rangle$$

Now we connect the Hermitian intersection form $(.,.)$ with the geometry of the complex Hodge bundle as follows. Inside the fibers $H^1(S,\mathbb{C})$ of the complex Hodge bundle $H^1_g(\mathbb{C})$ we have a natural lattices $H^1(S,\mathbb{Z}\oplus i\mathbb{Z})$. By declaring that the vectors of these lattices in nearby fibers (i.e., fibers associated to nearby Abelian differentials) are identified by parallel transport, we obtain the so-called *Gauss-Mannin connection* $D_{H^1_g(\mathbb{C})}$ on $H^1_g(\mathbb{C})$. By definition, the Gauss-Manin connection is a flat connection on $H^1_g(\mathbb{C})$ preserving the Hermitian intersection form $(.,.)$. However, the reader should be aware that the decomposition

$$H^1_g(\mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

is *not* equivariant (neither by Gauss-Manin connection nor by Teichmüller flow)! In any case, this decomposition defines an orthogonal projection $\pi_1 : H^1_g(\mathbb{C}) \to H^{1,0}$ of vector bundles.

Since $H^{1,0}$ is a Hermitian vector bundle (w.r.t. $(.,.)$), we can consider the unique (non-flat) connection $D_{H^{1,0}}$ on $H^{1,0}$ which is compatible with $(.,.)$ and the Gauss-Manin connection $D_{H^1_g(\mathbb{C})}$, namely,

$$D_{H^{1,0}} = \pi_1 \circ D_{H^1_g(\mathbb{C})}$$

See [GH, p.73] for more details. The second fundamental form

$$A_{H^{1,0}} := D_{H^1_g(\mathbb{C})} - D_{H^{1,0}} = (\mathrm{Id} - \pi_1) \circ D_{H^1_g(\mathbb{C})}$$

is a differential form of type $(1,0)$ with values on the bundle of complex-linear maps from $H^{1,0}$ to $H^{0,1}$. In particular, $A_{H^{1,0}}$ can be written as a matrix-valued differential form of type $(1,0)$. In the literature, $A_{H^{1,0}}$ is also called *Kodaira-Spencer map*.

The curvatures $\Theta_{D_{H^1_g(\mathbb{C})}}$ and $\Theta_{D_{H^{1,0}}}$ of the connections $D_{H^1_g(\mathbb{C})}$ and $D_{H^{1,0}}$, and the second fundamental form (Kodaira-Spencer map) are related by the following formula:

$$\Theta_{D_{H^{1,0}}} = \Theta_{D_{H^1_g(\mathbb{C})}}|_{H^{1,0}} + A^*_{H^{1,0}} \wedge A_{H^{1,0}}.$$

See [GH, p.78] for more details. Since the Gauss-Manin connection $D_{H^1_g(\mathbb{C})}$ is flat, its curvature vanishes, so that, in our context, the previous formula becomes:

$$\Theta_{D_{H^{1,0}}} = A^*_{H^{1,0}} \wedge A_{H^{1,0}}.$$

By taking an orthonormal basis $\{\omega_1, \dots, \omega_g\}$ of $H^{1,0}$ and observing that $\{\overline{\omega_1}, \dots, \overline{\omega_g}\}$ is an orthonormal basis of $H^{0,1}$, the previous equation becomes

$$\Theta = -A \cdot A^*$$

where $\Theta = \Theta_\omega$ and $A = A_\omega$ are the matrix forms of $\Theta_{D_{H^{1,0}}}$ and $A_{H^{1,0}}$ at the point (S, ω). In particular, the curvature of the Hermitian bundle $H^{1,0}$ (identified with the real Hodge bundle by Hodge's representation theorem) is *negative semi-definite*.

As it is well-known in Differential Geometry, one can use the second fundamental form to derive first order variational formulas along geodesics. This is the content of the following lemma:

LEMMA 1.1. — *The Lie derivative $\mathcal{L}(c_1,c_2)_\omega$ of the Hodge inner product of two parallel (i.e., locally constant) sections $c_1, c_2 \in H^1(S,\mathbb{R})$ in the direction of the Teichmüller flow can be written as*

$$\mathcal{L}(c_1,c_2)_\omega := \frac{d}{dt}(c_1,c_2)_{g_t(\omega)}|_{t=0} = 2\,Re(A_\omega(h_\omega(c)),\overline{h_\omega(c_2)})$$

Proof. — By definition, for any $c \in H^1(S,\mathbb{R})$, $c = [(h_\omega(c)+\overline{h_\omega(c)})/2]$ and

$$(c_1,c_2)_\omega = \mathrm{Re}\,(h_\omega(c_1),h_\omega(c_2)).$$

On the other hand, since the cohomology classes $c \in H^1(S,\mathbb{R})$ are interpreted as parallel (locally constant) sections of $H^1_g(\mathbb{C})$ with respect to the Gauss-Manin connection, we have

$$D_{H^1_g(\mathbb{C})}h_\omega(c) = -D_{H^1_g(\mathbb{C})}\overline{h_\omega(c)} = -\overline{D_{H^1_g(\mathbb{C})}h_\omega(c)}.$$

Since the Gauss-Manin connection is compatible with the Hermitian intersection form, one gets

$$\begin{aligned}
\mathcal{L}(c_1,c_2)_\omega &= (D_{H^1_g(\mathbb{C})}h_\omega(c_1),h_\omega(c_2)) + (h_\omega(c_1),D_{H^1_g(\mathbb{C})}h_\omega(c_2)) \\
&= -(\pi_1\overline{D_{H^1_g(\mathbb{C})}h_\omega(c_1)},h_\omega(c_2)) - (h_\omega(c_1),\pi_1\overline{D_{H^1_g(\mathbb{C})}h_\omega(c_2)}) \\
&= -(\overline{A_\omega(h_\omega(c_1))},h_\omega(c_2)) - (h_\omega(c_1),\overline{A_\omega(h_\omega(c_2))}) \\
&= \overline{(A_\omega(h_\omega(c_1)),\overline{h_\omega(c_2)})} + \overline{(\overline{h_\omega(c_1)},A_\omega(h_\omega(c_2)))}
\end{aligned}$$

Because $(h_\omega(c_1),\overline{h_\omega(c_2)}) = 0$ (as $H^{1,0}$ and $H^{0,1}$ are orthogonal), one also gets

$$\begin{aligned}
0 &= (D_{H^1_g(\mathbb{C})}h_\omega(c_1),\overline{h_\omega(c_2)}) + (h_\omega(c_1),\overline{D_{H^1_g(\mathbb{C})}h_\omega(c_2)}) \\
&= (A_\omega(h_\omega(c_1)),\overline{h_\omega(c_2)}) + (h_\omega(c_1),\overline{A_\omega(h_\omega(c_2))})
\end{aligned}$$

By putting these two equations together, one sees that the desired result follows. □

The relevance of this lemma to the study of derivative of the Teichmüller flow resides in the fact that, from the definitions, it is not hard to see that the Kontsevich-Zorich cocycle is simply the parallel transport with respect to Gauss-Manin connection of cohomology classes along the orbits of the Teichmüller flow. In other words, the previous lemma says that the infinitesimal change of inner-products (and/or norms) of cohomology classes under Kontsevich-Zorich cocycle is driven by the second fundamental form. In particular, it is important to be able to compute the second fundamental form $A = A_\omega$ in a more explicit way. This is the content of the following lemma due to Giovanni Forni [F1, Lemma 2.1, Lemma 2.1'].

LEMMA 1.2. — *Denote by $B = B_\omega$ the bilinear form*

$$B_\omega(\alpha,\beta) := \frac{i}{2}\int_S \frac{\alpha\beta}{\omega}\overline{\omega}, \quad \alpha,\beta \in H^{1,0}(S,\omega).$$

Then, the second fundamental form $A = A_\omega$ can be expressed in terms of $B = B_\omega$ as

$$(A_\omega(\alpha),\overline{\beta}) = -B_\omega(\alpha,\beta)$$

for all $\alpha,\beta \in H^{1,0}(S,\omega)$.

We refer to G. Forni's article for the proof of this lemma. At this stage, we dispose of all elements to study exhibit special ("totally degenerate") orbits of the Teichmüller flow (and actually $SL(2,\mathbb{R})$-action) on the moduli space of Abelian differentials.

2. Two totally degenerate $SL(2,\mathbb{R})$-orbits

We consider the family of curves (Riemann surfaces) defined by the algebraic equations

$$M_3 = M_3(x_1,\dots,x_4) = \{y^4 = (x-x_1)\dots(x-x_4)\}$$

and

$$M_4 = M_4(x_1,\dots,x_4) = \{y^6 = (x-x_1)^3(x-x_2)(x-x_3)(x-x_4)\}$$

with $x_1,x_2,x_3,x_4 \in \overline{\mathbb{C}}$. The family M_3 (resp. M_4) consists of genus 3 (resp. 4) Riemann surfaces. We equip the Riemann surface

$$\{y^4 = (x-x_1)\dots(x-x_4)\}$$

with the Abelian differential $\omega_3 = dx/y^2$, and the Riemann surface

$$\{y^6 = (x-x_1)^3(x-x_2)(x-x_3)(x-x_4)\}$$

with the Abelian differential $\omega_4 = (x-x_1)dx/y^3$.

A quick computation reveals that ω_3 is an Abelian differential with 4 simple zeroes while ω_4 is an Abelian differential with 3 double zeroes. In other words, $(M_3,\omega_3) \in \mathcal{H}(1,1,1,1)$ and $(M_4,\omega_4) \in \mathcal{H}(2,2,2)$.

We claim that the families (M_3,ω_3) and (M_4,ω_4) are $SL(2,\mathbb{R})$-orbits. Indeed, since $(\omega_3)^2 = p_3^*(q_0)$ and $(\omega_4)^2 = p_4^*(q_0)$, where $p_l : M_l \to \overline{\mathbb{C}}$, $p_l(x,y) = x$, $l = 3,4$ and $q_0 = \frac{dx^2}{(x-x_1)\dots(x-x_4)}$ is a quadratic differential with 4 simple poles, and the $SL(2,\mathbb{R})$-action commutes with p_l (as $SL(2,\mathbb{R})$ acts on moduli space of Abelian differentials by post-composition with charts), we have that (M_3,ω_3) and (M_4,ω_4) are $SL(2,\mathbb{R})$-loci. On the other hand, the Riemann surface structure of $M_3(x_1,\dots,x_4)$ and $M_4(x_1,\dots,x_4)$ are completely determined by the cross-ratio of $x_1,\dots,x_4\in\overline{\mathbb{C}}$ (that is, a single complex parameter), and the choice of an unit Abelian differential on $M_3(x_1,\dots,x_4)$ or $M_4(x_1,\dots,x_4)$ corresponds to a single real parameter, we also have that the loci (M_3,ω_3) and (M_4,ω_4) have the same real dimension as the real Lie group $SL(2,\mathbb{R})$ (namely 3). Putting these two facts together, we obtain the desired claim.

Now, we will proceed to understand the Kontsevich-Zorich cocycle over these two $SL(2,\mathbb{R})$-orbits. Firstly, as we already noticed, the Kontsevich-Zorich cocycle G_t^{KZ} always acts by the usual $SL(2,\mathbb{R})$ representation on $\mathbb{R}^2\simeq\mathbb{R}\cdot[\mathrm{Re}(\omega_l)]\oplus\mathbb{R}\cdot[\mathrm{Im}(\omega_l)]=: H^1_{st}(\omega_l)$, $l=3,4$. Therefore, since G_t^{KZ} preserves the symplectic intersection form on $H^1(S,\mathbb{R})$, our task is reduced to study the restriction of G_t^{KZ} to the symplectic orthogonal

$$H^1_{(0)}(\omega_l):=\{c\in H^1(S,\mathbb{R}): c\wedge[\mathrm{Re}(\omega_l)]=c\wedge[\mathrm{Im}(\omega_l)]=0\}$$

of $H^1_{st}(\omega_l)$, $l=3,4$.

In the case at hand, we dispose of *explicit basis* of $H^1_{(0)}(\omega_l)$ in both cases $l=3,4$. Indeed, we affirm that, after using Hodge's representation theorem to view $H^1_{(0)}(\omega_l)$ inside $H^{1,0}(M_l,\omega_l)$,

$$H^1_{(0)}(\omega_3)=\mathrm{span}\{dx/y^3, xdx/y^3\}$$

and

$$H^1_{(0)}(\omega_4)=\mathrm{span}\{(x-x_1)dx/y^4,(x-x_1)^2dx/y^5,(x-x_1)^3dx/y^5\}$$

In fact, a direct calculation shows that $\{\omega_3:=dx/y^2, dx/y^3, xdx/y^3\}$ is a basis of holomorphic differentials of the genus 3 Riemann surface M_3, and $\{\omega_4:=(x-x_1)dx/y^3,(x-x_1)dx/y^4,(x-x_1)^2dx/y^5,(x-x_1)^3dx/y^5\}$ is a basis of holomorphic differentials of the genus 4 Riemann surface M_4. Thus, it suffices to check that the dx/y^3 and xdx/y^3 are "symplectic orthogonal" to ω_3, and $(x-x_1)dx/y^4,(x-x_1)^2dx/y^5,(x-x_1)^3dx/y^5$ are symplectic orthogonal to ω_4. To do so, we observe that M_l, $l=3,4$, has a natural automorphism

$$T_l(x,y):=(x,\varepsilon_l y)$$

where $\varepsilon_l=\exp(\pi i/(l-1))$, $l=3,4$. Moreover, the holomorphic differentials listed above are eigenforms for the action of T^* whose eigenvalues have the form ε_l^j for some $j\geqslant(l-1)$. Hence, for $l=3$,

$$(dx/y^3)\wedge\omega_3=T_3^*(dx/y^3)\wedge T_3^*(\omega_3)=\frac{1}{\varepsilon_3^5}(dx/y^3)\wedge\omega_3,$$

$$(xdx/y^3)\wedge\omega_3=T_3^*(xdx/y^3)\wedge T_3^*(\omega_3)=\frac{1}{\varepsilon_3^5}(xdx/y^3)\wedge\omega_3$$

and, for $l = 4$,

$$\frac{(x-x_1)dx}{y^4} \wedge \omega_4 = T_4^* \left(\frac{(x-x_1)dx}{y^4} \right) \wedge T_4^*(\omega_4) = \frac{1}{\varepsilon_4^7} \frac{(x-x_1)dx}{y^4} \wedge \omega_4,$$

$$\frac{(x-x_1)^j dx}{y^5} \wedge \omega_4 = T_4^* \left(\frac{(x-x_1)^j dx}{y^5} \right) \wedge T_4^*(\omega_4) = \frac{1}{\varepsilon_4^8} \frac{(x-x_1)^j dx}{y^5} \wedge \omega_3,$$

for $j = 2, 3$. In particular, since $\varepsilon_3^5 = \varepsilon_3 = i \neq 1$, $\varepsilon_4^7 = \varepsilon_4 = \exp(\pi i/3) \neq 1$ and $\varepsilon_4^8 = \varepsilon_4^2 = \exp(2\pi i/3) \neq 1$, we conclude that all wedge products above vanish, and the affirmation is proved.

Finally, the behavior of the Kontsevich-Zorich cocycle G_t^{KZ} restricted to $H^1_{(0)}(\omega_l)$, $l = 3, 4$, is described by the following result:

THEOREM 4. — $G_t^{KZ}|_{H^1_{(0)}(\omega_l)}$, $l = 3, 4$, *is isometric with respect to the Hodge norm.*

This theorem was first proved by G. Forni [F2] in the case $l = 3$, and by G. Forni and the author [FMt] (see also [FMZ1]) in the case $l = 4$. The proof below follows the original arguments in these articles.

Proof. — By Lemmas 1.1 and 1.2, the first order variation (derivative) of the Hodge norm under Kontsevich-Zorich cocycle is controlled by the second fundamental form A_{ω_l}, and A_{ω_l} can be written in terms of an explicit bilinear form B_{ω_l}. In particular, from these lemmas one sees it suffices to show that bilinear form $B = B_{\omega_l}$ vanishes identically on $H^1_{(0)}(\omega_l)$.

To do so, we use a similar strategy to the one applied to prove that $H^1_{(0)}(\omega_l)$ is symplectically orthogonal to $H^1_{st}(\omega_l)$, namely, we use the natural automorphism T_l of M_l to change variables in the integral defining B_{ω_l}.

For sake of concreteness, we perform the calculation in the case $l = 3$ only (leaving the case $l = 4$ as an exercise to the reader). In this situation, we saw that $\alpha = dx/y^3$ and $\beta = xdx/y^3$ span $H^1_{(0)}(\omega_3)$. Hence, since $B = B_{\omega_3}$ is a bilinear form, it suffices to check that

$$B(\alpha, \alpha) = B(\alpha, \beta) = B(\beta, \alpha) = B(\beta, \beta) = 0.$$

By definition,

$$B(\alpha, \alpha) = \int \frac{\alpha\alpha}{\omega_3} \overline{\omega_3} = \int \frac{T_3^*\alpha T_3^*\alpha}{T_3^*\omega_3} \overline{T_3^*\omega_3} = \frac{\varepsilon_3^2}{\varepsilon_3^3 \varepsilon_3^3 \overline{\varepsilon_3}^2} B(\alpha, \alpha) = -B(\alpha, \alpha).$$

Similarly,

$$B(\beta, \beta) = \int \frac{\beta\beta}{\omega_3} \overline{\omega_3} = \int \frac{T_3^*\beta T_3^*\beta}{T_3^*\omega_3} \overline{T_3^*\omega_3} = \frac{\varepsilon_3^2}{\varepsilon_3^3 \varepsilon_3^3 \overline{\varepsilon_3}^2} B(\beta, \beta) = -B(\beta, \beta)$$

and

$$B(\alpha,\beta)=\int\frac{\alpha\beta}{\omega_3}\overline{\omega_3}=\int\frac{T_3^*\alpha T_3^*\beta}{T_3^*\omega_3}\overline{T_3^*\omega_3}=\frac{\varepsilon_3^2}{\varepsilon_3^3\varepsilon_3^3\overline{\varepsilon_3}^2}B(\alpha,\beta)=-B(\alpha,\beta).$$

Thus, $B(\alpha,\alpha)=B(\beta,\beta)=B(\alpha,\beta)=0$ and the argument for the case $l=3$ is complete. □

Actually, the content of this theorem was significantly improved in a subsequent work by J.-C. Yoccoz and the author [MY]: in fact, there it was shown that, in an appropriate basis, the Kontsevich-Zorich cocycle acts through an *explicit finite group* of matrices (whose order is 96 for $l=3$ and 72 for $l=4$) related to "symplectic" subgroups of the automorphism group of D_4 type root systems. In particular, since any finite group of matrices always preserves an inner-product, this allows to (re)derive the fact that the Kontsevich-Zorich cocycle restricted to $H^1_{(0)}(\omega_l)$ acts by isometries.

In general, the fact that the Kontsevich-Zorich cocycle is isometric (on the symplectic orthogonal of H^1_{st}) is very unusual: for instance, the previous theorem ensures, in particular, that the homological action of pseudo-Anosov elements (i.e., the Kontsevich-Zorich cocycle over periodic orbits of Teichmüller flow) contained in the $SL(2,\mathbb{R})$-orbits (M_l,ω_l), $l=3,4$, has only eigenvalues of norm 1 (in $H^1_{(0)}(\omega_l)$). This is in sharp contrast with the well-known fact (among specialists) that the homological action of "typical" pseudo-Anosov elements is highly non-trivial in the sense that its eigenvalues have multiplicity one and they don't lie in the unit circle (in particular no isometric behavior whatsoever).

Therefore, in this sense, the $SL(2,\mathbb{R})$-orbits (M_l,ω_l), $l=3,4$, are "totally degenerate" (as the Kontsevich-Zorich cocycle along them exhibit an unexpected behaviour).

In the literature, the $SL(2,\mathbb{R})$-orbit (M_3,ω_3) is commonly called *Eierlegende Wollmilchsau* due to its really unusual properties. In fact, it turns out that this example was also discovered by F. Herrlich, M. Möller and G. Schmithüsen [HS] (but the motivation [coming from Algebraic Geometry] was slightly different from Forni's one). More recently, after a suggestion of B. Weiss and V. Delecroix, the $SL(2,\mathbb{R})$-orbit (M_4,ω_4) is sometimes called *Ornithorynque*.

The curious reader maybe asking whether these examples can be generalized to provide more "totally degenerate" examples. Firstly, after the works of M. Bainbridge [Ba], and A. Eskin, M. Kontsevich and A. Zorich [EKZ1] that such examples can *not* exist in genus 2. Secondly, it was recently proved by M. Möller [Mo] that besides the previous two examples and maybe some potential new examples in genus 5, there are no further examples of "totally degenerate" (i.e., $G_t^{KZ}|_{H^1_{(0)}}$ is isometric) *closed* $SL(2,\mathbb{R})$! In other words, G. Forni and the author were somewhat lucky to find the sole two known examples of totally degenerate closed $SL(2,\mathbb{R})$-orbits. In the language of Algebraic Geometry, a totally degenerate closed $SL(2,\mathbb{R})$-orbit is called a family of *Shimura and Teichmüller curves* (because the total degeneracy property can be shown to be equivalent to the fact that the family of curves gives rise to totally geodesic curves in *both* moduli spaces of curves and Abelian varieties), or equivalent, the Jacobians of the (genus g) curves in such a family display a *fixed part* of maximal dimension (namely $g-1$). For more comments on this, we refer the reader to M. Möller's article [Mo].

On the other hand, these examples can be generalized to a class of Riemann surfaces giving rise to closed $SL(2,\mathbb{R})$-orbits called *square-tiled cyclic covers* after the works of G. Forni, A. Zorich and the author [FMZ1], and A. Eskin, M. Kontsevich and A. Zorich [EKZ2]. In particular, we know that, although they can't be totally degenerate in general, it is proven in these works that they can be partially degenerate and we can decompose *explicitly* $H^1_{(0)}$ into a direct sum of a degenerate part (where G_t^{KZ} is isometric) and a non-degenerate part. Actually, in these square-tiled cyclic cover examples, the degenerate part is very easy to identify: as a part of a joint work with G. Forni and A. Zorich [FMZ2], we show that this degenerate part is *exactly* the annihilator of the bilinear form $B = B_\omega$ and actually this degenerate part is equivariant with respect to parallel transport with respect to Gauss-Manin connection (in particular, $SL(2,\mathbb{R})$-invariant). In the language of Algebraic Geometry, it follows that, for square-tiled cyclic covers, the degenerate part corresponds exactly to the fixed part of the Jacobians of the associated family of curves.

We end these notes with the following remark:

Remark 2.1. — At the time the talk was delivered (November 25, 2010), it was an open question to know whether the annihilator of $B = B_\omega$ was always invariant under the $SL(2,\mathbb{R})$-action and/or parallel transport with respect to Gauss-Manin connection. About a month later, G. Forni, A. Zorich and the author [FMZ2] found an example of a 5-dimensional $SL(2,\mathbb{R})$-invariant locus of Abelian differentials in genus 10 such that the annihilator of B_ω is *not* $SL(2,\mathbb{R})$-invariant, and, more recently, A. Avila, J.-C. Yoccoz and the author generalized this example to construct a family of loci with interesting *dynamical* properties (e.g., the associated "neutral Oseledets bundle" is *not* continuous).

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BIBLIOGRAPHY

[Ba] M. Bainbridge, *Euler characteristics of Teichmüller curves in genus two*, Geometry and Topology, **11**, 2007, 1887-2073.

[BMW] K. Burns, H. Masur and A. Wilkinson, *The Weil-Petersson geodesic flow is ergodic*, Annals of Math. **175**, 2012, 835–908, `arXiv:1004.5343`.

[EKZ1] A. Eskin, M. Kontsevich, A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, `arxiv:1112.5872`, 2011, 1–106.

[EKZ2] ———, *Lyapunov spectrum of square-tiled cyclic covers*, J. Mod. Dyn., **5**, n. 2, 2011, 319–353.

[F1] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Annals of Math., **155**, no. 1, 2002, 1–103.

[F2] ———, *On the Lyapunov exponents of the Kontsevich–Zorich cocycle*, Handbook of Dynamical Systems v. 1B, B. Hasselblatt and A. Katok, eds., Elsevier, 2006, 549–580.

[FMt] G. Forni and C. Matheus, *An example of a Teichmuller disk in genus 4 with degenerate Kontsevich–Zorich spectrum*, `arXiv:0810.0023v1`, 2008, 1–8.

[FMZ1] G. Forni, C. Matheus and A. Zorich, *Square-tiled cyclic covers*, J. Mod. Dyn., **5**, n. 2, 2011, 285–318.

[FMZ2] G. Forni, C. Matheus and A. Zorich, *Lyapunov spectrum of equivariant subbundles of the Hodge bundle*, `arxiv:1112.0370`, 2011, 1–63.

[FMZ3] G. Forni, C. Matheus and A. Zorich, *Zero Lyapunov exponents of the Hodge bundle*, `arxiv:1201.6075`, 2012, 1–39.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978. xii+813 pp. ISBN: 0-471-32792-1

[H] J. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics*, Vol. 1, Teichmüller theory, Matrix Editions, Ithaca, NY, 2006. xx+459 pp. ISBN: 978-0-9715766-2-9; 0-9715766-2-9

[HS] F. Herrlich and G. Schmithüsen, *An extraordinary origami curve*, Math. Nachr. **281:2**, 2008, 219–237.

[KZ] M. Kontsevich, A. Zorich, *Connected components of the moduli spaces of Abelian differentials*, Inventiones Mathematicae, **153:3**, 2003, 631–678.

[DM] C. Matheus, *Lyapunov spectrum of Kontsevich-Zorich cocycle on the Hodge bundle of square-tiled cyclic covers I*, post at the mathematical blog "Disquisitiones Mathematicae" (http://matheuscmss.wordpress.com/)

[MY] C. Matheus and J.-C. Yoccoz, *The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis*, J. Mod. Dyn., **4**, n. 3, 2010, 453–486.

[MYZ] C. Matheus, J.-C. Yoccoz and D. Zmiaikou, *Homology of origamis with symmetries*, arxiv:1207.2423, 2012, 1–35.

[Mo] M. Möller, *Shimura and Teichmüller curves*, J. Mod. Dyn. **5**, 2011, 1–32, arXiv:math/0501333v2.

[Z] A. Zorich, *Flat surfaces*, in collection "Frontiers in Number Theory, Physics and Geometry. Vol. 1: On random matrices, zeta functions and dynamical systems"; Ecole de physique des Houches, France, March 9–21 2003, P. Cartier; B. Julia; P. Moussa; P. Vanhove (Editors), Springer-Verlag, Berlin, 2006, 439–586.

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