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LARGE-SCALE ISOPERIMETRY ON LOCALLY COMPACT GROUPS AND APPLICATIONS

Romain Tessera

ABSTRACT. — We introduce various notions of large-scale isoperimetric profile on a locally compact, compactly generated amenable group. These asymptotic quantities provide measurements of the degree of amenability of the group. We are particularly interested in a class of groups with exponential volume growth which are the most amenable possible in that sense. We show that these groups share various interesting properties such as the speed of on-diagonal decay of random walks, the vanishing of the reduced first L^p -cohomology, or the existence of proper isometric actions on L^p whose orbits are almost quasi-isometries.

RÉSUMÉ. — Nous introduisons différentes notions de profil isopérimétrique à grande échelle d'un groupe localement compact, compactement engendré, et moyennable. Ces quantités asymptotiques permettent de mesurer la moyennabilité du groupe. En particulier, nous nous intéressons à la classe des groupes moyennables à croissance exponentielle qui en ce sens sont les "plus moyennables possibles". Nous montrons que ces groupes partagent divers propriétés intéressantes, comme la vitesse de décroissance de la probabilité de retour des marches aléatoires, l'annulation de leur cohomologie- L^p réduite en degré 1, ou bien l'existence d'actions par isométries affines sur L^p dont les orbites sont presque des quasi-isométries.

1. Isoperimetric profiles

Let G be a locally compact, compactly generated group equipped with a left-invariant Haar measure μ . Let S be a compact symmetric generating subset of G , i.e. $\bigcup_{n \in \mathbb{N}} S^n = G$. Equip G with the left-invariant word metric⁽¹⁾ associated to S , i.e. $d_S(g, h) = \inf\{n, g^{-1}h \in S^n\}$.

Let λ be the action of G by left-translations on functions on G , i.e. $\lambda(g)f(x) = f(g^{-1}x)$. When we restrict to functions in $L^p(G)$, λ is called the left regular representation of G on $L^p(G)$.

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⁽¹⁾ To have a real metric we must assume that S is symmetric. However, this assumption does not play any role in the sequel

For any $1 \leq p \leq \infty$, and any subset A of G , define

$$J_p(A) = \sup_f \frac{\|f\|_p}{\sup_{s \in S} \|f - \lambda(s)f\|_p},$$

where f runs over functions in $L^p(G)$, supported in A . We can define two kinds of “ L^p -isoperimetric profile”, depending on if we want to optimize $J_p(A)$ fixing the volume of A , or its diameter. In the first case, we obtain what is often called the L^p -isoperimetric profile (see for instance [1]),

$$j_{G,p}(v) = \sup_{\mu(A)=v} J_p(A).$$

In the second case, we obtain what we call the L^p -isoperimetric profile inside balls since it is given by

$$J_{G,p}^b(n) = \sup_{x \in X} J_p(S^n).$$

As an easy generalization of Følner’s characterization of amenability, the group G is amenable if and only if

$$\lim_{n \rightarrow \infty} J_{G,p}^b(n) = \infty \quad (\text{resp.} \quad \lim_{n \rightarrow \infty} j_{G,p}(n) = \infty).$$

Hence, the growth of these functions can be interpreted as a measurement of the amenability of G . The results recalled in the next section support this interpretation. First, we need to introduce some notation.

Asymptotic behavior

We will be interested in the “asymptotic behavior” of these nondecreasing functions. Precisely, let $f, g : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be nondecreasing functions. We write respectively $f \preceq g$, $f \prec g$ if there exists $C > 0$ such that $f(t) = O(g(Ct))$, resp. $f(t) = o(g(Ct))$ when $t \rightarrow \infty$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation \approx .

2. Monotonicity of the isoperimetric profiles and invariance under quasi-isoperimetry

In [19], we proved the following result, inspired from [5]

THEOREM 2.1. — *Let (G, S) and (H, T) be two compactly generated, locally compact groups, equipped with symmetric generating subsets S and T respectively. Then, the asymptotic behaviours of $j_{G,p}$, $J_{G,p}^b$, for any $1 \leq p \leq \infty$ does not depend on S . Moreover, if G and H are both unimodular, and if G is quasi-isometric to H , then*

$$j_{G,p} \approx j_{H,p},$$

and

$$J_{G,p}^b \approx J_{H,p}^b.$$

Hence, $j_{G,p}$ and $J_{G,p}^b$ can be considered as “large scale properties of the group G ”. On the other hand, it is natural to think that closed subgroups and quotients are not “less amenable” than the group itself. This is illustrated by the following results, that we proved in [20].

THEOREM 2.2. — *Let H be a closed, compactly generated subgroup of G and let $1 \leq p \leq \infty$. Then,*

- *if H is unimodular, then*

$$j_{G,p} \preceq j_{H,p};$$

- *if H is not distorted in G , then*

$$J_{G,p}^b \preceq J_{H,p}^b.$$

The first statement was known for finitely generated groups [8]. The case of quotients is much easier to prove and was also known in the discrete case.

THEOREM 2.3. — *Let G and Q be two compactly generated locally compact groups and let $\pi : G \rightarrow Q$ be a surjective continuous homomorphism. Let $1 \leq p \leq \infty$. Then, $j_{G,p} \preceq j_{Q,p}$, and $J_{G,p}^b \preceq J_{H,p}^b$.*

3. Elementary solvable groups

By a theorem of Coulhon and Saloff-Coste [6], if G is a compactly generated, locally compact group with exponential growth, then $j_{G,p}(t) \preceq \log t$. On the other hand it is very easy to see that $J_{G,p}(t) \leq 2t$. Our main result in [20] is to prove that the converse inequalities are true for certain classes of groups. Note that $J_{G,p}(r) \leq j_{G,p}(V(r))$. So, in particular, if the group has exponential growth, $J_{G,p}^b(t) \succeq t$ implies $j_{G,p} \succeq \log t$.

DEFINITION 3.1. — *The class of elementary solvable groups ES is the class of all quotients of compactly generated closed subgroups of finite products of groups of triangular matrices $T(d, k)$ for any integer d and any local field k .*

Note that the class *ES* does not only contain linear groups as shown by the following example due to Hall [10]. Fix a prime q and consider the group of upper triangular 3 by 3 matrices:

$$G = \left\{ \begin{pmatrix} q^n & 0 & x & z \\ 0 & q^{-n} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbf{Z}[1/q]; n \in \mathbf{Z} \right\}.$$

Taking the quotient by the central infinite cyclic subgroup of unipotent matrices $I + mE_{1,4}$ where $m \in \mathbf{Z}$, we obtain an elementary solvable group which is non-residually finite since its center is isomorphic to $\mathbf{Z}[1/q]/\mathbf{Z}$.

DEFINITION 3.2. — *The class GES of geometrically elementary solvable groups is the smallest class of locally compact groups*

- *containing unimodular closed compactly generated subgroups of the group $T(d, k)$, for any integer d and any local field k ;*
- *stable under taking finite products, quotients, and unimodular closed compactly generated subgroups;*
- *stable under quasi-isometry.*

The class GES contains the (non necessarily solvable) lamplighter groups $F \wr \mathbf{Z} = F(\mathbf{Z}) \rtimes \mathbf{Z}$, where F is a finite groups. Namely, such a group is trivially quasi-isometric to any $F' \wr \mathbf{Z}$ where F' has same cardinality as F . If F' is a product of \mathbf{Z}/q for finitely many primes q , then $F' \wr \mathbf{Z}$ is a subgroup in a finite product of lamplighter groups $\mathbf{Z}/q \wr \mathbf{Z}$. On the other hand, one can easily embed $\mathbf{Z}/q \wr \mathbf{Z}$ as a discrete subgroup of the algebraic unimodular group $(k_q \times k_q) \rtimes k_q^*$ over the local field $k_q = \mathbf{Z}/q((X))$.

In [20], we prove

THEOREM 3.3. — *Let G be an geometrically elementary solvable group with exponential growth. Then, for every $1 \leq p \leq \infty$,*

$$j_{G,p}(t) \approx \log t.$$

This result was known for connected amenable Lie groups [15], for the lamplighter and other particular examples [3]. To prove Theorem 3.3, we establish a stronger result for the group of triangular matrices $T(d, k)$ over a local field k , i.e. that $J_{G,p}^b(t) \succeq t$.

Restricting to groups with exponential growth, we obtain

COROLLARY 3.4. — *Let G be an geometrically elementary solvable group with exponential growth. Then, for every $1 \leq p \leq \infty$,*

$$j_{G,p}(t) \approx \log t.$$

By a theorem of Mustapha [12], if k is a p -adic field, then closed subgroups of $T(d, k)$ whose Zariski closure is compactly generated are non-distorted. We therefore obtain

THEOREM 3.5. — *Let k be a p -adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d, k)$, whose Zariski closure is compactly generated. Then, for every $1 \leq p \leq \infty$,*

$$J_{p,G}^b(t) \approx t.$$

In particular these groups have controlled Følner sequences (see [17]).

In [17], we proved it for connected amenable Lie groups, lamplighter groups, and solvable Baumslag-Solitar groups.

4. Random walks

Let G be a locally compact, compactly generated group. Let (X, μ) be a quasi-transitive G -space, i.e. a locally compact Borel measure space on which G acts measurably, co-compactly, properly, and almost preserving the measure μ , i.e.

$$\sup_{g \in G} \sup_{x \in X} \frac{d(g \cdot \mu)}{d\mu}(x) < \infty.$$

For every $x \in X$, let ν_x be a probability measure on X which is absolutely continuous with respect to μ . We assume that there exist $S \subset S'$, two compact generating subsets of G , and a compact subset of X satisfying $GK = X$, such that for every $x \in X$, the support of ν_x is contained in $gS'K$, for some $g \in G$ such that $x \in gSK$. Let us also suppose that $\nu_x(y)$ is larger than a constant $c > 0$ for y in gSK . Denote by P the Markov operator on $L^2(X)$ defined by

$$Pf(x) = \int f(gy) d\nu_x(y).$$

DEFINITION 4.1. — *With the previous notation, we call (X, P) a quasi- G -transitive random walk. Moreover, if P is self-adjoint, then (X, P) is called a symmetric quasi- G -transitive random walk.*

By a slight abuse of notation, we write $dP_x(y) = d\nu_x(y) = p_x(y)d\mu(y)$ and $dP_x^n(y) = p_x^n(y)d\mu(y)$.

In [19], we introduced the notions of large-scale isoperimetric profiles on a metric measure space. We also introduce large-scale equivalences between metric measure spaces, which are general transformations preserving the isoperimetric profile. For instance a quasi-isometry between two locally compact compactly generated groups is a large-scale equivalence. With the previous notation, the measure space (X, μ) can be endowed with a metric that makes it large-scale equivalent to G . Our main result in [19] therefore implies that X and G have the same isoperimetric profile.

Again in [19], we prove a straightforward generalization of [1, Theorem 7.1], which relates $j_{X,2}$ with the asymptotic decay of $\sup_{x \in X} p_x^{2n}(x)$. As a consequence, we derive the following result.

THEOREM 4.2. — *Let G be a geometric elementary solvable group with exponential growth. Then for every symmetric quasi- G -transitive random walk (X, P) , we have*

$$\sup_{x \in X} p_x^{2n}(x) \approx e^{-n^{1/3}}.$$

Classically, we define a random walk directly on the group by taking P to be the convolution by a compactly supported, symmetric probability ν . For instance, take ν to be the uniform probability on S : when the group is discrete, it coincides with the simple random walk on the Cayley graph of G associated to S . In this context, the fact that $p_x^n(x) \approx e^{-n^{1/3}}$ was known for connected amenable Lie groups [15], for finitely generated torsion-free solvable groups with finite Prüfer rank [16], and for the lamplighter group $F \wr \mathbf{Z}$, where F is a finite group in [3]. In [16], they asked if this is also true for the algebraic group $SOL(\mathbf{Q}_p) = (\mathbf{Q}_p \times \mathbf{Q}_p) \rtimes \mathbf{Q}_p^*$, which is obviously a special case the previous Theorem (see also [13] for a purely probabilistic proof).

5. L^p -compression

5.1. Equivariant L^p -compression

Recall that the equivariant L^p -compression rate of a locally compact compactly generated group is the supremum of $0 \leq \alpha \leq 1$ such that there exists a proper isometric affine action σ on some L^p -spaces satisfying, for all $g \in G$,

$$\|\sigma(g).0\|_p \geq C^{-1}|g|_S^\alpha - C,$$

for some constant $C < \infty$, $|g|_S$ being the word length of g with respect to a compact generating subset S .

It follows from [17, Corollary 13], that a group with linear L^p -isoperimetric profile inside balls have equivariant L^p -compression $B_p(G) = 1$. Hence, we obtain

THEOREM 5.1. — *Let k be a q -adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d, k)$, whose Zariski closure is compactly generated. Then, $B_p(G) = 1$ for any $1 \leq p \leq \infty$.*

Non-equivariant L^p -compression

Recall that the L^p -compression rate of a metric space (X, d) is the supremum of all $0 \leq \alpha \leq 1$ such that there exists a map F from X to some L^p -space satisfying, for all $x, y \in X$,

$$C^{-1}d(x, y)^\alpha - C \leq \|F(x) - F(y)\|_p \leq d(x, y),$$

for some constant $C < \infty$.

Another theorem of Mustapha [12] says that an algebraic compactly generated subgroup of $GL(d, k)$, where k is a q -adic field, is non-distorted in $GL(d, k)$. As $T(d, k)$ is co-compact in $GL(d, k)$ and satisfies $B_p(T(d, k)) = 1$, we obtain

THEOREM 5.2. — *Let k be a q -adic field. Let G be an algebraic compactly generated subgroups of $GL(d, k)$. Then, the L^p -compression rate of G satisfies $R_p(G) = 1$ for any $1 \leq p \leq \infty$.*

In particular, symmetric spaces have L^p -compression 1.

6. L^p -cohomology

Recall that the first reduced cohomology of a compactly generated locally compact group G with values in a representation π on some Banach space E , is the space of affine actions of G , with linear part π , modulo those actions σ which admit a sequence v_n of almost fixed points, i.e.

$$\|\sigma(g)v_n - v_n\| \rightarrow 0,$$

uniformly on compact subsets.

In [18], we show that this definition is actually essentially equivalent (one just need to make some adjustments when the group is not unimodular)

to the usual definitions of first reduced L^p -cohomology for metric spaces (and that for connected Lie groups, it coincides with the usual notion for Riemannian manifolds).

In [18, Theorem 1], we proved that groups with controlled Følner sequences (i.e. $J_{G,1}^b(t) \approx t$) have trivial reduced cohomology with values in the left regular representation on $L^p(G)$, for $1 < p < \infty$. We therefore obtain

THEOREM 6.1. — *Let k be a p -adic field. Let G be a quotient of a closed compactly generated subgroups of $T(d, k)$, whose Zariski closure is compactly generated. Then for every $1 < p < \infty$, $\overline{H}^1(G, \lambda_{G,p}) = 0$.*

Note that Cheeger and Gromov proved [4] that the first reduced ℓ^2 -cohomology (i.e. the ℓ^2 -Betti numbers) vanish in all degrees for an amenable group. Gromov conjectures that this would extend to $p > 1$. In spite of groups with infinite center [11], this is still wide open.

In [18], we use the above theorem in the case of connected Lie groups, together with a result of Pansu [14, Théorème 1] to deduce a characterization of homogeneous Riemannian manifolds which are Gromov hyperbolic

THEOREM 6.2. — *Let M be a homogeneous Riemannian manifold. Then the following dichotomy holds.*

- *Either M is non-elementary Gromov hyperbolic, and then it has non-trivial first reduced L^p -cohomology for p strictly larger than a critical value $p_0 \geq 1$.*
- *or the first reduced L^p -cohomology is trivial for all $p > 1$.*

Questions

In the sequel, G denotes a locally compact compactly generated group.

Question 6.3. — We conjecture that all (geometrically) elementary solvable satisfy $J_{G,p}^b(t) \approx t$.

Question 6.4. — Is the class ES, (resp. GES, of groups having controlled Følner pairs) stable under extension?

Question 6.5. — Is every group satisfying $j_{G,p} \succeq \log t$ geometrically elementary solvable? Or better: is it quasi-isometric to an elementary solvable group?

Note also that all the known amenable groups with equivariant Hilbert compression rate $B_2(G) = 1$ are quasi-isometric to elementary solvable groups.

Gromov remarked (see [7]) that if G is amenable, then $B_2(G) = R_2(G)$. In particular, $B_2(G)$ is invariant under quasi-isometry among amenable groups.

Question 6.6. — Do we have $B_p(G) = R_p(G)$ for amenable groups, and for all $1 \leq p, \infty$?

Question 6.7. — Let H be a closed compactly generated subgroup of G . Do we have $B_p(G) \leq B_p(H)$ (resp. $R_p(G) \leq R_p(H)$) for all $1 \leq p, \infty$?

This would be especially interesting for $p = 2$ and for amenable groups, as $B_2(G)$ could be interpreted as a geometric measurement of the amenability of G .

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